Vectors, Parametric Equations and Three Dimensional Coordinates

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To the Student

In your mathematical career, you have been able to make good use of the formula $y = mx + b$ for a line in the plane. The main goal of the next two weeks will be to expand your repertoire of formulas for lines and planes in three dimensional space.

One of the main tools we will use is the idea of a “vector.” A vector is designed to represent quantities with both size and direction. Vectors are used to represent velocities and forces in many different applications: wind speeds, ocean currents, the forces acting on a bridge span, the motions of subatomic particles and galaxies. Vectors can also represent prices on the stock market, changes in political opinions, DNA sequences, and much much more. The operations on vectors that you learn in this module will have meaning in each of these different contexts. We hope that this short unit will give you a taste for these possibilities, and that you will seek to learn more.

Each day there will be a batch of problems for you to work on and think about before class. If you read a problem and don’t know how to do it, read it again more slowly. Sometimes drawing a picture of what it says can help. If you’ve thought about it for five minutes and are still stuck, go on to the next problem and come back to it later. During class you will have an opportunity to ask questions and find out the answer.
Chapter 1

Problem Sequence

1.1 Day 1

Work on the following problems outside of class, on your own. Bring your ideas and solutions to class to discuss.

**Problem 1.** Let \( f(t) = (t - 1, 3 - t) \). Sketch the points \( f(0), f(1), f(2), f(3), \) and \( f(4) \) in the plane. What do you notice?

**Problem 2.** Suppose that \( f(t) \) in the previous problem represents the position of a ferret at time \( t \).

(a) In what direction is the ferret traveling?

(b) What is the speed of the ferret?

**Problem 3.** Imagine you have a young cousin who doesn’t yet know about \((x,y)\) coordinates in the plane. How would you explain it to them? As an example, how do you describe plotting the point \((-2,3)\)?

**Problem 4.** We would like to introduce a similar way of locating points in three dimensions. We use three axes, which we call the \( x \), \( y \), and \( z \) axes. In mathematics we typically use the convention that the positive \( z \) axis points up, the positive \( y \) axis points to the right, and the positive \( x \) axis points out of the page towards us.

Label the axes in the picture below. (Imagine looking into the corner of a room.)
Problem 5. Plot the following points on the three-dimensional coordinate system above: (1, 4, 2), (−3, 0, −5), and (0, −3, 2).

Problem 6. Draw a new 3D coordinate system, label the axes, and sketch these planes.

(a) \( z = 5 \)

(b) \( x = 2 \)

(c) \( y = -3 \)

Problem 7. In two dimensions, two lines must either intersect or be parallel. Is the same thing true in three dimensions? What about two planes, or a line and a plane? Sketch and/or describe the different possibilities.

In your mathematical career, you have been able to make good use of the formula \( y = mx + b \) for a line in the plane. The main goal of the next two weeks will be to expand your repertoire of formulas for lines and planes in three dimensional space.
1.2 Day 2

Definition 1. A vector is an arrow with a direction and a length ("magnitude") but no fixed position. We can represent a vector in the plane as an ordered pair \( \langle a,b \rangle \).

Problem 8. The velocity of the ferret from Problem 2 is (fill in the blanks) \( \langle \quad , \quad \rangle \).

Problem 9. The vector from \( \langle 1,3 \rangle \) to \( \langle 5,-2 \rangle \) is (fill in the blanks) \( \langle \quad , \quad \rangle \). Illustrate with a sketch.

Problem 10. Why are vectors \( u \) and \( v \) in the picture above both \( \langle 2,1 \rangle \)?

Problem 11. The vector from \( \langle a,b \rangle \) to \( \langle c,d \rangle \) is (fill in the blanks) \( \langle \quad , \quad \rangle \).

Problem 12. The vector from \( \langle 0,0 \rangle \) to \( \langle a,b \rangle \) is (fill in the blanks) \( \langle \quad , \quad \rangle \). This is sometimes called the "position vector" of the point \( (a,b) \). Draw a sketch of the point \( (1,5) \) and its position vector.
Problem 13. Where does the plane $2x + 3y + z = 6$ intersect the three coordinate axes? Use this information to sketch the plane.

Problem 14. Sketch and describe the set of points in 3D where $x = 2$ and $y = -3$.

Problem 15. Sketch the triangle with corners $(0,0,0)$, $(1,3,0)$, and $(1,3,2)$. Explain why this is a right triangle, and use the Pythagorean Theorem to find the distance from $(0,0,0)$ to $(1,3,2)$.

Problem 16. Let $q(t) = (2\cos t, 2\sin t)$. Use graph paper to plot the points $q(0)$, $q(\pi/4)$, $q(\pi/2)$, $q(2\pi/3)$, $q(5\pi/6)$, $q(\pi)$, $q(3\pi/2)$, and $q(2\pi)$. What do you notice?

Problem 17. Suppose that a giraffe is traveling along the same path as the ferret from Problem 2, but going twice as fast in the opposite direction. Write an equation or set of parametric equations for the position of the giraffe at time $t$. 
1.3 Day 3

Consider the velocity of the ferret from Problem 2. If we call that \( v \), then the velocity of the giraffe from Problem 17 will be \( -2v \). This is an example of multiplying a vector by a number.

**Problem 18.** Just to make sure we're all together on this. The velocity of the ferret was \( v = \langle a, b \rangle \) and the velocity of the giraffe was \( -2v = \langle -2a, -2b \rangle \).

When we multiply a vector by a number, the number acts by scaling the size of the vector, and if the number is negative, by changing the direction. Because of this scaling effect, numbers in this context are often called *scalars*. When things get complicated, it is often very useful to keep track of which of the things you’re talking about are vectors and which are scalars. Multiplying a vector by a scalar is called *scalar multiplication*.

**Problem 19.** If \( u = \langle 3, 5 \rangle \), then \( 4u = \langle 12, 20 \rangle \).

**Problem 20.** If \( v = \langle a, b \rangle \), then \( kv = \langle ka, kb \rangle \).

**Problem 21.** If \( w = \langle a, b, c \rangle \), then \( kw = \langle ka, kb, kc \rangle \).

**Problem 22.** If \( v = \langle -2, 1, 7 \rangle \), then \( -5v = \langle 10, -5, -35 \rangle \).

**Problem 23.** Sketch and describe the shape with parametric equations \( x = -2 + t, y = 1, \) and \( z = -3 + t \).

**Problem 24.** Does the equation \( z = x - 1 \) describe a line or a plane in three dimensions?

We can write the three equations from Problem 23, as one equation:

\[
(x, y, z) = (-2, 1, -3) + t(1, 0, 1).
\]

This is sometimes called the “vector form” of the equation.

**Problem 25.** Label the point \((-2, 1, -3)\) on your graph for Problem 23. Sketch the vector \(\langle 1, 0, 1 \rangle\), starting from that point.

**Problem 26.** Sketch and describe the set of points in 3D where \( x + y + z = 1 \).

**Problem 27.** (a) What is the distance from \((-1, -2, 4)\) to \((2, 3, -1)\)?

(b) What is the distance from \((a, b, c)\) to \((p, q, r)\)?

**Problem 28.** (a) What is the distance from \((0, 0, 0)\) to \((a, b, c)\)?

(b) What is the length of the vector \(\langle a, b, c \rangle\)?

We use the notation \(|v|\) for the length (or “magnitude”) of a vector.

**Problem 29.** What is \(|\langle 2, -5, 1 \rangle|\)? What is \(|\langle -3, -4 \rangle|\)?
1.4 Day 4

Problem 30. An aircraft carrier is traveling NW at 5 mph. Meanwhile a sailor is walking briskly across the deck from port to starboard, coincidentally also at 5 mph. What is the net (“resultant”) velocity of the sailor? Explain with a picture.

We define vector addition geometrically as the following: If we want to add two vectors, move them so that one of them (say, “v”) starts where the other (call it “w”) ends. Then draw the arrow from the beginning of w to the end of v: that new arrow is \( w + v \).

\[ \text{Problem 31. (a)} \text{ On graph paper, drawing carefully with a ruler, draw two separate arrows and label them } v \text{ and } w. \text{ You can make them whatever vectors you want.} \]

\[ \text{(b)} \text{ Now draw another copy of } v \text{ starting at the end of } w. \text{ (Make sure you don’t change the length or direction!)} \]

\[ \text{(c)} \text{ Now draw } w + v. \]

\[ \text{(d)} \text{ Based on your drawing: What is } |w + v|? \]

Problem 32. (a) Add the vectors \( \langle 2,4 \rangle \) and \( \langle 3,-1 \rangle \) by drawing them as in Problem 31.

\[ \text{(b)} \text{ The vector you get is } \langle 2,4 \rangle + \langle 3,-1 \rangle = \langle \text{ } , \text{ } \rangle. \]

Problem 33. Find a vector in the same direction as \( \langle 1,2,-2 \rangle \) but with length 15.
Definition 2. A unit vector is a vector whose length (a.k.a. magnitude) is 1.

Problem 34. Find a unit vector in the same direction as $\langle 1, 2, -2 \rangle$.

Problem 35. Find a unit vector in the same direction as $\langle -2, 5, 2 \rangle$.

Definition 3. • The unit vector pointing in the positive $x$ direction is called $\mathbf{i}$.

• The unit vector pointing in the positive $y$ direction is called $\mathbf{j}$.

• The unit vector pointing in the positive $z$ direction is called $\mathbf{k}$.

Problem 36. Sketch $\mathbf{i}$ and $\mathbf{j}$ in the plane.

Problem 37. Sketch $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ on a three dimensional coordinate system.

Problem 38. Sketch $-2\mathbf{i}$, $5\mathbf{j}$, and $-2\mathbf{i} + 5\mathbf{j}$ in the plane.

The following is a theorem you may have learned in your trigonometry or precalculus class.

Theorem 1. Law of Cosines Given any triangle with sides of lengths $a$, $b$, and $c$, and having an angle of measure $\alpha$ opposite the side of length $a$, the following equation holds: $a^2 = b^2 + c^2 - 2bc \cos(\alpha)$.

Problem 39. Sketch the vectors $\langle 3, 1 \rangle$ and $\langle 1, 5 \rangle$. Use the Law of Cosines to find the angle between them.

Problem 40. Write parametric equations for the line through the point $(4, 0, -3)$ in the direction of the vector $\langle 5, -2, 1 \rangle$. 

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Problem 41. Sketch the vectors \(\langle -1, -2, 4 \rangle\) and \(\langle 2, 3, -1 \rangle\). Use the Law of Cosines to find the angle between them.

Problem 42. In general, is it true that \(\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle\)? Why or why not?

Problem 43. In general, is it true that \(v + w = w + v\)? Why or why not? Illustrate with a picture.

Problem 44. Is it always true that \(|v + w| = |v| + |w|\)? If so, explain why; if not, give a counterexample.

Problem 45. How should vector subtraction be defined?

Problem 46. How are the vectors \(\langle 3, -7 \rangle\) and \(3i - 7j\) related?

Problem 47. How are the vectors \(\langle a, b, c \rangle\) and \(ai + bj + ck\) related?

Problem 48. If you drop a 10g magnet near a refrigerator, it will not fall straight down but be pulled towards the fridge as it falls. Suppose that the magnetic force is about 4 Newtons, and that acceleration due to gravity is \(9.8 \text{ m/s}^2\).

(a) Draw a diagram representing the two forces on the magnet.

(b) Add the forces to get the total (“resultant”) force on the magnet.

Problem 49. Does the line \(P(t) = (2, 5, 1) + t\langle 3, -1, 2 \rangle\) intersect the plane \(x + 2y - 5z = 10\)? If so where?

Definition 4. If \(v = \langle v_1, v_2, v_3 \rangle\) and \(w = \langle w_1, w_2, w_3 \rangle\) then the dot product of \(v\) and \(w\) is defined by
\[
\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3.
\]

Problem 50. Compute the following dot products.

(a) \(\langle 1, 2, 3 \rangle \cdot \langle 2, -1, 4 \rangle\)

(b) \(\langle 8, -2 \rangle \cdot \langle 1, 5 \rangle\)

(c) \((3i + 2j - k) \cdot (2i - 2j + k)\)

Problem 51. Suppose that \(v\) and \(w\) are vectors. Is \(v \cdot w\) a vector, or a number?

Definition 5. We say the vectors \(v\) and \(w\) are orthogonal if \(v \cdot w = 0\).

Problem 52. Which of these pairs of vectors are orthogonal?
(a) $\langle 3, 4 \rangle$ and $\langle 4, -3 \rangle$

(b) $\langle -1, -1, 4 \rangle$ and $\langle 2, 3, -1 \rangle$

(c) $i + 3j$ and $2i - 6j$

(d) $i$ and $k$
1.6  Day 6

Problem 53.  Find a vector orthogonal to $\langle 1,3,1 \rangle$.

Problem 54.  Is it always true that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$? If so, explain why; if not give a counterexample.

Problem 55.  Is it always true that $\mathbf{u}(\mathbf{w} \cdot \mathbf{v}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{u} \cdot \mathbf{v})$? If so, explain why; if not give a counterexample.

Problem 56.  Show that if $\mathbf{v} = \langle v_1,v_2,v_3 \rangle$ and $\mathbf{w} = \langle w_1,w_2,w_3 \rangle$, then

$$|\mathbf{v} - \mathbf{w}|^2 = |\mathbf{v}|^2 - 2 \mathbf{v} \cdot \mathbf{w} + |\mathbf{w}|^2.$$  

Problem 57.  Use the Law of Cosines (Theorem 1) and Problem 56 to show that for any two vectors $\mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos \theta$$

where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Problem 58.  Use the dot product to find the angle between the vectors $4\mathbf{i} - 2\mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j}$. Illustrate with a sketch.

Problem 59.  If two vectors are orthogonal, what is the angle between them?  

This is also a very important fact!

Problem 60.  Find two vectors orthogonal to $\langle 1,2 \rangle$. How many are there?

Problem 61.  There were many possible answers to Problem 53. Geometrically, what do they represent?

Problem 62.  Write an equation for the line through the points $(5,-3,4)$ and $(2,1,-2)$.

Problem 63.  Is the line through $(-3,-10,5)$ and $(-1,-2,3)$ parallel to the line through $(4,12,1)$ and $(-1,-8,6)$?

Problem 64.  (a) Find two points on the plane $3x - 2y + 5z = 5$ and one point that is not on the plane.

(b) Find a vector in (or parallel to) the plane $3x - 2y + 5z = 5$ and another vector that is not in (or parallel to) the plane.

(c) Is either of these vectors orthogonal to $\langle 3,-2,5 \rangle$?

Definition 6.  If $a$, $b$, $c$, and $d$ are numbers, the $2 \times 2$ determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$  

Problem 65.  Compute the following $2 \times 2$ determinants.
Problem 66. I hung a decorative light on my wall the other day by hanging two hooks on the wall, attaching two ribbons to the light, then tying each ribbon to a hook. It was a little crooked, so one of the ribbons is at an angle of 37 degrees from horizontal and the other is at an angle of 43 degrees from horizontal. The light did not fall down, so the force of tension in the two ribbons must be exactly counteracting the force of gravity on the light (which is about 4 Newtons). Draw a diagram of these three force vectors, and use the fact that their sum is zero to compute the tensions in the two ribbons.
1.7 Day 7

Problem 67. Sketch the two vectors \(5\mathbf{i} - \mathbf{j} + \mathbf{k}\) and \(\mathbf{i} + 5\mathbf{j}\). Are they perpendicular?

Problem 68. Use the dot product to find the angles between the vectors in Problems 39 and 41.

Problem 69. Is it always true that \(c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w}\)? If so, explain why; if not give a counterexample.

Problem 70. Is it always true that \(\mathbf{u} + (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{w})\)? If so, explain why; if not give a counterexample.

Problem 71. Is it always true that \(|\mathbf{u}||\mathbf{v}| = |\mathbf{u} \cdot \mathbf{v}|\)? If so, explain why; if not give a counterexample.

Problem 72. Is it always true that \(\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2\)? If so, explain why; if not give a counterexample.

Problem 73. Suppose that \(P = (p_1, p_2, p_3)\) and \(Q = (q_1, q_2, q_3)\) are points in the plane \(3x - 2y + 5z = 7\).

\(a\) What is \(3q_1 - 2q_2 + 5q_3\)?

\(b\) Write a formula for the vector from \(P\) to \(Q\): \(\overrightarrow{PQ} = \langle \quad, \quad, \quad \rangle\).

\(c\) What is \(\langle 3, -2, 5 \rangle \cdot \overrightarrow{PQ}\)?

\(d\) What is the angle between \(\langle 3, -2, 5 \rangle\) and the plane \(3x - 2y + 5z = 7\)? Illustrate with a sketch.

Problem 74. (a) Looking back at the previous problem, explain why the vector \(\langle a, b, c \rangle\) is always orthogonal to the plane \(ax + by + cz = d\).

\(b\) Is this consistent with the planes in Problems 6, 13, 24, and 26? Draw pictures to illustrate.

Problem 75. (a) Write the equation of a plane perpendicular to the vector \(\langle 5, 1, 2 \rangle\). (There should be many possible choices. Why?)

\(b\) Which plane, perpendicular to the vector \(\langle 5, 1, 2 \rangle\), contains the point \(\langle 1, 2, 2 \rangle\)? Explain your reasoning.

Definition 7. The cross product of the vectors \(\mathbf{v} = \langle v_1, v_2, v_3 \rangle\) and \(\mathbf{w} = \langle w_1, w_2, w_3 \rangle\) is

\[\mathbf{v} \times \mathbf{w} = \left| \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_2 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_2 & w_2 \end{vmatrix} \right| \]

Problem 76. Compute the following cross products.
(a) \( \langle 3, 5, -2 \rangle \times \langle 5, -1, 0 \rangle \)

(b) \( \mathbf{i} \times \mathbf{j} \)

(c) \( (-4\mathbf{i} + 3\mathbf{k}) \times (2\mathbf{i} - 5\mathbf{j}) \)

(d) \( \langle 5, -1, 0 \rangle \times \langle 3, 5, -2 \rangle \)

**Problem 77.** (a) *Is the cross product of two vectors a vector, or a scalar?*

(b) *Which of these makes sense? \((\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}\), or \(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\)?*
1.8 Day 8

Problem 78. Does the line with parametric equations \(x = 2 - t, y = -5 + 4t,\) and \(z = 2 + 3t\) intersect the line with parametric equations \(x = 5 + 2t, y = -3 - t,\) and \(z = 9 + 2t?\) If so, where? If not, why not?

Problem 79. Which, if either, of the lines in Problem 78 contains the point \((1, -1, 5)?\)

Problem 80. Suppose two toy airplanes are flying along the paths given in Problem 78. Do they collide? What do \(x, y, z,\) and \(t\) represent in this situation?

Problem 81. Write the equation for the plane which is perpendicular to the line \((2, -1, 6) + t(-3, 4, 2),\) and contains the point \((1, 4, -7).\) Sketch the line and the plane.

Problem 82. Write an equation for the line through the point \((1, 2, 4)\) which is perpendicular to the plane \(z = -2x - 4y + 12.\) Sketch the line and the plane.

Problem 83. Prove that the cross product of two vectors is perpendicular to the two original vectors. In other words, \(v \times w\) is perpendicular to \(v\) and to \(w.\)

Problem 84. Find a vector perpendicular to both \((5, -3, 1)\) and \((-4, 2, -1).\)

Problem 85. Given two vectors \(v\) and \(w\) in three dimensional space, how many vectors are there that are perpendicular to both \(v\) and \(w?\)

Problem 86. (a) Is it always true that \(v \times w = w \times v?\) Prove or give a counterexample.

(b) Is it always true that \((k v) \times w = v \times (k w) = k(v \times w)?\) Prove or give a counterexample.

Problem 87. (a) Is it always true that \((u + v) \times w = u + (v \times w)?\) Prove or give a counterexample.

(b) Is it always true that \(u \times (v + w) = (u \times v) + (u \times w)?\) Prove or give a counterexample.

Claim 1. The Right Hand Rule. Suppose you have two vectors \(v\) and \(w\) in three dimensional space. If you point the fingers of your right hand along \(v,\) then curl your fingers towards \(w\) (rotate your hand so you can do this without hurting yourself?), then your thumb will point in the direction of \(v \times w.\)

Problem 88. Sketch the vectors from Problem 76 and check that the Right Hand Rule is true in these cases.
1.9 Day 9

Problem 89. Find the equation of the plane containing \((2,3,4), (1,2,3)\) and \((6,-2,5)\).

Problem 90. Find the equation of the plane containing the line \((x,y,z) = (1+2t, -1+3t, 4+t)\) and the point \((1,-1,5)\).

**Definition 8.** It often happens that we have a vector \(v\), and another direction (represented by another vector \(u\)), and we want to write \(v\) as a sum of two vectors, one of which is in the given direction (parallel to \(u\)) and the other of which is perpendicular to \(u\).

For example, when an object is sliding down an inclined plane, the force of gravity pulls the object directly downwards. This force can be divided into a component parallel to the plane (direction of motion) and another component perpendicular to the plane (which is involved in friction).

We call these two pieces components of \(v\). In particular, one is the component of \(v\) in the direction of \(u\) and the other is the component of \(v\) perpendicular to \(u\).

Problem 91. Draw a sketch (in the plane!) of the vectors \(v = -i + 4j\) and \(u = i + j\), and the components of \(v\) parallel and perpendicular to \(u\).

Problem 92. (a) Use your sketch from Problem 91 and some trigonometry to write a formula for the length of the component of \(v\) parallel to \(u\).

(b) Now that you know the length, write the component of \(v\) parallel to \(u\) as a scalar multiple of \(u\). (Hint: Remember Problem 33? )

Problem 93. (a) Compare your formulas in the previous problem with the right hand side of the formula from Problem 57.

(b) Rewrite your formulas using the dot product.

Problem 94. In particular, when we calculate work, only the component of the force that is in the direction of the motion matters. Therefore, in the vector version of “Work equals force times distance,” when we say “times” we mean the dot product.

Find the work done by a (constant!) force of 20N directly downward, in sliding a box a distance of 5m along a plane which is at an angle of 45 degrees.

Problem 95. Consider the intersection of the two planes \(3x - 2y + 6z = 1\) and \(3x - 4y + 5z = 1\).

(a) What kind of shape will this be? How can we write an equation or equations for that kind of shape? What information do we need to do that? Can we get that information from the two plane equations?
(b) Find the equation(s) for the intersection.

**Problem 96.** Draw a sketch of two vectors $\mathbf{v}$ and $\mathbf{w}$ in the plane, based at the same point. Imagine the two vectors are two sides of a parallelogram, and draw the other two sides. Use some trigonometry to write a formula for the area of this parallelogram.

Yesterday we discussed the significance of the direction of the cross product. But the cross product is a vector, which has both direction and magnitude. What does the magnitude represent? Here is the answer, in five complicated steps.

**Problem 97.** (a) Suppose that $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Write out $|\mathbf{a} \times \mathbf{b}|^2$, in gory detail.

(b) Now write out $|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$, in gory detail, and compare with your answer in part a.

(c) Explain why $|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. (Hint: Use Problem 57!)

(d) Putting these three problems together, what can we conclude about $|\mathbf{a} \times \mathbf{b}|$?

**Problem 98.** Find the area of the parallelogram with corners at $(0,0,0)$, $(3,-1,5)$, $(2,2,1)$, and $(5,1,4)$. (Is this a parallelogram?)

**Problem 99.** Find the area of the triangle with corners at $(0,0,0)$, $(3,-1,5)$, and $(2,2,1)$.
Chapter 2

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