Topology

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Throughout this outline, $S$ is a set of points in which there is determined a collection of subsets called \textit{neighborhoods}, satisfying the following two axioms:

\textbf{Axiom 1:} Each point of $S$ is contained in a neighborhood.

\textbf{Axiom 2:} If $A$ is a neighborhood of a point $p$, and $B$ is a neighborhood of $p$, then there exists a neighborhood $C$ of $p$ such that $A \cap B$ contains $C$.

\textbf{Definition 3:} $S$ is called a topological space (or simply a space).

\textbf{Definition 4:} $A$ is a \textit{neighborhood} of $p$ if $A$ is a neighborhood and $p \in A$.

\textbf{Definition 5:} A point $p$ is an \textit{interior point} of a set $H$ if there exists a neighborhood $A$ of $p$ such that $A \in H$.

\textbf{Definition 6:} A point set $H$ is open if every point of $H$ is an interior point.

\textbf{Proposition 7:} Neighborhoods are open.

\textbf{Proposition 8:} The union of any collection of open sets is open.

\textbf{Proposition 9:} The intersection of two open sets is open.

\textbf{Proposition 10:} $S$ is open.

\textbf{Definition 11:} A subset $H$ of $S$ is closed if $S - H$ is open.

\textbf{Proposition 12:} The union of two closed sets is closed.

\textbf{Proposition 13:} The intersection of two closed sets is closed.

\textbf{Question 14:} Do Propositions 3, 5, and 6 extend to arbitrary collections?

\textbf{Definition 15:} A point $p$ is a \textit{limit point} of $H$ if every open set containing $p$ contains a point of $H$ distinct from $p$.

\textbf{Proposition 16:} A point set is closed if and only if it contains each limit point of the set.

\textbf{Proposition 17:} $S$ is closed.

\textbf{Proposition 18:} The empty set is both open and closed.

\textbf{Proposition 19:} A point $p$ is a limit point of $H$ if and only if each neighborhood of $p$ contains a point of $H$ distinct from $p$.

\textbf{Definition 20:} The closure $\overline{H}$ of a point set $H$, is the union of $H$ and the set of all limit points of $H$.

\textbf{Proposition 21:} $\overline{H}$ is the intersection of all closed supersets of $H$.

\textbf{Corollary 21.1:} $\overline{H}$ is closed.
Definition 22: The interior (Int $H$) of a point set $H$ is the set of interior points of $H$.

Proposition 23: Int $H$ is the union of all open subsets of $H$.

Definition 24: The boundary ($\text{Bd} \ H$) of $H$ is $\overline{H} - \text{Int} \ H$.

Proposition 25: $\text{Bd} \ H = \overline{H} \cap \overline{S - H}$

Proposition 26: $\text{Bd} \ H$ is empty if and only if $H$ is both open and closed.

Definition 27: Two non-empty point sets $A$ and $B$ are separated if $A \cap \overline{B} = B \cap \overline{A} = \emptyset$.

Definition 28: A point set is connected if it is not the union of two separated sets.

Proposition 29: The unit interval $[0, 1]$ (with the standard topology) is connected. (Assume every nonempty subset of $[0, 1]$ has a least upper bound.)

Proposition 30: $S$ is not connected if and only if there exists a nonempty proper subset $H$ of $S$ such that $\text{Bd} \ H = \emptyset$.

Proposition 31: If $H$ is connected, then so is $H \cup H'$ where $H'$ is in $\overline{H} - H$.

Proposition 32: If each of $H$ and $K$ is connected and $H \cap K \neq \emptyset$, then $H \cup K$ and $H \cap K$ are connected.

Definition 33: A collection $G$ of points sets is coherent if $G$ is not the union of two nonempty subcollections $E$ and $F$ such that $E^* \cap F^* = \emptyset$. ($E^* = \cup \{H : H \in E\}$)

Proposition 34: If $G$ is a coherent collection of connected sets, then $G^*$ is connected.

Proposition 35: If $C$ is a connected set and $G$ is a collection of connected sets no member of which is separated from $C$, then $C \cup G^*$ is connected.

Corollary 35.1: If $C$ is connected and $G$ is a collection of connected sets such that $C \cap H \neq \emptyset$ for each element $H$ of $G$, then $C \cup G^*$ is connected.

Corollary 35.2: $R^n$ (Euclidean n-space) is connected.

Definition 36: A collection $G$ of open sets is an open covering of a set $H$ if $H$ is a subset of $G^*$.

Definition 37: A set $H$ is bicomact if each open covering of $H$ has a finite subcovering.

Proposition 38: The unit interval $[0, 1]$ (with the standard topology) is bicomact.
Proposition 39: A closed subset of a bicom pact space is bicom pact. The union of a finite family of bicom pact subsets of a space is bicom pact.

Corollary 39.1: The bicom pact subsets of $R^3$ are precisely the closed sets with finite diameters.

Definition 40: A family of sets has the finite intersection property (f.i.p) provided each of its finite subfamilies has nonempty intersection.

Proposition 41: A space is bicom pact if and only if each family of closed sets having the f.i.p. has nonempty intersection.

Proposition 42: A subset of $R^n$ is bicom pact if and only if it is closed and has finite diameter.

Definition 43: A topological space is regular if for each point $p$ and each neighborhood $N$ of $p$, there exists a neighborhood $M$ of $p$ such that $\overline{M}$ is a subset of $N$.

Proposition 44: $S$ is regular if and only if for each point $p$ and each closed set $H$ not containing $p$, there exist open sets $V$ and $W$ such that $H \subset V \subset S - W \subset S - \{p\}$.

Definition 45: A point set $K$ is nowhere dense if $\overline{K}$ does not contain a nonempty open set.

Proposition 46: No bicom pact regular space is the union of countably many nowhere dense sets.

Definition 47: $S$ is a Hausdorff space if for each pair $x, y$ of distinct points of $S$, there exist disjoint open sets $X$ and $Y$ such that $x$ belongs to $X$ and $y$ belongs to $Y$.

Proposition 48: In a Hausdorff space, a set is bicom pact if and only if it is closed.

Proposition 49: A space is Hausdorff if and only if it is regular.

Definition 50: $S$ is normal if for each pair $E, F$ of closed disjoint sets, there exist open sets $V$ and $W$ such that $E \subset V \subset S - W \subset S - F$.

Proposition 51: Every bicom pact Hausdorff space is regular.

Proposition 52: Every regular bicom pact space is normal.

Definition 53: A space is Lindelof if each open covering of the space has a countable subcovering.

Proposition 54: Every Lindelof space is normal.

Definition 55: A component of a point set $X$ is a connected subset of $X$ that is not a proper subset of a connected subset of $X$. 
Proposition 56: Each point of a set $H$ belongs to one and only one component of $H$.

Proposition 57: Is each component of an open set open?

Definition 58: The $p$-component of a set $X$ is the component of $X$ that contains $p$.

Proposition 59: The $p$-component of a point set $X$ is the union of all connected subsets of $X$ that contain $p$.

Proposition 60: If $H$ and $K$ are connected sets, $K \subset H$, and $H - K = A \cup B$ where $A, B$ are separated sets, then $A \cup K$ is connected.

Question 61: Is $K \cup C$ connected when $C$ is a component of $H - K$?

Definition 62: A continuum is a bicomponent connected Hausdorff space.

Proposition 63: Every nondegenerate continuum is uncountable.

Proposition 64: Every nondegenerate connected regular Hausdorff space is uncountable.

Question 65: Does there exist a nondegenerate countable connected Hausdorff space?

Proposition 66: No continuum is the union of a countable ($> 1$) family of nonempty disjoint closed sets.

Proposition 67: Suppose $p$ and $q$ are distinct points of a bicomponent Hausdorff space $M$, and suppose $\{ H_a : a \in A \}$ is an indexed collection of closed sets in $M$ such that for each $a \in A$, $H_a \subset H_{a+1}$, and $H_a$ cannot be separated between $p$ and $q$. Then $\cap \{ H_a : a \in A \}$ cannot be separated between $p$ and $q$.

Definition 68: A continuum $C$ is irreducible between two disjoint sets if $C$ intersects each set and no proper subcontinuum of $C$ intersects both sets.

Proposition 69: Each pair of points of a continuum lies in a subcontinuum irreducible between the two points.

Definition 70: A continuum $H$ is irreducible about a set $K$ if $H$ contains $K$ and no proper subcontinuum of $H$ contains $K$.

Proposition 71: If $K$ is any subset of a continuum $C$, then $C$ contains a subcontinuum irreducible about $K$.

Proposition 72: Suppose $X$ is a bicomponent Hausdorff space, and $p$ and $q$ belong to different components of $X$. Then there exist separated sets $H$ and $K$ such that $p$ belongs to $H$, $q$ belongs to $K$, and $X = H \cup K$.

Question 73: Does Proposition 72 hold when $X$ is not bicomponent?
Proposition 74: Suppose $C$ is a continuum, and $D$ is an open proper subset of $C$ that contains a point $p$. Then $\text{Bd} \ D$ contains a limit point of the $p$-component of $D$.

Definition 75: Suppose that $\{X_n\}$ is a sequence of subsets of $S$. The set of all points $x$ in $S$ such that every open set containing $x$ intersects all but a finite number of elements of $\{X_n\}$ is called the limit inferior of $\{X_n\}$ ($\liminf X_n$). The set of all points $y$ in $S$ such that every open set containing $y$ intersects infinitely many elements of $\{X_n\}$ is called the limit superior of $\{X_n\}$ ($\limsup X_n$).

Proposition 76: $\liminf X_n = \limsup X_n$.

Proposition 77: $\limsup X_n \neq \emptyset$.

Proposition 78: If $\liminf X_n \neq \emptyset$, then $\limsup X_n \neq \emptyset$.

Proposition 79: $\liminf X_n = \liminf \overline{X_n}$ and $\limsup X_n = \limsup \overline{X_n}$.

Proposition 80: $\liminf X_n$ and $\limsup X_n$ are both closed sets.

Proposition 81: If $\{X_n\}$ is a sequence of connected sets in a continuum, and if $\liminf X_n$ is not empty, then $\limsup X_n$ is a continuum.

Definition 82: A function $f$ of $S$ into a space $T$ is continuous provided that if $p$ is a point of the closure of a subset $X$ of $S$, then $f(p)$ belongs to $\overline{f[X]}$. A continuous function is sometimes called a map.

Proposition 83: For real-valued functions of one real variable, this definition is equivalent to the standard $\delta - \epsilon$ definition.

Proposition 84: Suppose $f$ is a function of $S$ into a space $T$. Then $f$ is continuous if and only if for each open set $G$ of $T$, $f^{-1}[G]$ is open in $S$.

Question 85: Does Proposition 84 hold when the word “open” is everywhere replaced by the word “closed”?

Proposition 86: Every continuous image of a connected space is connected.

Proposition 87: Every continuous image of a bicom pact space is connected.

Question 88: Is the Lindelof property a continuous invariant?

Proposition 89: Every continuous image of a Hausdorff space is Hausdorff.

Proposition 90: The composition of two continuous functions is continuous.

Proposition 91: Suppose $X$ is a bicom pact space, $Y$ is a Hausdorff space, and $f$ is a one-to-one map of $X$ onto $Y$. Then $f^{-1}$ is continuous.

Question 92: Does Proposition 91 hold when $X$ is not required to be bicom pact?
Proposition 93: Every line interval $[a, b]$ is an image of $[0, 1]$ under a one-to-one map.

Proposition 94: $S$ is a normal space if and only if for each pair $A, B$ of disjoint closed subsets of $S$, there exists a map $f$ of $S$ into a line interval $[a, b]$ such that $f[A] = a$ and $f[B] = b$.

Proposition 95: For each positive integer $n$, let $f_n$ be a map of $S$ into the real line. Suppose there exists a convergent series of positive numbers $\sum_{n=1}^{\infty} M_n$, such that for each point $x$ of $S$ and each $n$, $|f_n(x)| \leq M_n$. Then for each point $x$ of $S$, the infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges to a number $f(x)$, and the function $f$ so defined is continuous.

Definition 96: $S$ has the continuous extension property provided that for each map $f$ of any closed subset $H$ of $S$ into an interval $[a, b]$, there exists a map $f^*$ of $S$ into $[a, b]$ such that $f^*(x) = f(x)$ for each point $x$ of $H$.

Proposition 97: $S$ is normal if and only if $S$ has the continuous extension property.

Proposition 98: Suppose $S$ is normal, and $f$ is a map of a closed subset $H$ of $S$ into $I^n$ (the Cartesian product of $[0, 1]$ with itself $n$ times). Then there exists an extension $f^*$ of $f$ to all of $S$ (i.e., $f^*: S \to I^n$ and $f^*(x) = f(x)$ for each point $x$ of $H$).

Definition 99: A collection $C$ of open sets is a base for $S$ if for each point $p$ of $S$, and each open set $U$ of $S$ that contains $p$, there exists an element $G$ of $C$ such that $p \in G \subset U$.

Definition 100: The Cartesian product of the indexed collection of sets $\{X_a : a \in A\}$ is the set of functions $\times \{X_a : a \in A\} = \{f : A \to \bigcup\{X_a : a \in A\}\}$ and $f(a) \in X_a$.

Definition 101: The $a$-th projection map is the function $P_a: \times \{X_a : a \in A\} \to X_a$ such that $P_a(x) = x_a$ for every point $x$ of $\times \{X_a : a \in A\}$.

To define the standard topology given to the Cartesian product, called the product topology, we define the base.

Definition 102: A subset $D$ of $\times \{X_a : a \in A\}$ belongs to $D$ if there is a finite subset $\{a_1, a_2, \cdots, a_n\}$ of $A$, and open subsets $D_1, D_2, \cdots, D_n$ of $X_{a_1}, X_{a_2}, \cdots, X_{a_n}$, respectively, such that $D = \{x \in \times \{X_a : a \in A\} : x_{a_i}$ belongs to $D_i$ for $i = 1, 2, \cdots, n\}$. The collection forms a base for the product topology.

Proposition 103: If $D$ is open in $\times \{X_a : a \in A\}$, then $P_a[D]$ is open for every $a$ belonging to $A$.

Proposition 104: A function $f: S \to \times \{X_a : a \in A\}$ is continuous if and only if $P_a f$ is continuous for each $a$ belonging to $A$.
**Proposition 105:** The space $\times \{X_a : a \in A\}$ is Hausdorff if and only if each $X_a$ is a Hausdorff space.

**Proposition 106:** The product of two connected spaces is connected.

**Proposition 107:** The product of two normal spaces is normal.

**Proposition 108:** The product of two bimetric spaces is bimetric.

**Question 109:** Do Propositions 106, 107, and 108 extend to all Cartesian products?

**Definition 110:** A function $d : S \times S \rightarrow [0, +\infty)$ is a metric for $S$ if for every $x$, $y$, and $z$ belonging to $S$:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, y) + d(y, z) \geq d(x, z)$.

**Definition 111:** $S$ is a metrizable space (metric space) if there exists a metric $d$ for $S$ such that the collection $\{N(x, r) : x \in S$ and $r > 0\}$ is a base for the topology of $S$ where $N(x, r) = \{y : d(x, y) < r\}$.

**Proposition 112:** The function $d$ (defined above) is continuous.

**Definition 113:** $S$ is first countable if for each point $p$ of $S$ there exists a countable collection $C$ of open sets (each containing $p$) such that every open set in $S$ that contains $p$ also contains an element of $C$.

**Proposition 114:** $S$ is metrizable if and only if $S$ is first countable.

**Definition 115:** A set is said to be of type $G_\delta$ if it is the intersection of countably many open sets.

**Proposition 116:** Every closed subset of a metric space is of type $G_\delta$.

**Proposition 117:** $S$ is second countable if $S$ has a countable base.

**Proposition 118:** Every bimetric metric space is second countable.

**Definition 119:** A sequence $\{X_n\}$ is a convergent sequence if $\lim \inf X_n = \lim \sup X_n$.

**Proposition 120:** If $M$ is a bimetric metric space, then every sequence of subsets of $M$ has a convergent subsequence.

**Definition 121:** A set $D$ is a dense subset of a set $E$ if $D \subseteq E \subseteq \overline{D}$.

**Definition 122:** $S$ is a separable space if there exists a countable dense subset of $S$. 

Proposition 123: A metric space is separable if and only if it is second countable.

Proposition 124: Proposition 123 holds for all spaces.

Proposition 125: Proposition 120 holds for separable metric spaces.

Proposition 126: A metric space \( M \) is bicontinuous if and only if every infinite subset of \( M \) has a limit point.

Proposition 127: Proposition 126 holds for all spaces.

Proposition 128: If every uncountable subset of a metric space \( M \) has a limit point, then every uncountable subset of \( M \) has a limit point.

Proposition 129: A metric space \( M \) is Lindelöf if and only if every uncountable subset of \( M \) has a limit point.

Proposition 130: Proposition 129 holds for all spaces.

Definition 131: A sequence of points \( p_i \) in a metric space \( (M, d) \) is a Cauchy sequence if for each real number \( r > 0 \), there exists an integer \( N \) such that \( m, n > N \) implies \( d(p_m, p_n) < r \). The metric space \( (M, d) \) is complete if each Cauchy sequence in \( M \) has a limit point in \( M \).

Definition 132: A homeomorphism is a one-to-one map whose inverse is continuous.

Proposition 133: Suppose \((M_1, d_1)\) and \((M_2, d_2)\) are metric spaces, and there exists a homeomorphism of \( M_1 \) onto \( M_2 \) (i.e., \( M_1 \) and \( M_2 \) are homeomorphic). Then \((M_2, d_2)\) is complete if \((M_1, d_1)\) is complete.

Proposition 134: Every closed subspace of a complete metric space is a complete metric space (with respect to the restricted metric).

Proposition 135: Suppose \( D \) is a dense type \( G_δ \) subset of a complete metric space \((M, d)\). Then \( D \) is not the union of countably many nowhere-dense subsets of \( M \).

Question 136: Is \( D \) in Proposition 135 a complete metric space?

Proposition 137: The rationals are not a \( G_δ \) subset of \( R^1 \).

Definition 138: Let \# denote the first uncountable ordinal, and let \([1, \#]\) denote the set of all ordinals up to and including \# (similarly, \([1, \#)\) denotes those up to but not including \#). For every pair of ordinals \( a \) and \( b \) \((a < b)\) in \((1, \#)\), the segment \((a, b) = \{ x : a < x < b \}\) and the sects \([1, a)\) and \((a, \#]\) are neighborhoods.

Proposition 139: If \( C \) is a collection of neighborhoods that cover \([1, \#)\), then there exists \( x < \# \) such that \( \bigcup \{ G \in C : x \in G \} \) covers \([x, \#)\).
Proposition 140: Suppose $f$ is map of $[1, \#)$ into $\mathbb{R}^1$. Then a point $x$ of $[1, \#)$ and a real number $r$ exist such that $f([x, \#)) = r$.

Proposition 141: $[1, \#)$ and $[1, \#]$ are normal.

Proposition 142: $[1, \#]$ is metrizable.

Proposition 143: $[1, \#)$ is metrizable.

Proposition 144: $[1, \#]$ is bicomplete.

Definition 145: A metric space $S$ is complete if there exists a metric $k$ for $S$ such that $(S, k)$ is complete.

Proposition 146: $[1, \#]$ is bicomplete.

Proposition 147: Every infinite subset of $[1, \#)$ has a limit point.

Proposition 148: The product of countably many metric spaces is metrizable.

Proposition 149: The product obtained by crossing $[0, 1]$ with itself countably many times is metrizable.

Proposition 150: Proposition 148 is true if the word “countable” is replaced by the word “uncountable”.

Proposition 151: Every regular second countable Hausdorff space is metrizable.