Topology

Stuart Anderson
Introduction

These are notes from a Texas-style topology course that I took from Dr. Stuart Anderson in Spring 1990 at Texas A & M, Commerce. This was the first Texas-style course I had ever taken.

The course is fairly self-contained, except as noted at the beginning. Near the end of the course, Dr. Anderson spent about a week lecturing on the Axiom of Choice, its equivalence to Zorn’s Lemma and Zermelo’s Theorem, and its relation to Tychonoff’s Theorem.

As with many Texas-style courses, the biggest challenge for the instructor is to keep all the students involved. I have observed Texas-style classes where a few students end up proving all the theorems, or where a few students never present any proofs. One way to prevent this is to split the proof of a theorem into several lemmas. This provides some easier proofs for weaker students. If all else fails, the instructor can always ask a student to have a proof ready for the next class.

Finally, a set of notes for a Texas-style course should be used only as an outline for a course. The instructor should feel free to make changes to suit his class. I have never used the same set of notes twice.

Good luck. These are a great set of notes!

James Ochoa

1 Preliminaries

Basic notation and terminology of sets and functions are assumed.

Theorem 1 (DeMorgan’s Laws): Let $A$ and $B$ be subsets of the set $X$. Then

1. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and
2. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 2: Let $I$ be an indexing set. For each $\delta \in I$, let $A_\delta$ be a set. We define the following two sets:

1. $\bigcup_{\delta \in I} A_\delta = \{x \mid \text{there exists } \delta \in I \text{ such that } x \in A_\delta\}$ and
2. $\bigcap_{\delta \in I} A_\delta = \{x \mid \text{for all } \delta \in I, x \in A_\delta\}$

Theorem 3 (Generalized DeMorgan’s Laws): Let $\{A_\delta \mid \delta \in I\}$ be a collection of subsets of the set $X$. Then

1. $X \setminus \left( \bigcup_{\delta \in I} A_\delta \right) = \bigcap_{\delta \in I} (X \setminus A_\delta)$ and
2. \[ X \setminus (\bigcap_{\delta \in I} A_\delta) = \bigcup_{\delta \in I} (X \setminus A_\delta) \]

**Theorem 4:** Let \( A \) and \( B \) be subsets of the set \( X \). Then

\[ A \setminus B = A \bigcap (X \setminus B) \]

**Theorem 5:** Let \( f : X \to Y \) be a function, and let \( A \) and \( B \) be subsets of \( Y \). Then

1. \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \),
2. \( f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \),
3. \( f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \), and
4. \( f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B) \).

**Theorem 6:** Let \( f : X \to Y \) be a function. Let \( A \) be a subset of \( X \), and let \( B \) be a subset of \( Y \). Then

1. \( A \subset f^{-1}(f(A)) \),
2. \( f(f^{-1}(B)) \subset B \), and
3. \( f(X) \setminus f(A) \subset f(X \setminus A) \).

2 **Theorem Sequence**

**Definition 7:** A topological space \((X, \tau)\) is a set \( X \) and a family of sets \( \tau \) satisfying the following three conditions:

1. the empty set \( \emptyset \), and \( X \), are members of \( \tau \);
2. if \( A \) and \( B \) are in \( \tau \), then \( A \cap B \) is in \( \tau \); and
3. if \( I \) is an indexing set, and \( A_\delta \) is in \( \tau \) for each \( \delta \) in \( I \), then \( \bigcup_{\delta \in I} A_\delta \) is in \( \tau \).

The members of \( \tau \) are called **open sets** and \( \tau \) is called the **topology on** \( X \).

**Definition 8:** Let \((X, \tau)\) be a topological space. A subset \( A \) of \( X \) is called a **closed set** if \( X \setminus A \) is open.

**Theorem 9:** The union of finitely many closed sets is closed. The intersection of an arbitrary family of closed sets is closed.

**Theorem 10:** For any topological space \((X, \tau)\), the sets \( \emptyset \) and \( X \) are closed.

**Theorem 11:** Let \( A \) be a subset of \( X \). Then \( A \) is open if and only if for each \( x \) in \( A \), there is an open set \( O_x \) such that \( x \) is a member of \( O_x \) and \( O_x \) is a subset of \( A \).
Definition 12: Let $(X, \tau)$ be a topological space, and let $A$ be a subset of $X$. The interior of $A$, notated $\text{int}(A)$, is the union of all open subsets of $A$. The exterior of $A$, notated $\text{ext}(A)$, is the union of all open sets not intersecting $A$.

Theorem 13: The interior and exterior operators satisfy the following:

1. $\text{int}(\emptyset) = \emptyset$ and $\text{ext}(\emptyset) = X$,
2. $\text{int}(X) = X$ and $\text{ext}(X) = \emptyset$,
3. $\text{int}(\text{int}(A)) = \text{int}(A)$,
4. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$,
5. $\text{ext}(A \cup B) = \text{ext}(A) \cup \text{ext}(B)$,
6. $\text{int}(A) \subseteq A$ and $\text{ext}(A) \subseteq X \setminus A$, and
7. if $A \subseteq B$, then $\text{int}(A) \subseteq \text{int}(B)$ and $\text{ext}(B) \subseteq \text{ext}(A)$.

Definition 14: Let $x$ be a member of $X$, and let $A$ be a subset of $X$. Then $x$ is said to be a boundary point of $A$ if every open set containing $x$ intersects both $A$ and $X \setminus A$. The set of all boundary points of $A$, notated $\partial A$, is called the boundary of $A$.

Theorem 15: For every subset $A$ of $X$, the sets $\text{int}(A)$, $\text{ext}(A)$, and $\partial A$, are mutually disjoint and their union is $X$. Moreover, $\text{int}(A)$ and $\text{ext}(A)$ are open sets, and $\partial A$ is a closed set.

Theorem 16: A set $A$ is closed if and only if $\partial A \subseteq A$. A set $A$ is open if and only if $\partial A \subseteq X \setminus A$.

Definition 17: The closure of a set $A$, notated $\overline{A}$, is the intersection of all closed sets containing $A$.

Theorem 18: A set $A$ is closed if and only if $\overline{A} = a$.

Theorem 19: If $A$ is a set, then $\overline{A} = \text{int} A \cup \partial A$.

Theorem 20: The closure operator satisfies the following:

1. $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$;
2. $A \subseteq \overline{A}$;
3. $\overline{\overline{A}} = \overline{A}$;
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$; and
5. if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Theorem 21: A point $x$ is in $\overline{A}$ if and only if every open set containing $x$ intersects $A$. 

Definition 22: A point \( x \) is a limit point (also called cluster point or accumulation point) of a set \( A \) if every open set containing \( x \) contains a point of \( A \) different from \( x \). The derived set of a set \( A \), notated \( A' \), is the set of all limit points of \( A \).

Theorem 23: A set \( A \) is closed if and only if \( A' \subseteq A \).

Theorem 24: For any set \( A \), \( \overline{A} = A \cup A' \).

Definition 25: Let \( \tau \) and \( \sigma \) be topologies on \( X \). We say that \( \tau \) is finer (or larger) than \( \sigma \) if \( \sigma \subseteq \tau \). We say that \( \tau \) is coarser (or smaller) than \( \sigma \) if \( \tau \subseteq \sigma \). If \( \tau \subseteq \sigma \) or \( \sigma \subseteq \tau \), then the topologies are said to be comparable. Otherwise, they are not comparable.

Definition 26: A family \( B \) of subsets of a set \( X \) is a base for a topology on \( X \) if the following two conditions are satisfied:

1. for each \( x \) in \( X \), there is a \( B \in B \) such that \( x \in B \); and
2. if \( A \) and \( B \) are in \( B \) and \( x \in A \cap B \), then there is a \( C \) in \( B \) such that \( x \in C \) and \( C \subseteq A \cap B \).

Theorem 27: Let \( B \) be a base for a topology on a set \( X \). Let

\[ \tau = \{ U \mid U \text{ is the union of members of } B \} \]

Then \( \tau \) is a topology on \( X \).

Definition 28: The topology \( \tau \) defined in Theorem 27 is called the topology generated by \( B \).

Theorem 29: The topology generated by the base \( A \) is finer than the topology generated by the base \( B \) if and only if for any \( B \in B \) and any \( x \in B \), there exists an \( A \in A \) such that \( x \in A \) and \( A \subseteq B \).

Theorem 30: A family \( B \) of subsets of \( X \) is a base for a given topology \( \tau \) on \( X \) if and only if the following two conditions are true:

1. for each \( U \) in \( \tau \) and \( x \in U \), there is a \( B \in B \) such that \( x \in B \) and \( B \subseteq U \), and
2. \( B \subseteq \tau \).

Problem 31: (double-check this one!) Let \( X \) be a set. Prove or disprove the following statements.

1. If \( I \) is a set and \( \{ \tau_\delta \mid \delta \in I \} \) is a collection of topologies on \( X \), then \( \bigcap_{\delta \in I} \tau_\delta \) is a topology on \( X \).
2. If \( \tau \) and \( \sigma \) are topologies on \( X \), then \( \tau \cup \sigma \) is a topology on \( X \).
3. If \( \{ \tau_\delta \mid \delta \in I \} \) is a collection of topologies on \( X \), then there exist unique topologies \( \tau \) and \( \sigma \) on \( X \) such that \( \tau \subseteq \tau_\delta \subseteq \sigma \) for all \( \delta \in I \).
Definition 32: A metric space $(X, d)$ is a set $X$ together with a function $d : X \times X \to \mathbb{R}$ which satisfies the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$ for all $x, y \in X$; and
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The function $d$ is called a metric on $X$.

Definition 33: Let $(X, d)$ be a metric space. For $x \in X$ and $r > 0$, the set $B(x, r) = \{ y \mid y \in X \text{ and } d(x, y) < r \}$ is called the $r$-neighborhood (r-ball) about $x$.

Theorem 34: Let $(X, d)$ be a metric space. The collection of all sets $B(x, r)$ such that $x \in X$ and $r > 0$, is a base for a topology on $X$.

Definition 35: The topology generated by $r$-neighborhoods in Theorem 34 is called the metric topology on $X$ generated by $d$.

Definition 36: A topological space $(X, \tau)$ is called metrizable if there is a metric $d$ on $X$ such that the metric topology on $X$ generated by $d$ is $\tau$.

Theorem 37: Let $(X, \tau)$ be a topological space, and $Y \subseteq X$. Let $\tau_Y = \{ Y \cap U \mid U \in \tau \}$. Then $\tau_Y$ is a topology on $Y$.

Definition 38: The topological space $(Y, \tau_Y)$ in Theorem 37 is called the relative (or induced) topology on $Y$. Sets in $\tau_Y$ are called open in $Y$ or open relative to $Y$. Similar terminology is used for closed sets.

Theorem 39: Let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$ and $A \subseteq Y$. Then

1. $A$ is $\tau_Y$-closed if and only if $A = Y \cap F$, where $F$ is a $\tau$-closed subset of $X$;
2. a member $x$ of $Y$ is a $\tau_Y$-limit point of $A$ if and only if $x$ is a $\tau$-limit point of $A$;
3. the $\tau_Y$-closure of $A$ is the intersection of $Y$ and the $\tau$-closure of $A$; and
4. the intersection of $Y$ and the $\tau$-interior of $A$ is a subset of the $\tau_Y$-interior of $A$.

Theorem 40: Let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$ and $A \subseteq Y$. Then

1. if $A$ is closed in $Y$ and $Y$ is closed in $X$, then $A$ is closed in $X$; and
2. if $A$ is open in $Y$ and $Y$ is open in $X$, then $A$ is open in $X$.

Definition 41: A topological space $(X, \tau)$ is connected if $X$ is not the union of two nonempty disjoint open sets. A subset $Y$ of $X$ is connected if $(Y, \tau_Y)$ is connected.
Theorem 42: The space $X$ is connected if and only if the only subsets of $X$ which are both open and closed are $\emptyset$ and $X$.

Theorem 43: Let \( \{ A_\delta \mid \delta \in I \} \) be a collection of subsets of the set $X$. If \( A_\alpha \cap A_\beta \neq \emptyset \), then \( \bigcup \limits_{\delta \in I} A_\delta \) is connected.

Theorem 44: Let $A$ be a subset of $X$. If $A$ is connected and $A \subseteq B \subseteq \overline{A}$, then $B$ is connected.

Definition 45: Let $A$ and $B$ be subsets of $X$. We say that $A$ and $B$ are separated if $A \cap B = \emptyset$.

Theorem 46: If $A$ and $B$ are both closed or both open, then the sets $A \setminus B$ and $B \setminus A$ are separated sets.

Theorem 47: A space $X$ is connected if and only if $X$ is not the union of two nonempty separated sets.

Definition 48: A nonempty subset $C$ of $X$ is said to be a **component** of $X$ if

1. $C$ is connected, and
2. if $A$ is any connected subset of $X$ and $A \cap C \neq \emptyset$, then $A \subseteq C$.

If $x$ is a member of $X$ and $C$ is the component of $X$ such that $x \in C$, then we write $C = C(x)$.

Theorem 49: Let $x$ be a member of $X$. Then the component $C(x)$ is the union of all connected subsets of $X$ containing $x$.

Theorem 50: Let $(X, \tau)$ be a topological space. Then

1. each component of $X$ is closed, and
2. if $A$ and $B$ are distinct components of $X$, then $A$ and $B$ are separated.

We will accept the following theorem without proof.

Theorem 51: The components of $X$ form a partition of $X$ into maximal connected subsets.

Definition 52: A space $X$ is **locally connected** if it has a basis consisting of connected sets.

Theorem 53: If $X$ is locally connected, then the components of open sets are open.

Theorem 54: A space $X$ is locally connected if and only if for each $x$ in $X$ and each neighborhood $U$ of $x$, there exists an open connected set $V$ such that $x \in V$ and $V \subseteq U$. 

Definition 55: Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces, \(f : X \to Y\) be a function, and \(x\) be a member of \(X\). We say that \(f\) is continuous at \(x\) if the inverse image of every open set containing \(f(x)\) is an open set containing \(x\). That is, \(f\) is continuous at \(x\) if for each open set \(V\) containing \(f(x)\), there is an open set \(U\) such that \(x \in U\) and \(f(U) \subseteq V\). We say that the function \(f\) is continuous if \(f\) is continuous at every point in \(X\).

Theorem 56: Let \(f : (X, \tau) \to (Y, \sigma)\). The following six conditions are equivalent:

1. \(f\) is continuous,
2. the inverse image of each open subset of \(Y\) is open in \(X\),
3. the inverse image of each closed subset of \(Y\) is closed in \(X\),
4. the inverse image of each member of a base for \(\sigma\) is open in \(X\),
5. for every subset \(A\) of \(X\), \(f(C) \subseteq f(A)\), and
6. for every subset \(B\) of \(Y\), \(f^{-1}(A) \subseteq f^{-1}(B)\).

Definition 57: A function \(F : (X, \tau) \to (Y, \sigma)\) is a homeomorphism if \(f\) is one-to-one (notated \(1 - 1\)) and onto, and both \(f\) and \(f^{-1}\) are continuous. In this case, \((X, \tau)\) and \((Y, \sigma)\) are said to be topologically equivalent. Any property which when possessed by a space is possessed by all homeomorphic images of that space, is called a topological property or a topological invariant.

Theorem 58: If \(X\) and \(Y\) are topological spaces and \(f\) is a \(1 - 1\) function from \(X\) onto \(Y\), then the following are equivalent:

1. \(f\) is a homeomorphism;
2. if \(G\) is a subset of \(X\), then \(f(G)\) is open in \(Y\) if and only if \(G\) is open in \(X\);
3. if \(F\) is a subset of \(X\), then \(f(F)\) is closed in \(Y\) if and only if \(F\) is closed in \(X\); and
4. if \(E\) is a subset of \(X\), then \(f(E) = f(E)\).

Theorem 59: Let \(X\), \(Y\), and \(Z\) be topological spaces \(f : X \to Y\) and \(g : Y \to Z\). If \(f\) and \(g\) are continuous, then \(g \circ f : X \to Z\) is continuous.

Theorem 60: If \(A\) is a subset of \(X\) and \(f : X \to Y\) is continuous, then \(f|_A : A \to Y\) is continuous.

Theorem 61: If \(X = A \cup B\) where \(A\) and \(B\) are both open (or both closed) in \(X\), and \(f : X \to Y\) is a function such that both \(f|_A\) and \(f|_B\) are continuous, then \(f\) is continuous.

Theorem 62: The continuous image of an connected set is connected. That is, if \(X\) is connected and \(f : X \to Y\) is continuous, then \(f(X)\) is connected.
Definition 63:
1. A space \((X, \tau)\) is called a \(T_0\)-space if for each pair of distinct members of \(X\), there is an open set \(U\) containing one of the members but not the other.
2. A space \((X, \tau)\) is called a \(T_1\)-space if for each pair of distinct members \(x\) and \(y\) of \(X\), there is an open set \(U\) containing \(x\) but not \(y\).
3. A space \((X, \tau)\) is called a Hausdorff \((T_2)\) space if for each open pair of members \(x\) and \(y\) in \(X\), there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\).
4. A space \((X, \tau)\) is called regular if for each closed subset \(K\) of \(X\) and \(x\) in \(X\), there exist disjoint open sets \(U\) and \(V\) such that \(K \subseteq V\) and \(x \in V\).
5. A space is called a \(T_3\)-space if it is both regular and \(T_1\).
6. A space \((X, \tau)\) is called normal if for each pair \(E\) and \(F\) of disjoint closed subsets of \(X\), there exist disjoint open sets \(U\) and \(V\) such that \(E \subseteq U\) and \(F \subseteq V\).
7. A space is called a \(T_4\)-space if it is both normal and \(T_1\).

Theorem 64: A space \((X, \tau)\) is a \(T_1\)-space if and only if singleton sets are closed.

Theorem 65: If \((X, \tau)\) is a Hausdorff space, then
1. each finite set is closed; and
2. \(x\) is a limit point of a subset \(A\) of \(X\) if and only if each open set containing \(x\) contains infinitely many members of \(A\).

Definition 66: A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be open (respectively closed) if the image of each open (respectively closed) set in \(X\) is open (respectively closed) in \(Y\).

Theorem 67: If \((X, \tau)\) is Hausdorff and \(f : (X, \tau) \rightarrow (Y, \sigma)\) is a closed, one-to-one, and onto, then \((Y, \sigma)\) is Hausdorff.

Theorem 68: A space \((X, \tau)\) is regular if and only if for each \(x\) in \(X\) and each open set \(U\) containing \(x\), there exists an open set \(V\) such that \(x \in V\) and \(V \subseteq U\).

Theorem 69: A space \((X, \tau)\) is normal if and only if for each closed set \(K\) and open set \(U\) containing \(K\), there exists an open set \(V\) such that \(K \subseteq V \subseteq \overline{V} \subseteq U\).

Theorem 70: Every metric space is normal.
Theorem 71 (Urysohn’s Lemma): A space $X$ is normal if and only if for each pair of disjoint closed sets $A$ and $B$ in $X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Theorem 72 (Tietze’s Extension Theorem): A space $X$ is normal if and only if whenever $A$ is a closed subset of $X$ and there is a continuous function $f : A \to \mathbb{R}$, there exists a continuous extension of $f$ to all of $X$; that is, there is a continuous function $F : X \to \mathbb{R}$ such that $F|_A = f$. Moreover, if $f$ is bounded, then $F$ may chosen to be bounded also.

Definition 73: A collection of sets $\Phi = \{A_\delta : \delta \in \Delta\}$ is called a covering (cover) of $X$ if $X \subseteq \bigcup_{\delta \in \Delta} A_\delta$. Any subcollection of $\Phi$ which is also a cover of $X$ is called a subcover.

Definition 74: A cover $\Phi$ of the space $X$ is called an open cover of $X$ if each member of $\Phi$ is an open subset of $X$.

Definition 75 (need to think about this one!): A space $(X, \tau)$ is compact if every open cover of $X$ has a finite subcover. A subset $Y$ of $X$ is said to be compact if $(Y, \tau|_Y)$ is compact.

Definition 76: A family of sets $\Psi = \{A_\delta : \delta \in \Delta\}$ has the finite intersection property if the intersection of each finite subfamily of $\Phi$ is nonempty; that is, if $\Psi$ is a finite subset of $\Phi$, then $\bigcap_{\delta \in \Psi} A_\delta$ is a nonempty set.

Theorem 77: A space $X$ is compact if and only if each family of closed subsets of $X$ which has the finite intersection property has a nonempty intersection; that is, if $\Phi$ is a collection of closed subsets of $X$, then $\bigcap_{\delta \in \Phi} A_\delta$ is a nonempty set.

Theorem 78: The continuous image of a compact set is compact.

Theorem 79: A compact subset of a Hausdorff space is closed.

Theorem 80: Disjoint compact subsets of a Hausdorff space have disjoint neighborhoods. That is, if $A$ and $B$ are disjoint compact subsets of a Hausdorff space, then there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Definition 81: If $X$ is compact, $Y$ is Hausdorff, and $f : X \to Y$ is continuous, then $f$ is a closed map.

Theorem 82: A one-to-one continuous function from a compact space onto a Hausdorff space is a homeomorphism.

Definition 83: A family $\varphi$ of subsets of a set $X$ is a subbase for a topology on $X$ if for each $x$ in $X$, there is an $S$ in $\varphi$ such that $x \in S$. 
**Theorem 84:** Let \( \varphi \) be a base for a topology on \( X \). Let \( \mathcal{B} \) be the set of all finite intersections of members of \( \varphi \). Then \( \mathcal{B} \) is a base for a topology on \( X \).

**Definition 85:** Let \( Y \) be a set and let \( \{(X_\alpha, \tau_\alpha) \mid \alpha \in A\} \) be a collection of topological spaces. For each \( \alpha \) in \( A \), let \( f_\alpha \) be a function from \( Y \) into \( X_\alpha \). The smallest topology \( w \) on \( Y \) such that for all \( \alpha \) in \( A \), \( f_\alpha : Y \rightarrow X_\alpha \) is continuous, is called the weak topology on \( Y \).

**Theorem 86:** Let \( Y \) be a set and let \( \{(X_\alpha, \tau_\alpha) \mid \alpha \in A\} \) be a collection of topological spaces. For each \( \alpha \) in \( A \), let \( f_\alpha \) be a function from \( Y \) into \( X_\alpha \). The family \( \varphi = \{f^{-1}(U) \mid U \in T_\alpha \text{ and } \alpha \in A\} \) is a subbase for the weak topology \( w \) on \( Y \).

**Definition 87:**

1. The *Cartesian product* of the collection of sets \( \{X_\alpha \mid \alpha \in A\} \) is the set

\[
\prod_{\alpha \in A} X_\alpha = \{x : A \rightarrow \bigcup_{\alpha \in A} X_\alpha \mid x(\alpha) \in X_\alpha \text{ for each } \alpha \in A\}
\]

2. The set \( X_\alpha \) is called the \( \alpha \)-th coordinate space, and \( x(\alpha) \) is called the \( \alpha \)-th coordinate of \( x \).

3. Let \( \beta \in A \). The function \( P_\beta : \prod_{\alpha \in A} X_\alpha \rightarrow X_\beta \) defined by \( P_\beta(x) = x(\beta) \) is called the \( \beta \)-th projection function.

4. The *product topology* on \( \prod_{\alpha \in A} X_\alpha \) is the weak topology determined by the functions \( P_\alpha \).

**Theorem 88:** Each projection function is an open function.

**Theorem 89:** A product space is connected if and only if each coordinate space is connected.

**Theorem 90 (Alexander’s Sub-base Theorem):** Let \( X \) be a topological space, and let \( \varphi \) be a subbase for the topology on \( X \). If every open cover of \( X \) by members of \( \varphi \) has a finite subcover, then \( X \) is compact.

**Theorem 91 (Tychonoff’s Theorem):** A product space is compact if and only if each coordinate space is compact.