Calculus I - Single Variable Calculus

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To The Student

This class will be taught in a way that may be different from many mathematics classes you have encountered in the past. In this course you will learn about calculus by solving a carefully designed sequence of problems. Solving a problem in this course will always have two components:

(a) Find a solution.

(b) Explain how you know that your solution is correct.

This will help prepare you to use mathematics in the future, when an underlying expectation will likely be that it will be up to you to determine the validity of a mathematical argument.
To The Instructor

This calculus class is taught through problem solving. Wherever possible we have structured the notes so that students work through examples, notice patterns, create conjectures, and prove the resulting theorems. This structure is summarized in the diagram below:

\[
\text{Examples} \xrightarrow{\text{pattern}} \text{Conjecture} \xrightarrow{\text{proof}} \text{Theorem}
\]

Each day, the instructor assigns some problems for students to work on before the next class. The expectation is that students will solve all (or nearly all) of the assigned problems prior to that class. At this next class, the students present their solutions.

And that is basically it! In what follows we will provide more detail about the implementation of this process, as well as suggestions for grading and other course components.

0.1 Institutional Context

Our typical class size is 20-35 students. Most of the students major in math, science, or health-related fields. However, there are usually several students who are majoring in the arts, humanities, or business. It may be worth noting that we do not have engineering or physics programs.

Our classes consist of a mix of students who have already taken Calculus in high school and those seeing the material for the first time. When dealing with students who have seen the material before, the instructor must take care to ensure students are following the instructions in the notes, and not using theorems or techniques before they appear in the notes.

In creating these notes, we intentionally pared down the standard Calculus I curriculum (if there is such a thing!) as far as we thought we could. We think it is important to engage deeply with this material, and in our experience students can fill in the missing pieces later, as needed. This course
concludes with the Fundamental Theorem of Calculus and a brief introduction to the method of substitution. However, an in-depth treatment of integration techniques (including substitution) is reserved for Calculus II.

We used earlier versions of these notes to teach a 4-credit Calculus I course, and had sufficient time at the end of the semester for the instructor to supplement these notes with the topics they felt most important.

We now use these notes to teach a 3-credit Calculus I course, and we are able to cover the entirety of these notes, leaving sufficient time for weekly quizzes, two midterms, and some review.

0.2 Pacing

In order to get through all of the problems in these notes, the class must move at a very brisk pace compared to a typical IBL class. While these numbers can vary depending on the precise problems under consideration, we would ordinarily expect about 10 problems to be presented during a 50-minute class period, or about 15 problems during a 75-minute class period.

The instructor will need to keep this pace in mind at all times. Many problems can be presented with little or no discussion as these problems typically serve to provide data from which to conjecture a general theorem. It is after the first of these sorts of problems and when theorems are conjectured or proved that more in-depth classroom discussions are warranted.

Depending on the mathematical background of students, an instructor considering adopting these notes may find it helpful to supplement these notes with mini-lectures to set the stage for a topic, or to review something from high school (such as trigonometry) before diving into the notes. The instructor can also use mini-lectures to introduce topics or ideas not formally touched on in these notes.

0.3 Classroom Considerations

We prefer to teach this class in rooms that have both abundant white board space and additional portable white boards: this allows over a dozen students to be writing up problems simultaneously. We usually bring extra markers and erasers each day so that we are never short of supplies.

We have also taught in classrooms with inadequate whiteboard space. In these instances we make sure to bring a significant number of portable
whiteboards to the classroom. There are a variety of different portable whiteboard solutions available, and there should be a workable solution that fits virtually any budget constraints.

0.4 The First Day

In our experience, it is good to spend the majority of the first day of class doing mathematics. We typically spend only a few minutes going over logistics (attendance, highlights of the syllabus) and having the students introduce themselves. We leave a detailed reading of the syllabus as homework, and give time on the second day to go over additional questions if necessary.

We hand out the notes, and give students a chance to work on the first few problems alone or in groups. After a while students are chosen (either by volunteering, or by the instructor walking around and seeing who has completed the problems) and they write the solutions up on the board and present them in the order of the problem numbers.

The problems in the first chapter are intended to go quickly and successfully since many students are nervous about presenting in front on their peers. We try to move quickly through this material, and usually end up assigning the rest of the first chapter as the homework to prepare for the second day of class.

We photograph student solutions after the presentations and post them on the course management system. This is described in more detail a little later.

0.5 A Typical Day

A typical day starts by quickly recapping the main results of the previous class, and then assigning who will write up the problems for the day. This can be done in the following way:

- Put the problem numbers on the board: 1.1, 1.2, 1.3, etc. These may be split up (1.10abc, 1.10de) or, in rare cases, two problems may be combined. It is most efficient to decide on how to split up the problems ahead of time, although it can be done on the fly.

- Call on students in order of who has done the fewest presentations to pick an unclaimed problem. Next to each problem, write the name of the student who chose that problem.
• If there are still unclaimed problems after all students have been called on, go through the list of students again.

• As students choose problems, they go to the board and write them up. This takes 10-15 minutes total.

One variation is to call on the students in groups instead of in order (students who have presented fewer than 3 times, then those who have presented 3-5 times, etc.), and have them go up to the board and write their name by an unclaimed problem. This has the advantage of obscuring who has the fewest presentations, but also takes longer at the start of class.

Another variation is to have all students write their name by all problems they can do, and if more than one person signs up for the same problem, the person with the fewest presentations can present it, or in case of a tie, a random number generator can be used.

While students are writing up the problems, the instructor can walk around and do a quick check that problems are correct (if desired - sometimes it is good if that happens in the presentation) or check with students who didn’t volunteer to present to give them some additional instruction with difficulties they may have had.

After the problems have been written up, students take turn presenting their results in the order of the problem numbers. The mathematics should be correct, and the explanations clear. Other students should be encouraged to ask questions, but if there is a mistake and no one notices, the instructor should point it out.

If a proof has small errors, they can be corrected on the spot. This can be done directly, or done indirectly with a comment such as “There is a computation error in the fourth line,” or “The proof is missing an equal sign.”

If the error is more significant, it can be corrected as a class (which may take a while), or the student can make corrections and re-present later in the class or the next day. Waiting and re-presenting later generally works well for problems where the result isn’t needed immediately, while taking time to make corrections as a class works well for problems that are particularly deep and/or challenging.

If time runs out before students have finished presenting, those problems are saved for the next class. If the work is on a portable board it can be saved as-is; otherwise the instructor can photograph the work or students can rewrite it next time.
On the other hand, if there is additional time remaining after all presentations are complete, the instructor can recap the main ideas of that day and the students can begin work on the next batch of problems. A few of those problems may be presented that day if desired.

As the instructor, we not only make sure to point out errors that no one else has caught, but we provide some additional information about writing up the mathematics (e.g. the appropriate use of equal signs) and about mathematical connections to other problems from the course.

0.6 Recording the Information

We take photos of the work that was presented and post these photos in our course management system: whenever possible we take photos shortly after people present, finishing up if necessary at the end of class.

The work usually needs to be erased at the end of class, and we have found that it works best to have a number of students help erase; otherwise, the photographing and erasing can take several extra minutes.

0.7 Grades and Assessment

We have found that there is a great deal of flexibility in how we determine grades. Typically Presentations (or a combination of attendance and presentations) count for 25-50% of the course, and exams make up the majority of the rest.

We have tried a number of different methods for assigning grades for the overall Presentation grade in the course. The most important commonality is that we give full credit for a presentation even if there are errors. We want students to feel comfortable presenting, and also want to acknowledge that mistakes are a normal part of doing mathematics. The only caveat is that if we sense that students are winging it, we reserve the right to see their written work. In such a case, the student will not receive credit.

We all compute the overall presentation grade differently, but it’s typically related to the proportion of times that a person went to the board compared to how often they were called on or to the class average.

We typically give two midterms and a final exam. These exams may be similar to those given in a lecture course or may be more conceptual; likewise, the individual grading schemes vary.
We also usually give weekly quizzes. This seems to help students regularly review material by studying for the quiz, and also provides an ongoing way for them to gauge their progress in the course.

The primary homework is for students to prepare solutions to the assigned problems; however, we have also tried different ways of assigning additional homework:

- No additional homework.
- Assigning occasional additional problems to be turned in as homework.
- Requiring students to carefully write up problems from this packet (that may or may not have been presented) on a daily or weekly basis.
- Using a textbook with an online homework system and regularly assigning online nuts-and-bolts homework (although less than we had with a lecture course)

What we found was that all of these methods worked reasonably well. The one caveat was that if we assigned too much additional homework, students started seeing it (rather than the problems for presentation) as their main responsibility, so less seems to be better than more.

### 0.8 Meeting outside of Class

We have found that students visit our offices a lot more than they did with a lecture course, both in and out of office hours.

### 0.9 Use of Additional Resources

We have encouraged students to work together. Some students prefer to work alone, but many end up forming regular study groups and are better off for it: the individual explanations give individual accountability.

We have not encouraged but also not forbidden the use of books or the internet, as long as students are using it for help rather than just copying answers. We have found that the students who look up information in videos seem to have a hard time explaining their solutions in class, and end up abandoning that strategy on their own.
Chapter 1

Preliminaries

Instructor Note: We recommend printing out a copy of Chapter 1 and handing it out on day 1. We have the students spend half of the first class working together on problems, and the second half presenting solutions. Students leave with the expectation that they should complete the remainder of Chapter 1 for homework, and solutions will be presented on day 2.

We will begin by reviewing some concepts which you may or may not have seen before.

Instructor Note: We are deliberately beginning the course with relatively simple problems. The goal here is simply to get as many students as possible to the board on days 1 and 2, setting the tone for the rest of the semester. Formal definitions will come later as needed. Note also that the problems focusing on domain and range give an excellent opportunity to discuss the various notations for sets.

Problem 1.1. Draw the graph of $y = x^2$.

Problem 1.2. What is the domain of the function $y = x^2$?

Problem 1.3. What is the range of the function $y = x^2$?

Problem 1.4. Draw the graph of $z = x^3$.

Problem 1.5. What is the domain of the function $z = x^3$?

Problem 1.6. What is the range of the function $z = x^3$?

Problem 1.7. Draw the graph of $y = 2x + 1$.

Problem 1.8. What is the domain of the function $y = 2x + 1$?

Problem 1.9. What is the slope of $y = 2x + 1$?
Instructor Note: We mention both domain and range here, because they are closely related. However we will focus mainly on domain, as that is what is central in what follows.

**Problem 1.10.** Consider the function $f(x) = \frac{1}{x}$.

(a) Evaluate $f(1)$.
(b) Evaluate $f(2)$.
(c) Evaluate $f(3)$.
(d) Evaluate $f(1/2)$.
(e) Evaluate $f(1/3)$.

**Problem 1.11.** Let’s continue considering the function $f(x) = \frac{1}{x}$.

(a) Evaluate $f(-1)$.
(b) Evaluate $f(-2)$.
(c) Evaluate $f(-3)$.
(d) Evaluate $f(-1/2)$.
(e) Evaluate $f(-1/3)$.

**Problem 1.12.** Now, draw the graph of $f(x) = \frac{1}{x}$.

**Problem 1.13.** Finally, what is the domain of $f(x) = \frac{1}{x}$?

**Instructor Note:** The function $g(x) = \frac{2x^2 + x}{x}$ below, along with $y = 2x + 1$ from above, will be used again in Chapter 3 in an introduction to limits.

**Problem 1.14.** Consider the function $g(x) = \frac{2x^2 + x}{x}$.

(a) Evaluate $g(1)$.
(b) Evaluate $g(2)$.
(c) Evaluate $g(3)$.
(d) Evaluate $g(-1)$.
(e) Evaluate $g(-2)$.
Problem 1.15. Explain why 0 is not in the domain of the function $g$ above.

Instructor Note: In the following problem, be sure to contrast with the graph of $y = 2x + 1$ from before.

Problem 1.16. Sketch the graph of the function $g$ from above.

Instructor Note: In our experience, the notation of piecewise functions gives students great difficulty, and they may also take much longer to work through than you might expect. However, they are not absolutely necessary for the remainder of the course, and may be excluded if so desired. The only places where piecewise functions come up again are in the context of limits, critical points, and local extrema.

Problem 1.17. Let $h(x) = \begin{cases} x^2 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$

(a) Evaluate $h(-3)$.

(b) Evaluate $h(-2)$.

(c) Evaluate $h(-1)$.

(d) Evaluate $h(0)$.

(e) Evaluate $h(1)$.

(f) Evaluate $h(2)$.

(g) Evaluate $h(3)$.

Problem 1.18. Sketch the graph of $h$ from above.

Problem 1.19. Let $f(x) = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$

(a) Evaluate $f(-3)$.

(b) Evaluate $f(-2)$.

(c) Evaluate $f(-1)$.

(d) Evaluate $f(0)$.

(e) Evaluate $f(1)$.

(f) Evaluate $f(2)$.

(g) Evaluate $f(3)$.
Problem 1.20. Sketch the graph of $f$ from above.

Instructor Note: Our experience is that students are familiar with the absolute value function and its associated definition. However they are not accustomed to thinking about it as a piecewise function. The following problem is meant to elicit that connection. If the wording of the problem causes students some difficulty, it may be appropriate for the instructor to help the students make this connection through a class discussion instead.

Problem 1.21. The function $f$ described above is often referred to by a particular name. What is it? In addition, it has a particular notation. What is it?

Instructor Note: The following problem is simply to get students used to the idea that the break between pieces of a piecewise function does not necessarily occur at 0, and also that functions need not be continuous.

Problem 1.22. Let $w(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ 2x + 1 & \text{if } x > 1. \end{cases}$

(a) Evaluate $w(-3)$.
(b) Evaluate $w(-2)$.
(c) Evaluate $w(-1)$.
(d) Evaluate $w(0)$.
(e) Evaluate $w(1)$.
(f) Evaluate $w(2)$.
(g) Evaluate $w(3)$.
(h) Evaluate $w(0.5)$.
(i) Evaluate $w(1.5)$.
(j) Evaluate $w(0.9)$.
(k) Evaluate $w(1.1)$.

Problem 1.23. Sketch the graph of $w$ from above.

We will conclude this chapter by looking at lines.

Problem 1.24. Find the slope of the line through the points $(1, 3)$ and $(3, 6)$. 

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Problem 1.25. Find the equation of the line through the points \((1, 3)\) and \((3, 6)\).

Problem 1.26. Find the slope of the line through the points \((-2, -4)\) and \((1, -10)\).

Problem 1.27. Find the equation of the line through the points \((-2, -4)\) and \((1, -10)\).

Problem 1.28. Find the slope of the line through the points \((2, 1/2)\) and \((5, 1/3)\).

Instructor Note: The final problem, which obviously has multiple correct answers, is meant to give the class an opportunity for more discussion than might occur in the problems above with a single correct answer. At this point in the course it is important to get students in the habit of sharing and discussing their ideas in class.

Problem 1.29. Suppose a line has slope 3 and passes through the point \((1, 2)\). Give examples of 2 other points the line passes through.
Chapter 2

The Derivative

In this chapter we will use the average rate of change of a function \( f \) to develop a definition for what is known as the derivative of \( f \). This is one of the central concepts in calculus.

**Problem 2.1.** Consider the function defined by \( f(x) = x^2 \).

(a) Sketch the graph of \( f \) over the interval \([0, 6]\).

(b) Draw the line that goes through the points on the graph where \( x = 0 \) and where \( x = 2 \). (This is called the **secant line** between those points.) Calculate the slope of this secant line.

(c) Draw the secant line for \( f \) through the points on the graph where \( x = 2 \) and where \( x = 4 \). Calculate the slope of this secant line.

(d) Draw the secant line for \( f \) through the points on the graph where \( x = 4 \) and where \( x = 6 \). Calculate the slope of this secant line.

The formula for the slope of a (secant) line comes up frequently, and has other interpretations as well. One of them is given below.

**Definition 2.2.** Suppose \( f \) is a function and \( a \) and \( b \) are two numbers in the domain of \( f \). The **average rate of change of \( f \) between \( a \) and \( b \)** is the slope of the secant line between the points \((a, f(a))\) and \((b, f(b))\). That is, the average rate of change over the interval \([a, b]\) is given by

\[
\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}.
\]

**Problem 2.3.** Consider the function defined by \( g(x) = x^3 \).

(a) What is the average rate of change of \( g \) over the interval \([0, 2]\)?

(b) What is the average rate of change of \( g \) over the interval \([2, 4]\)?

(c) What is the average rate of change of \( g \) over the interval \([4, 6]\)?
Problem 2.4. Consider the function defined by $h(x) = 2x + 1$.

(a) What is the average rate of change of $h$ over the interval $[0, 2]$?
(b) What is the average rate of change of $h$ over the interval $[2, 4]$?
(c) What is the average rate of change of $h$ over the interval $[4, 6]$?

Problem 2.5. Consider the function defined by $k(x) = 1 - 4x$.

(a) What is the average rate of change of $k$ over the interval $[0, 2]$?
(b) What is the average rate of change of $k$ over the interval $[2, 4]$?
(c) What is the average rate of change of $k$ over the interval $[4, 6]$?

Instructor Note: The next problem may cause troubles as students may not know how to articulate a good explanation. It is okay to expect better explanations as the course progresses, and not get bogged down at this point.

Problem 2.6. You may have noticed that in some of the cases above, the average rate of change remains the same over different intervals, and in other cases the it changes over different intervals. For which kinds of functions does the average rate of change remain the same regardless of the interval used? Explain your reasoning.

A large focus of first semester calculus is to measure how functions are changing, not just over intervals but at specific points. In terms of lines, this means we will look at something called the tangent line instead of secant lines.

Instructor Note: Without a formal definition of the tangent line, answers to the following problem may be vague or somewhat off-base. This is okay at this point, as the notion of a tangent line will become clearer as students progress through the notes.

Problem 2.7. Consider the graph of $y = f(x)$ below. Which line most closely matches the curve at the point $x = 2$ and at points close to $x = 2$? This line is called the tangent line to $f$ at $x = 2$. 
Instructor Note: In the following problems, students approximate the slopes of tangent lines to \( f(x) = x^2 \) at various points. In Chapter 4 students will compute the exact values of these slopes and be asked to reflect back on these approximations.

**Problem 2.8.** Graph the function \( f(x) = x^2 \) from \( x = -3 \) to \( x = 3 \). Draw the tangent line to \( f \) at \( x = 1 \).

**Problem 2.9.** Still using \( f(x) = x^2 \) from \( x = -3 \) to \( x = 3 \), draw the tangent lines to \( f \) at the following points: \( x = -2, x = -1, x = 0, \) and \( x = 2 \).

**Problem 2.10.** Consider the function \( f(x) = x^2 \) above and the five tangent lines you have now drawn.

(a) Is the slope of the tangent line at \( x = -2 \) less than \(-1\), around \(-1\), between \(-1\) and \(0\), around \(0\), between \(0\) and \(1\), around \(1\), or greater than \(1\)?

(b) Is the slope of the tangent line at \( x = -1 \) less than \(-1\), around \(-1\), between \(-1\) and \(0\), around \(0\), between \(0\) and \(1\), around \(1\), or greater than \(1\)?

(c) Is the slope of the tangent line at \( x = 0 \) less than \(-1\), around \(-1\), between \(-1\) and \(0\), around \(0\), between \(0\) and \(1\), around \(1\), or greater than \(1\)?

(d) Is the slope of the tangent line at \( x = 1 \) less than \(-1\), around \(-1\), between \(-1\) and \(0\), around \(0\), between \(0\) and \(1\), around \(1\), or greater than \(1\)?

(e) Is the slope of the tangent line at \( x = 2 \) less than \(-1\), around \(-1\), between \(-1\) and \(0\), around \(0\), between \(0\) and \(1\), around \(1\), or greater than \(1\)?

We would like to calculate the exact slope of the tangent line, but the fact that we know only one point on that line is problematic.
Problem 2.11. Consider the tangent line to \( f(x) = x^2 \) at \( x = 2 \) from above. There is only one point on that line for which we actually know both the \( x \)- and \( y \)-coordinates. What is that point?

Problem 2.12. Given that you need two points to calculate the slope of a line, and we only know one point on the tangent line, how might we be able to find an approximate value of the slope? Think of as many different ways as you can, and use one to approximate the slope of the tangent line to \( f(x) = x^2 \) at \( x = 2 \). Is your approximation consistent with what you found in Problem 2.10 (e)?

Although there are many ways to approximate the slope of a tangent line, we will now focus on one systematic way that will generalize well.

Problem 2.13. Let’s continue working with \( f(x) = x^2 \).

(a) Compute the slope of the secant line passing through the points on the graph when \( x = 2 \) and when \( x = 3 \).

(b) Compute the slope of the secant line passing through the points on the graph when \( x = 2 \) and when \( x = 2.5 \).

(c) Compute the slope of the secant line passing through the points on the graph when \( x = 2 \) and when \( x = 2.1 \).

The process in the previous problem is can get tedious very quickly. Let’s find a better way!

Problem 2.14. In the previous problems we performed the same calculation 3 times with slightly different \( x \)-values. In preparation for a more general formula, we will examine the interval \([2, 2 + \Delta x] \).

(a) What value of \( \Delta x \) will produce the interval \([2, 3] \)?

(b) What value of \( \Delta x \) will produce the interval \([2, 2.5] \)?

(c) What value of \( \Delta x \) will produce the interval \([2, 2.1] \)?

(d) What interval would \( \Delta x = 0.01 \) produce?

Problem 2.15. Let’s return to working with \( f(x) = x^2 \).

(a) Write down, but do not simplify, the expression for the average rate of change of \( f \) over the interval \([2, 2 + \Delta x] \). Your answer should include the symbol \( \Delta x \).

(b) Simplify your expression from the previous part as far as possible.

(c) Why is it important that \( \Delta x \) be nonzero?
Problem 2.16. Let’s continue to consider \( f(x) = x^2 \). Use the simplified formula from Problem 2.15 and the appropriate value of \( \Delta x \) from Problem 2.14 to compute the following.

(a) Compute the average rate of change of \( f \) over the interval \([2, 3]\). (What value of \( \Delta x \) did you use?)

(b) Compute the average rate of change of \( f \) over the interval \([2, 2.5]\).

(c) Compute the average rate of change of \( f \) over the interval \([2, 2.1]\).

(d) Are the answers here the same as the slopes of the secant lines you found in Problem 2.13 even though the process is different?

Problem 2.17. Once again, let \( f(x) = x^2 \). Look again at the formula you came up with in 2.15. As \( \Delta x \) gets closer and closer to 0, the secant lines through the points where \( x = 2 \) and \( x = 2 + \Delta x \) look more and more like the tangent line to \( f \) at \( x = 2 \). What value does the slope of these secant lines get closer to as \( \Delta x \) gets closer to 0? (This turns out to be the precise value of the slope of the tangent line at \( x = 2 \).)

Problem 2.18. Compare your answer in the previous problem to that in Problem 2.10 (e) and Problem 2.12.

Problem 2.19. Let’s now compute the slope of the tangent line to \( f(x) = x^2 \) at \( x = 1 \).

(a) First, find the average rate of change of \( f \) over the interval \([1, 1 + \Delta x]\). Simplify this expression as much as possible.

(b) Now, what value does your expression in part (a) approach as \( \Delta x \) approaches 0?

(c) What is the slope of the tangent line to \( f \) at \( x = 1 \)?

(d) Is your answer above consistent with your answer to Problem 2.10 (d)?

Instructor Note: The definition of the derivative below mentions existence. Note that we do not discuss situations where derivatives do not exist until later in the notes. At this point students are still just sorting this idea out and we do not feel it would be helpful to wrestle with this subtle idea yet. Of course if it comes up naturally in conversation we would encourage the instructor to foster that discussion.

Definition 2.20. Let \( f \) be a function with the number \( x = a \) in its domain. We say the \textbf{derivative of \( f \) at \( x = a \) exists} if there is a line tangent to the graph of \( f \) at \( x = a \). In this case, we call the slope of this line the \textbf{derivative of \( f \) at \( x = a \)}. 

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Problem 2.21. Let \( f(x) = x^2 \).

(a) What is the derivative of \( f \) at \( x = 1 \)?
(b) What is the derivative of \( f \) at \( x = 2 \)?

We will now examine the same concepts from above with a new function.

Problem 2.22. Consider the function defined by \( g(x) = \frac{1}{x} \).

(a) Sketch the graph of the function \( g \) over the interval \([-4, 4]\).
(b) Now sketch the tangent lines to \( g \) at each of \( x = -3, x = -1, x = -1/2, x = 1/2, x = 1, \) and \( x = 3 \).
(c) Is the slope of the tangent line at \( x = 3 \) less than -1, around -1, between -1 and 0, around 0, between 0 and 1, around 1, or greater than 1?

Problem 2.23. Let \( g(x) = \frac{1}{x} \).

(a) Compute the average rate of change of \( g \) over the interval \([3, 4]\). Keep your answer as a fraction.
(b) Compute the average rate of change of \( g(x) = \frac{1}{x} \) over the interval \([3, 3+\Delta x]\). Simplify this expression as much as possible.

Problem 2.24. Let \( g(x) = \frac{1}{x} \), and consider your expression for the average rate of change of \( g \) over the interval \([3, 3+\Delta x]\) from the previous problem.

(a) If we let \( \Delta x = 1 \), over what interval would we be computing the average rate of change?
(b) Plug \( \Delta x = 1 \) into your simplified expression for the average rate of change for \( g \).
(c) Is your answer the same as what you found in Problem 2.23 (a) even though the process is different?

Problem 2.25. Once again, let \( g(x) = \frac{1}{x} \), and consider your expression for the average rate of change of \( g \) over the interval \([3, 3+\Delta x]\) from the previous 2 problems.

(a) Consider the interval \([3, 3.5]\). What value of \( \Delta x \) gives this interval? What is that average rate of change of \( g \) over this interval?
(b) Consider the interval \([3, 3.02]\). What value of \( \Delta x \) gives this interval? What is that average rate of change of \( g \) over this interval?
(c) What number does the average rate of change of \( g \) over the interval \([3, 3+\Delta x]\) approach as \( \Delta x \) approaches 0?
Problem 2.26. Let’s continue with \( g(x) = 1/x \). Recall that the average rate of change is also giving you the slope of secant lines.

(a) What is the derivative (that is, the slope of the tangent line) of \( g \) at \( x = 3 \)? In this context, the derivative is sometimes referred to as the \textbf{instantaneous rate of change} at \( x = 3 \).

(b) Is your answer consistent with what you found in Problem 2.22 (c)?

The next few problems will be dealing with the idea of velocity. Note that many people use the terms speed and velocity interchangeably. However, there is an important distinction: speed is always positive while velocity can be negative. In broader terms, velocity incorporates both the speed and direction of travel.

Recall that both the slope of a secant line and the average rate of change of a function are found by the change in \( y \) divided by the change in \( x \). Similarly, the \textbf{average velocity} of an object is the change in distance divided by the change in time. Note that for any problem about distance, speed, acceleration, etc., you need to include units in your answer.

Problem 2.27. Suppose the height of a hot air balloon (in meters) is given by the function \( h(t) = \sqrt{t} \) after \( t \) seconds. What is the average velocity of the balloon from \( t = 4 \) to \( t = 9 \)?

Problem 2.28. Consider the balloon described in the previous problem.

(a) Find a formula for the average velocity of the balloon over the interval \([4, 4 + \Delta t]\).

(b) Calculate the average velocity of the balloon over the interval \([4, 4 + \Delta t]\) for smaller and smaller values of \( \Delta t \).

(c) As \( \Delta t \) gets smaller, what number does the average velocity get close to? In this context, this number is sometimes referred to as the \textbf{instantaneous velocity} of the balloon at \( t = 4 \), or even just the \textbf{velocity} of the balloon at \( t = 4 \). Note that this is the same process we use to find the derivative.

We conclude the chapter by tying together the various concepts that were discussed.

\textbf{Instructor Note: The problems below are intended to make clear that all three concepts have the same formula, and to reinforce the connection between the average and the instantaneous.}

Problem 2.29. Consider a function \( f \) over the interval \([a, b]\).
(a) What is a formula for the slope of the secant line for $f$ over this interval?

(b) What is a formula for the average rate of change for $f$ over this interval?

(c) If $f$ represents the distance of an object from some point, what is a formula for the average velocity of the object over this interval?

**Instructor Note:** The next problem is meant to summarize the ideas from the previous few problems. It is not expected for them to provide a precise definition of the derivative. This simply sets the stage for limits (the topic of the next chapter) and the limit definition that appears in Chapter 4.

**Problem 2.30.** For any given function $f$, describe how you would find the derivative of $f$ at $x = a$.

**Problem 2.31.**

(a) Instead of the slope of a secant line, the derivative gives the slope of the ____________.

(b) Instead of the average rate of change, the derivative gives the ____________ rate of change.

(c) Instead of the average velocity, the derivative gives the ____________ velocity.
Chapter 3

Limits

In this chapter we will introduce the notion of limits, which we will use to compute derivatives in later chapters.

Instructor Note: In this chapter we treat limits very briefly. We are trying to simply establish the knowledge required to evaluate derivatives in the following chapters. We suggest working through these problems as quickly as possible.

The goal of the next few problems, in addition to introducing the idea of a limit, is to show that two functions which are identical everywhere except at a single point can have the same limit even if the function values at that point disagree. It will be important to make note of the different domains of the functions involved. In particular, \( f \) is defined at 0, but \( g \) is not.

Problem 3.1. Consider the function \( f \) defined by \( y = f(x) = 2x + 1 \), which you encountered in Chapter 1.

(a) State the domain of \( f \), and sketch the graph of \( f \).

(b) As the values of \( x \) get closer to 0, is there a number that the corresponding \( y \)-values get closer to? If such a number exists, it is called the limit as \( x \) approaches 0 of \( f(x) \), and is denoted by \( \lim_{x \to 0} f(x) \).

(c) Does \( f(0) \) exist? If so, what is \( f(0) \)?

Problem 3.2. Consider the function \( g \) defined by \( y = g(x) = \frac{2x^2 + x}{x} \), which you also encountered in Chapter 1.

(a) State the domain of \( g \), and sketch the graph of \( g \). How is this graph similar to and how is it different from the graph of \( f \) above?

(b) As the values of \( x \) get closer to 0, is there a number that the corresponding \( y \)-values get closer to? That is, find \( \lim_{x \to 0} g(x) \).
(c) Does $g(0)$ exist? If so, what is $g(0)$?

**Problem 3.3.** Consider the functions $f(x) = 2x + 1$ and $g(x) = \frac{2x^2 + x}{x}$, from above.

(a) Is $f(0)$ equal to $g(0)$?
(b) Are $f$ and $g$ the same function?
(c) Is $\lim_{x \to 0} f(x)$ equal to $\lim_{x \to 0} g(x)$?

Note that we may be interested in limits where $x$ approaches a number different from 0. The following definition generalizes our notion of limit from above.

**Definition 3.4.** We say the limit of $f$ as $x$ approaches $a$ is equal to $L$ if, as $x$ gets closer to $a$, the corresponding values of $f(x)$ get closer to $L$. We write $\lim_{x \to a} f(x) = L$.

(This definition is slightly informal, but works well in Calculus. A more formal definition is used in the course Real Analysis.)

**Problem 3.5.** Now, let $f(x) = \frac{x^2 - 9}{x - 3}$.

(a) Explain why $f$ is not defined at $x = 3$.
(b) Find a simplified function which is equivalent to $f$ at every point except $x = 3$. Call this function $g$.
(c) What is $g(3)$?
(d) What is $\lim_{x \to 3} g(x)$?
(e) What is $\lim_{x \to 3} f(x)$?

**Problem 3.6.** Evaluate the following limits.

(a) $\lim_{x \to -5} x^2 - 1$
(b) $\lim_{x \to -5} \frac{x^2 + 6x + 5}{x + 5}$
(c) $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x^2 - 25}$
(d) $\lim_{x \to -5} 8$

**Problem 3.7.** Let us summarize what we have noticed thus far about limits.
(a) If \( f(a) \) exists, how would you find \( \lim_{x \to a} f(x) \)?

(b) If \( f(a) \) does not exist, how would you find \( \lim_{x \to a} f(x) \)?

Note that the methods for computing limits you described above work for most functions you will encounter, but they are not always true. Think of them as guidelines.

**Problem 3.8.** Evaluate the following limits.

(a) \( \lim_{u \to 4} \frac{u + 2}{u^2 + u + 1} \)

(b) \( \lim_{t \to 4} \frac{t^2 - 16}{t - 4} \)

(c) \( \lim_{t \to -4} \frac{t^2 - 16}{t - 4} \)

(d) \( \lim_{s \to 9} \sqrt{s} \)

**Instructor Note:** We do not deal formally with continuity in these notes. However the following problem hints at the concept, and the instructor could give a mini-lecture on continuity if desired.

**Instructor Note:** Our experience is that the previous problems go fairly smoothly, but the next problem causes a lot of difficulty. This is a good time to emphasize the use of graphing software such as Desmos.

**Problem 3.9.** This problem will be a bit different from the previous ones, and is meant to solidify your understanding of limits.

Let \( f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

(a) Sketch a graph of \( f \) between \( x = -2 \) and \( x = 2 \).

(b) Looking at the graph or a table of values, is there a number that \( f(x) \) gets closer to as \( x \) gets closer to 0?

(c) Explain why \( \lim_{x \to 0} f(x) \neq 1 \) even though \( f(0) = 1 \).

Now that we have evaluated a number of limits using the guidelines we established above, the following problems investigate functions for which they simply do not work.
Instructor Note: In the following problems the instructor can emphasize the non-existence of the limit versus saying that it equals infinity as they see fit. More generally, the following problems are an opportunity to discuss the ideas and notation around limits involving infinity and one-sided limits. These ideas are not central in what follows, and can be treated as seen fit by the instructor.

**Problem 3.10.** Let \( f(x) = \frac{1}{x^2} \).

(a) Sketch the graph of \( f \).

(b) As \( x \) gets closer and closer to 0, is there a number that the values of \( f(x) \) get closer and closer to?

(c) What does that say about \( \lim_{x \to 0} f(x) \)?

**Problem 3.11.** Let \( g(x) = \frac{1}{x} \).

(a) Sketch the graph of \( g \).

(b) As \( x \) gets closer and closer to 0, is there a number that the values of \( g(x) \) get closer and closer to?

(c) What does that say about \( \lim_{x \to 0} g(x) \)?

**Problem 3.12.** Let \( h(x) = \frac{|x|}{x} \).

(a) Sketch the graph of \( h \).

(b) As \( x \) gets closer and closer to 0, is there a number that the values of \( h(x) \) get closer and closer to?

(c) What does that say about \( \lim_{x \to 0} h(x) \)?

**Problem 3.13.** In the problem above, it can be useful to look separately at \( h(x) \) when \( x > 0 \) and when \( x < 0 \). Let \( h(x) = \frac{|x|}{x} \) as above.

(a) If \( x > 0 \), what is the single value that \( h(x) \) gets closer to as \( x \) gets closer to 0 (while remaining greater than 0)? This is called the **limit of** \( h(x) \) **as** \( x \) **approaches** 0 **from the right** and is written as \( \lim_{x \to 0^+} h(x) \).

(b) If \( x < 0 \), what is the single value that \( h(x) \) gets closer to as \( x \) gets closer to 0 (while remaining less than 0)? This is called the **limit of** \( h(x) \) **as** \( x \) **approaches** 0 **from the left** and is written as \( \lim_{x \to 0^-} h(x) \).

**Problem 3.14.** Let \( w(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ 2x + 1 & \text{if } x > 1. \end{cases} \)

This is the same function we saw in Chapter 1.
(a) Compute \( \lim_{x \to 1^+} w(x) \).

(b) Compute \( \lim_{x \to 1^-} w(x) \).

(c) Explain why \( \lim_{x \to 1} w(x) \) does not exist.

There are many properties of limits that will help us both now with the computation of limits as well as later in the development of the derivative. We have actually been using these properties for some time without explicitly stating them. We will summarize these properties in the following theorem.

**Instructor Note:** Most theorems in these notes require students to fill in some blanks and/or provide a proof. However, there are a handful of instances (this being the first) where we provide a theorem without expectation of proof. The goal here is to not get bogged down in the minutiae of limits and forge ahead quickly to derivatives in the following chapter.

**Theorem 3.15.** Suppose that

\[
\lim_{x \to a} (f(x)) = L \quad \text{and} \quad \lim_{x \to a} (g(x)) = M.
\]

Then

(a) \( \lim_{x \to a} (f(x) + g(x)) = L + M \),

(b) \( \lim_{x \to a} (f(x) - g(x)) = L - M \),

(c) \( \lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M \),

(d) \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{L}{M} \) as long as \( M \neq 0 \),

(e) for any real number \( k \), \( \lim_{x \to a} (k \cdot f(x)) = k \cdot L \).

**Problem 3.16.** Suppose you know that \( \lim_{x \to 3} f(x) = 5 \) and \( \lim_{x \to 3} g(x) = -2 \). Evaluate the following limits.

(a) \( \lim_{x \to 3} (f(x) + g(x)) \)

(b) \( \lim_{x \to 3} (3f(x) - 4g(x)) \)

(c) \( \lim_{x \to 3} (f(x) + f(x)g(x)) \)
(d) $\lim_{x \to 3} \left( \frac{f(x)}{g(x)} \right)$

(e) $\lim_{x \to 3} \left( \frac{g(x)}{f(x)} \right)$
Chapter 4

Polynomials And The Power Rule

We have seen that the derivative of a function $f$ at a point $x = a$ represents the slope of the tangent line to $f$ at that point. We have also seen that we can compute the derivative of $f$ at $x = a$ by looking at the slopes of secant lines over the interval $[a, a + \Delta x]$, and evaluating this as $\Delta x$ approaches 0.

The language of limits from Chapter 3 now allows us to make this more precise.

**Instructor Note:** Student answers to part (c) below may be vague, and that is likely fine, as the point of the problem is to simply make students think about the definition carefully.

**Problem 4.1.** Suppose $f(x)$ is a function.

(a) Write a formula for the slope of the secant line between the points $(x, f(x))$ and $(x + h, f(x + h))$. Note that we are using the variable $h$ in place of $\Delta x$ because that is the conventionally used variable by mathematicians for calculating the derivative, probably because it is a little shorter.

(b) We can use the formula above to find the slope of the tangent line to $f(x)$ at the point $(x, f(x))$. To do this, we would use the formula for the slope of the secant line as $h$ approaches a particular number. What number?

(c) The **derivative** of $f$, written as $f'(x)$ or $\frac{df}{dx}$, is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists. Explain how this formula is connected to your answers for parts (a) and (b).
Problem 4.2. If the function $f$ has a derivative at $x$, we say the function is differentiate at $x$. If a function is differentiable at every point in a particular set (e.g., its domain), we say the function is differentiable on that set. Is the function $f(x) = x^2$ differentiable on the set $-\infty < x < \infty$?

Problem 4.3. Let $f(x) = x^2$. Compute $f'(x)$ using the definition.

Problem 4.4. Let’s continue with $f(x) = x^2$.

(a) Compute $f'(-2)$, $f'(-1)$, $f'(0)$, $f'(1)$, and $f'(2)$ by plugging these specific $x$-values into your answer to Problem 4.3.

(b) Are your answers here consistent with those you found in Problem 2.10?

(c) How does your answer for $f'(2)$ compare with your answer from 2.12?

Problem 4.5. Compute the derivative of each of the following functions.

(a) $f(x) = 7x - 4$
(b) $g(x) = 5 - 10x$
(c) $h(x) = 4x^2 - 7x + 8$
(d) $k(x) = 1 - 10x^2$

We are now going to turn our attention to finding shortcuts to computing derivatives.

Problem 4.6. Look back at Problem 4.5. Which functions had a single number as the derivative? With that in mind, what would you conjecture is the derivative of $y = 3x + 1$?

Problem 4.7. Suppose $f$ is the linear function $f(x) = mx + b$ for some constants $m$ and $b$.

(a) What do you think $f'$ will be? Explain why that is your guess.

(b) Use the definition above to calculate the derivative $f'$.

We will now come up with a general formula for quadratic equations.

Problem 4.8. Suppose $g$ is the quadratic function $g(x) = ax^2 + bx + c$ for some constants $a, b$ and $c$. Compute the derivative $g'$.

Instructor Note: We chose to find the derivative of a constant last because in our experience this is computationally the easiest, but conceptually the most difficult general derivative to find.

Problem 4.9. Suppose $f(x) = k$, where $k$ is a constant.
(a) What does the function \( y = j(x) \) look like? (The exact graph would depend upon the specific value of \( k \), but you can still give a general description.)

(b) What do you think \( j'(x) \) will be? Explain your reasoning.

(c) Use the definition of the derivative to compute the derivative \( j' \).

**Problem 4.10.** Use the shortcuts above or the limit definition of the derivative to find the derivatives of the following functions.

(a) \( f(x) = x^2 \)

(b) \( g(x) = x^3 \)

**Problem 4.11.** Follow the pattern from the previous problem to make a conjecture about the derivatives of the following functions.

(a) \( y = x^4 \)

(b) \( y = x^5 \)

(c) \( y = x^{100} \)

**Theorem 4.12. The Power Rule.** Let \( f(x) = x^n \) for some positive integer \( n \). Then \( f'(x) = ??? \). (The intent here is for you to simply write something in place of the question marks.)

We are now going to work toward a proof of this theorem. To do this we will first review ways to multiply polynomials.

**Problem 4.13.** Some of you may remember Pascal’s Triangle, where each entry is obtained by adding the two entries above. The first few rows are displayed below.

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
\end{array}
\]

Write out the next 2 lines of Pascal’s Triangle.

**Problem 4.14.** Multiply out \( (x+h)^2 \). Which line of Pascal’s Triangle corresponds to the coefficients in your answer? Do the same for \( (x+h)^3 \).

**Problem 4.15.** Using the pattern from the previous problem, expand \( (x+h)^4 \) and \( (x+h)^5 \) without actually multiplying them out.

**Problem 4.16.** Use the limit definition of the derivative to find the derivative of \( f(x) = x^4 \). Note that the difficult part of this problem is usually determining what \( (x+h)^4 \) is, but you have already sorted that out in the previous problem! Does your answer match your conjecture from Problem 4.11 (a)?
**Problem 4.17.** Use the limit definition of the derivative to differentiate \( g(x) = x^5 \). (Differentiate simply means to compute the derivative.) Does your answer match your conjecture from Problem 4.11 (b)?

**Problem 4.18.** Let \( f(x) = x^{100} \). We are going to use the limit definition to differentiate \( f \). No one wants to expand \((x+h)^{100}\), but fortunately, we do not have to pay attention to all of the terms.

(a) If we were to expand \((x+h)^{100}\), what would the first 2 terms be?

(b) What is the lowest power of \( h \) in the rest of the terms?

(c) If we simplified \( \frac{f(x+h) - f(x)}{h} \) in a way similar to what we did with \( x^4 \) and \( x^5 \), just as before, there will only be one term left without an \( h \). What is it?

(d) Evaluate \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

(e) Compare your answer to your conjecture from Problem 4.11 (c).

**Instructor Note:** The problem below is intended to help students recognize that this method of computing the derivative only works if \( n \) is a positive integer. This limitation is particularly important for students who have seen the general version of the Power Rule before, and who know that the rule does apply to other kinds of exponents. Even if the first answer is not perfect, the instructor can moderate a good discussion.

**Problem 4.19.** It turns out that the Power Rule will work for any real value of \( n \), however the technique from Problem 4.18 only works for positive integer values. Where does the technique break down if the exponent is not a positive integer?

**Problem 4.20.** Following the same technique as in Problem 4.18, prove the Power Rule for a positive integer \( n \). This may be a little trickier than in Problem 4.18 because we are using a letter \( n \) in place of the number 100. Do your best!

**Problem 4.21.** According to the Power Rule, what is the derivative of \( f(x) = x^{73} \)?

**Problem 4.22.** Let \( f(x) = x^n \). What does the Power Rule say \( f'(x) \) is if \( n = 1 \)? Does this make sense?

**Instructor Note:** This next problem is a good opportunity to discuss the variety of derivative notations.
Problem 4.23. For more practice using the Power Rule, differentiate the following.

(a) \( y = x^{23} \)
(b) \( y = t^{27} \)
(c) \( w = r^{10} \)

Our next goal is to determine what happens when you take the derivative of a sum or difference of functions.

Problem 4.24. Suppose \( f(x) = x^3 + x^2 \).

(a) Use the limit definition of the derivative to find \( f'(x) \).
(b) How does your answer for \( f'(x) \) relate to the derivatives of \( x^3 \) and \( x^2 \), respectively?
(c) Based on this, make a conjecture about what the derivative of \( y = x^7 + x^4 \) would be.

We will now come up with the general rule for differentiating the sum of two functions.

Theorem 4.25. Suppose \( f \) and \( g \) are differentiable functions, and let \( s(x) = f(x) + g(x) \). Then \( s'(x) = ??? \) (The intent here is for you to simply write something in place of the question marks.)

Instructor Note: It may be useful to refer back the the theorem in Chapter 3 about properties of limits (particularly that they are additive).

Problem 4.26. We will now prove Theorem 4.25.

(a) Write down the limit definition for the derivative of \( s(x) \). You do not need to simplify this at all, yet.
(b) What is \( \frac{s(x+h) - s(x)}{h} \) in terms of \( f(x+h), g(x+h), f(x), \) and \( g(x) \)?
(c) Rewrite the expression for \( \frac{s(x+h) - s(x)}{h} \) as a sum of two fractions, one with \( f \) terms in the numerator, and one with \( g \) terms in the numerator.
(d) Use your rewritten expression for \( \frac{s(x+h) - s(x)}{h} \) and the limit definition of the derivative to evaluate \( s'(x) \) in terms of \( f'(x) \) and \( g'(x) \).

Problem 4.27. Let \( f(x) = x^7 + x^3 \). Find \( f'(x) \).
Problem 4.28. Find the slope of the tangent line to \( y = x^2 + x^9 \) at \( x = -1 \).

Problem 4.29. Consider the two functions \( y = x^{15} + x^7 \) and \( y = x^{10} + x^{11} \).

(a) Which function has a larger \( y \)-value at \( x = 1 \) (or are they the same)?

(b) Which function has a steeper tangent line at \( x = 1 \) (or are they the same)?

There is an analogous rule for the difference of functions.

Theorem 4.30. Suppose \( f \) and \( g \) are differentiable functions, and let \( d(x) = f(x) - g(x) \). Then \( d'(x) = \ldots \)

Problem 4.31. Prove Theorem 4.30 in a similar way to which you proved Theorem 4.25.

Problem 4.32. Use the Power Rule, Theorem 4.25, and Theorem 4.30 to differentiate the following functions.

(a) \( y = x^9 - x^5 + x \)

(b) \( y = x^8 + x^{14} - 3 \)

We will now turn our attention to differentiating constant multiples of functions.

Problem 4.33. Let \( f(x) = 2x^3 \).

(a) Compute the derivative of \( f \). You can either use the limit definition of the derivative or find a way to rewrite \( f \) as a sum of functions whose derivatives you already know.

(b) How does the derivative of \( f \) relate to the derivative of \( x^3 \)?

Problem 4.34. Let \( g(x) = 5x^2 \).

(a) Compute the derivative of \( g \).

(b) How does the derivative of \( g \) relate to the derivative of \( x^2 \)?

Problem 4.35. Make a conjecture as to what the derivative of \( 27x^{81} \) would be.

We will now come up with the general rule for differentiating a constant multiple of a function.

Theorem 4.36. Suppose \( f \) is a differentiable function, \( k \) is any real number, and let \( m(x) = kf(x) \). Then \( m'(x) = \ldots \)

Problem 4.37. We will now prove Theorem 4.36. This proof will be shorter than the previous ones, which ironically makes it somewhat more difficult conceptually.
(a) What is \(m(x+h) - m(x)\) in terms of \(f(x+h), f(x),\) and \(k?\) (You need not simplify your answer at this point.)

(b) Simplify the expression for \(\frac{m(x+h) - m(x)}{h}\) by factoring out a single term.

(c) Use your simplified expression for \(\frac{m(x+h) - m(x)}{h}\) and the limit definition of the derivative to evaluate \(m'(x)\) in terms of \(f'(x)\) and \(k.\)

We now have all of the tools necessary to differentiate any polynomial.

**Problem 4.38.** Differentiate the following polynomials.

(a) \(y = 3x^4 + 5x + 2\)

(b) \(z = 11 - 6x^8 + 10x^{12}\)

(c) \(f(r) = \pi r^2\)

*Instructor Note: The instructor may want to draw attention to the way these shortcuts would deal with \(ax^2 + bx + c,\) and how this relates to the answer they obtained using the limit definition of the derivative.*
Chapter 5

Applications Of The Derivative

We now know how to differentiate polynomial functions. Before we see the techniques for finding the derivative of the other types of functions, we pause to examine some applications of the derivative.

Instructor Note: In the problems below, we have the students graph the tangent line to several functions. This is intended to reinforce the graphical connection between derivatives and tangent lines, to emphasize visually what it means when a derivative is zero, and to introduce the idea of why a function might not be differentiable at a point. The last two ideas will be used almost immediately in the discussion of extrema.

Instructor Note: In several of the problems below the student is asked “What do you notice?” after zooming in on a point. The intent is that the student notices that the function appears more linear after zooming in and/or that the function and the tangent line get more and more alike. As of the typing of these notes (March 2017) Desmos seems like the best tool to do this.

Problem 5.1. Let \( f(x) = x^2 \). We are now going to work through finding the equation of the tangent line to \( f \) at \( x = 1 \).

(a) What is the \( y \) value of the tangent line when \( x = 1 \)? (Hint: think about where the tangent line touches the function.)

(b) What is the slope of the tangent line when \( x = 1 \)? (Hint: think about the derivative.)

(c) Now that you have a point on the tangent line, and its slope, what is the equation of the tangent line?

(d) Use technology to graph both \( f \) and the tangent line you found above. Zoom in in the point of tangency (where they touch). What do you notice as you zoom in?
Problem 5.2. Continue to consider the function \( f(x) = x^2 \).

(a) Use the same process as above to find the equation of the tangent line to \( f \) at \( x = -1 \). As in the previous problem, use technology to graph both \( f \) and this tangent line. Zoom in on the point of tangency. Again, what do you notice?

(b) Repeat the process above at \( x = 2 \).

(c) Repeat the process above at \( x = 0 \).

Problem 5.3. Let \( g(x) = x^3 \).

(a) Find the equation of the tangent line to \( g \) at \( x = -1 \). As in the previous problems, use technology to graph both \( g \) and this tangent line. Zoom in on the point of tangency. Again, what do you notice?

(b) Repeat the process above at \( x = 0 \).

(c) Repeat the process above at \( x = 1 \).

Problem 5.4. Consider the function \( l(x) = 3x + 1 \).

(a) Find the equation of the tangent line to \( l \) at \( x = 0 \).

(b) Find the equation of the tangent line to \( l \) at \( x = 1 \).

(c) Find the equation of the tangent line to \( l \) at \( x = 2 \).

(d) Explain, possibly using a graph of \( l(x) \), why you got the answers that you did.

Problem 5.5. Consider the line \( y = mx + b \). What do you think the slope of the tangent line will be at different values of \( x \)? What do you think the equation of the tangent line will be?

Problem 5.6. Let \( j(x) = |x| \).

(a) Draw a graph of \( j \), carefully marking the scales on the \( x \)-axis and \( y \)-axis.

(b) Use the graph to find \( j'(1) \).

(c) What is the equation of the tangent line to \( j \) at \( x = 1 \)?

(d) Repeat this process to find the equation of the tangent line to \( j \) at \( x = 3.5 \).

(e) What is \( j'(x) \) if \( x \) is positive?

Problem 5.7. Let \( j(x) = |x| \), as above.

(a) Use the graph of \( j \) that you drew above to find \( j'(-1) \).
(b) What is the equation of the tangent line to \( j \) at \( x = -1 \)?

(c) Repeat this process to find the equation of the tangent line to \( j \) at \( x = -3.2 \).

(d) What is \( j'(x) \) if \( x \) is negative?

Instructor Note: In the problem below, the student is expected to notice that there is still a corner at \( (0, 0) \) even after zooming in; this is in contrast to the graphs of \( y = x^2 \) and \( y = x^3 \) which appear flat after zooming in. This, combined with the calculations above for \( j'(x) \), should help students understand why \( j \) is not differentiable at zero, since our experience is that students expect \( j'(0) \) to be zero.

Problem 5.8. Let \( j(x) = |x| \), as above.

(a) Use technology to graph \( j \). Zoom in on the point \( (0, 0) \). What do you notice?

(b) Explain why \( j \) is not differentiable at \( x = 0 \). That is, explain why \( j'(0) \) doesn’t exist even though \( x = 0 \) is part of the domain of \( j \).

Problem 5.9. Based on the examples above, under what circumstances in general might there be a place \( x = c \) within the domain of \( f(x) \) where \( f'(c) \) does not exist?

We will now turn our attention to finding minimum and maximum values of a function.

Definition 5.10. The point \( (c, f(c)) \) is a local minimum for the function \( f \) if \( f(c) \leq f(x) \) for all values of \( x \) that are “near” \( c \). On the other hand, the point \( (c, f(c)) \) is a local maximum for the function \( f \) if \( f(c) \geq f(x) \) for all values of \( x \) that are “near” \( c \). Local minima and local maxima are sometimes called local extrema of a function.

Definition 5.11. The point \( (c, f(c)) \) is a global minimum for the function \( f \) if \( f(c) \leq f(x) \) for all values of \( x \) that are in the domain of \( f \). On the other hand, the point \( (c, f(c)) \) is a global maximum for the function \( f \) if \( f(c) \geq f(x) \) for all values of \( x \) that are in the domain of \( f \). Global minima and global maxima are sometimes called global extrema of a function.

Instructor Note: In the problems below, we are using points for extrema. The instructor may wish to distinguish between the location of an extremum and its value.

Problem 5.12. Label all local minima, local maxima, global minima, and global maxima for the function shown below.
Problem 5.13. Label all local minima, local maxima, global minima, and global maxima for the function shown below.

Problem 5.14. We will now look at a few more examples.

(a) Find the local and global extrema of \( f(x) = x^2 \).
(b) Find the local and global extrema of \( g(x) = x^3 \).
(c) Find the local and global extrema of \( j(x) = |x| \).
(d) Find the local and global extrema of \( k(x) = \sin(x) \).

Instructor Note: The questions below are intended to elicit discussion about some of the subtleties extrema, including the difference between local and global extrema and possibly the distinction between the \( y \)-value of a minimum or maximum and the \( x \)-value where that extrema occurs.

Problem 5.15. Based on the examples above, answer the following questions. They are all phrased in terms of local or global minima, but similar questions could apply to local or global maxima.
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(a) Does a function have to have a local minimum?

(b) Is it possible for a function to have more than one local minimum? If so, is it possible for the local minima to have different y-values?

(c) Does a function have to have a global minimum?

(d) Is it possible for a function to have more than one global minimum? If so, is it possible for the global minima to have different y-values?

(e) How is a global minimum different from a local minimum?

In the next few problems we will examine connections between the local and global extrema and the derivative.

**Definition 5.16.** Suppose \(c\) is in the domain of the function \(f\). We say \((c, f(c))\) is a **critical point** for \(f\) if either \(f'(c) = 0\) or if \(f'(c)\) does not exist.

**Problem 5.17.** Find all the critical points for the function \(f(x)\) shown below. Also mark the intervals where \(f' > 0\), and the intervals where \(f' < 0\).

![Graph of function f(x)](image)

**Problem 5.18.** Find all the critical points for the function shown below. Also mark the intervals where \(f' > 0\), and the intervals where \(f' < 0\).

![Graph of function f(x)](image)
Instructor Note: Students may use a variety of methods for finding the critical points in the problem below. However, they will need to use a graph for part (d).

Problem 5.19. In each of the problems below, find all the critical points.

(a) \( f(x) = x^2 \)

(b) \( g(x) = x^3 \)

(c) \( j(x) = |x| \)

(d) \( k(x) = \sin(x) \)

Problem 5.20. If \( f \) has a local minimum (or maximum) at \( x = c \) and \( f'(c) \) exists, then what must be true about the value of \( f'(c) \)?

Problem 5.21. Is it possible for a function to have a local minimum or maximum at \( x = c \) but not have a derivative at \( x = c \)?

Problem 5.22. If \( f \) has a critical point at \( x = c \), is there necessarily a local or global extrema at \((c, f(c))\)?

The following question is intended to summarize what we have found so far. We are restricting ourselves to functions that are defined over all the real numbers; we will look at some other examples later in the chapter.

Problem 5.23. Suppose that \( f(x) \) is defined over of all the real numbers. Without relying on a graph of \( f \), how can you find the potential places where \( f \) might have local or global extrema?

Problem 5.24. Determine all critical points for the function

\[
f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 - \frac{10}{3}x^3.
\]

Problem 5.25. Determine all critical points for the function

\[
f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 + \frac{10}{3}x^3.
\]

Problem 5.26. Determine all critical points for the function

\[
g(x) = \begin{cases} 
x^2 & \text{for } x \leq 1 \\
3 - 2x & \text{for } x > 1
\end{cases}
\]

The problems below are intended to lead to a process for determining whether a critical point might be a minimum or maximum of a function.

Problem 5.27. Consider the function \( f(x) = x^3 + 2x \). Answer the following questions without looking at a graph of \( f \) (until the end).

(a) Verify that this function has no critical points.
(b) Where is \( f' \) positive and where is it negative?

(c) Using the information above, explain how you can use the derivative to tell whether the original function \( f \) is increasing or decreasing.

(d) Look at a graph of \( f \). Does the graph match what you found?

**Problem 5.28.** Suppose that \( f \) is differentiable everywhere, and that there is some interval on which \( f \) has no critical points.

(a) Explain why \( f' \) can’t be positive on part of the interval and negative on a different part of the interval. (This means that it must be positive everywhere or negative everywhere on the interval.)

(b) Explain how to decide whether \( f' \) is positive or negative on the interval.

(c) Explain how you can use this knowledge of \( f' \) to determine if \( f \) is increasing or decreasing on this interval.

**Problem 5.29.** Consider the function \( g(x) = x^3 - 3x \). Answer the following questions without looking at a graph of \( g \).

(a) Find the two critical points of \( g \). We will call them \( a \) and \( b \). These two points divide the real numbers up into three intervals: \( x < a \), \( a < x < b \), and \( x > b \). On each of these intervals, determine whether \( g' \) is positive or negative.

(b) Use the information above to determine whether \( g \) is increasing or decreasing on each of these intervals.

(c) Use part (b) to draw a really rough sketch of what \( g \) looks like. Don’t bother labeling any points or even drawing the \( x \)- or \( y \)-axes: just focus on the overall shape of the function.

(d) Use the information above to determine whether \( g \) has any local minima or maxima.

(e) Can you tell whether \( g \) has any global minima or maxima?

(f) Graph the function \( g \) using technology. Does the graph match what you found above?

**Problem 5.30.** Earlier you found all critical points for the function \( f(x) = \frac{1}{5} x^5 - \frac{3}{4} x^4 - \frac{10}{3} x^3 \). Answer parts (a) through (d) below without looking at a graph.

(a) Use \( f' \) to determine where \( f \) is increasing or decreasing.

(b) Use part (a) to draw a really rough sketch of what \( f \) looks like, again focusing only on the overall shape of the function.
(c) Use the information above to determine whether $f$ has any local maxima or minima.

(d) Can you tell whether $f$ has any global maxima or minima?

(e) Graph the function $f$. Does the graph match what you found above?

**Problem 5.31.** Repeat the process above for $f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 + \frac{10}{3}x^3$.

Once again, do not look at a graph to answer the questions until the last part.

The following problem summarizes the process used above. This process is sometimes called the **first derivative test** for minima and maxima because it uses the first derivative of $f$.

**Problem 5.32.** Suppose that $f$ is defined everywhere.

(a) Explain how you can determine whether $f$ has local minima and maxima without looking at a graph of the function?

(b) Does this process also help you to determine whether or not $f$ has global extrema?

In all of the examples above we have looked at functions that are defined on all of the real numbers. In some cases, the domain of the function is a closed interval. Consider the example below:

**Problem 5.33.** Consider the function $f(x) = x^2$ on the interval $[-3, 2]$.

(a) Graph this function.

(b) Find all local and global extrema.

(c) Why are there some additional points that weren’t there in Problem 5.14?

**Problem 5.34.** Consider the function $g(x) = x^3$ on the interval $[-3, 2]$.

(a) Graph this function.

(b) Find all local and global extrema.

(c) Why are there some additional points that weren’t there in Problem 5.14?

**Problem 5.35.** If $f$ is a differentiable function on a closed interval $[a, b]$, then it turns out $f$ must have both a global minimum and a global maximum. Describe where these global extrema must occur. This fact is known as the **Extreme Value Theorem**.
We will finish this chapter with seeing how something called the second derivative of a function can also give information about the function.

**Problem 5.36.** Consider the graph of a function $f$ below.

(a) Draw tangent lines to $f$ at $x = 1$, $x = 2$, and $x = 3$.

(b) Are the slopes of the tangent lines you drew above positive or negative?

(c) Based on this answer, is $f'$ positive or negative?

(d) Are the slopes increasing or decreasing? That is, as $x$ increases, how are the slopes of the tangent lines changing?

(e) Based on this answer is $f'$ increasing or decreasing?

(f) Based on the answers above, is the derivative of $f'$ positive or negative? We call the derivative of $f'$ the **second derivative** of $f$ and it is written as $f''$, which is a shorthand for $(f')'$.

**Problem 5.37.** Consider the graph of a function $f$ below.
Applications Of The Derivative

(a) Draw tangent lines to $f$ at $x = 1$, $x = 2$, and $x = 3$.
(b) Are the slopes of the tangent lines you drew above positive or negative?
(c) Based on this answer, is $f'$ positive or negative?
(d) Are the slopes increasing or decreasing?
(e) Based on this answer is $f''$ increasing or decreasing?
(f) Based on your answers above, is $f'''$ positive or negative?

Problem 5.38. Consider the graph of a function $f$ below.

Problem 5.39. Consider the graph of a function $f$ below.
(a) Draw tangent lines to $f$ at $x = 1$, $x = 2$, and $x = 3$.
(b) Are the slopes of the tangent lines you drew above positive or negative?
(c) Based on this answer, is $f'$ positive or negative?
(d) Are the slopes increasing or decreasing?
(e) Based on this answer is $f'$ increasing or decreasing?
(f) Based on your answers above, is $f''$ positive or negative?

Problem 5.40. Consider the function $f(x) = x^2$.

(a) For what values of $x$ is $f'$ positive?
(b) For what values of $x$ is $f'$ negative?
(c) For what values of $x$ is $f''$ positive? Where $f''$ is positive, $f$ is said to be concave up.
(d) For what values of $x$ is $f''$ negative? Where $f''$ is negative, $f$ is said to be concave down.

Problem 5.41. Consider the function $f(x) = -x^2$.

(a) For what values of $x$ is $f'$ positive?
(b) For what values of $x$ is $f'$ negative?
(c) For what values of $x$ is $f$ concave up?
(d) For what values of $x$ is $f$ concave down?

Problem 5.42. Consider the function $f(x) = x^3$.

(a) For what values of $x$ is $f'$ positive?
(b) For what values of \( x \) is \( f' \) negative?
(c) For what values of \( x \) is \( f \) concave up?
(d) For what values of \( x \) is \( f \) concave down?

The next problem summarizes what have learned about the relationships between a function \( f \), \( f' \), and \( f'' \).

**Problem 5.43.** Fill in the blanks below with an appropriate choice from the following words: increasing, decreasing, concave up, concave down.

(a) If \( f' > 0 \) then \( f \) is

(b) If \( f' < 0 \) then \( f \) is

(c) If \( f'' > 0 \) then \( f' \) is \( ________________ \), and \( f \) is \( ________________ \).

(d) If \( f'' < 0 \) then \( f' \) is \( ________________ \), and \( f \) is \( ________________ \).

We have used the notation \( f \), \( f' \), and \( f'' \) above. Recall that we may use the notation \( \frac{df}{dx} \) for the derivative of \( f \). In a similar way we may use \( \frac{d^2f}{dx^2} \) for the second derivative of \( f \). This is a shorthand for \( \frac{d}{dx} \left( \frac{df}{dx} \right) \).

**Instructor Note:** In the problem below, students may indicate the intervals on the graph, or may write them out formally. Either is fine. The instructor can decide to emphasize formality in labeling, if they prefer.

**Problem 5.44.** Consider the graph of \( f \) below.

(a) Find the intervals on which the function is concave up and the intervals on which the function is concave down.

(b) Label the point(s) where the graph changes concavity. These points are called **inflection points**.
Problem 5.45. Let’s keep considering the graph of \( f \) above.

(a) Label the critical points on the graph.

(b)

(c) Where \( f \) is concave up, are the critical points local maxima or are they local minima?

(d) Where \( f \) is concave down, are the critical points local maxima or are they local minima?

This leads us to a new theorem.

**Theorem 5.46. (The Second Derivative Test).** Let \( f \) be a differentiable function and let \( c \) be a point where \( f'(c) = 0 \). Then there is a local maximum at \( x = c \) if ??? and there is a local minimum at \( x = c \) if ???. Fill in the ??? with information about the second derivative

Problem 5.47. Consider the function \( g(x) = x^3 - 3x \).

(a) Compute \( g' \) and \( g'' \).

(b) Find the critical points of \( g \) and use the second derivative test to determine if these points are local maxima or local minima. There may be critical points where you can’t use the second derivative test; in that case, explain why.

(c) Use technology to graph the function \( g \). Does the graph match what you found above?

Problem 5.48. Repeat the process above for \( f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 - \frac{10}{3}x^3 \).

Problem 5.49. Repeat the process above for \( f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 + \frac{10}{3}x^3 \).
Chapter 6

Trigonometric Functions, The Product Rule, And The Quotient Rule

In this chapter, we will examine the derivatives of trigonometric functions. We’ll start with a quick review.

Note that we will always measure angles in radians, and you may recall that there are $2\pi$ radians in $360^\circ$.

**Problem 6.1.** Write the following angles in radians:

(a) $90^\circ$
(b) $45^\circ$
(c) $30^\circ$

**Instructor Note:** We start by reviewing some basic trigonometry ideas partly as a gentle review of trig but also because we’ve found that many students have forgotten about where some of the early results came from. Although the goal in the next several problems is to use the diagrams to compute trig functions, students may just state the answers from memory. In that case, the connections to diagram can be mentioned as part of a class discussion, but in the end not too much time should be spent on re-deriving trigonometry.

Angles like those above come up frequently because they are relatively easy to work with, as reviewed below.

**Problem 6.2.** Consider the square below with sides of length 1.
(a) What is the length of the diagonal?

(b) What is the measure of angles $\alpha$, $\beta$, and $\gamma$?

(c) Use this diagram to calculate $\sin(\pi/4)$ and $\cos(\pi/4)$.

**Problem 6.3.** Consider the equilateral triangle below with sides of length 1.

(a) What is the length of $a$?

(b) What is the length of $b$?

(c) What is the measure of angles $\alpha$, $\beta$, and $\gamma$?

(d) Use this diagram to calculate $\sin(\pi/3)$ and $\cos(\pi/3)$.

(e) Similarly, use this diagram to calculate $\sin(\pi/6)$ and $\cos(\pi/6)$.

Another way of viewing trigonometric functions uses the unit circle.

**Problem 6.4.** Consider the triangle inscribed in the unit circle below.
(a) The hypotenuse of the triangle has length 1. What are the lengths of the other two sides in terms of \( x \) and/or \( y \)?

(b) Evaluate \( \sin(\theta) \) in terms of \( x \) and/or \( y \).

(c) Evaluate \( \cos(\theta) \) in terms of \( x \) and/or \( y \).

(d) Use the diagram, and your formulas above, to explain why \( \sin(0) = 0 \) and \( \cos(0) = 1 \).

(e) Similarly, explain why \( \sin(\pi/2) = 1 \) and \( \cos(\pi/2) = 0 \).

**Problem 6.5.** One advantage of the circle diagram hinted at above is that it can be useful for computing trig functions of more difficult angles.
Use the diagram above and information from the previous problems to compute the following:

(a) What are the values of $x$ and $y$?

(b) Compute $\sin(3\pi/4)$ and $\cos(3\pi/4)$.

**Instructor Note:** This may be a good spot for the instructor to remind students of the identity $\sin^2(x) + \cos^2(x) = 1$, and could give a derivation.

**Problem 6.6.** We will now turn our attention to a third way to view the sine function.

(a) Compute $\sin(\pi)$, $\sin(3\pi/2)$, $\sin(2\pi)$, and $\sin(-\pi/2)$, using diagrams if necessary.

(b) Plot all 10 of the sine values you have computed in this problem as well as in problems 6.2-6.5. Connect the dots to form a sketch of $y = \sin(\theta)$. Note that your horizontal axis will be the $\theta$-axis, and your vertical axis will be the $y$-axis.

**Problem 6.7.** Likewise, we’ll finish up a review of the cosine function.

(a) Compute $\cos(\pi)$, $\cos(3\pi/2)$, $\cos(2\pi)$, and $\cos(-\pi/2)$, using diagrams if necessary.

(b) Plot all 10 of the cosine values you have computed in this problem as well as in 6.2-6.5. Connect the dots to form a sketch of $y = \cos(\theta)$. 
Now that we have reviewed sine and cosine, we will now work toward finding their derivatives. We begin this by looking at slopes of tangent lines.

**Problem 6.8.** Let \( f(x) = \sin(x) \). Use technology to graph \( f \) and the lines \( y = x \) and \( y = 2x \) on the same set of axes.

(a) Which of the two lines looks like it is the tangent line to \( f \) at \( x = 0 \)?

(b) With that in mind, find \( f'(0) \).

**Problem 6.9.** Once again, let \( f(x) = \sin(x) \). Keeping the graph of \( f \) in mind make a conjecture about the following values.

(a) \( f'(-\pi/2) \)

(b) \( f'(\pi/2) \)

(c) \( f'(\pi) \)

(d) \( f'(3\pi/2) \)

(e) \( f'(2\pi) \)

**Problem 6.10.** Still using \( f(x) = \sin(x) \), and using the values from the previous problem, sketch a graph of \( f'(x) \). Make a conjecture about what function \( f'(x) \) is.

We will now turn our attention to using the limit definition of the derivative to verify our conjecture about the derivative of \( \sin(x) \).

**Problem 6.11.** Let \( f(x) = \sin(x) \).

(a) Write down the limit definition for \( f'(x) \).

(b) Why is it not possible to simplify and evaluate this limit the way we did with polynomials?

The following theorem will help us to simplify the above limit.

**Theorem 6.12.** For any \( x, y \in \mathbb{R} \), the following identity holds.

\[
\sin(x + y) = \sin(x) \cos(y) + \sin(y) \cos(x)
\]

**Problem 6.13.** We now return to the limit definition for the derivative of \( f(x) = \sin(x) \) from Problem 6.11.

(a) Use Theorem 6.12 above to rewrite \( \sin(x + h) \).

(b) Now substitute your answer from part (a) into the limit definition for the derivative of \( \sin(x) \) from Problem 6.11.
(c) Rewrite your answer from part (b) as the sum of two limits, one involving \( \frac{\sin(h)}{h} \), and one involving \( \frac{\cos(h) - 1}{h} \).

(d) Evaluate \( \lim_{h \to 0} \frac{\sin(h)}{h} \) by examining the graph of \( y = \frac{\sin(x)}{x} \), using technology.

(e) Evaluate \( \lim_{h \to 0} \frac{\cos(h) - 1}{h} \) by examining the graph of \( y = \frac{\cos(x) - 1}{x} \), using technology.

(f) Finally, evaluate \( f'(x) \). Does this match what you found in Problem 6.10?

**Problem 6.14.** Let \( f(x) = 3 \sin(x) + 5 - x^2 \). Compute \( f'(x) \).

We will now turn our attention to the derivative of \( \cos(x) \).

**Problem 6.15.** Draw \( f(x) = \cos(x) \) and use the graph of \( f \) to make a conjecture about the following values.

(a) \( f'(-\pi/2) \)

(b) \( f'(0) \)

(c) \( f'(\pi/2) \)

(d) \( f'(\pi) \)

(e) \( f'(3\pi/2) \)

(f) \( f'(2\pi) \)

**Problem 6.16.** Still using \( f(x) = \cos(x) \), and using the values from the previous problem, sketch a graph of \( f'(x) \). Make a conjecture about what function \( f'(x) \) is.

As before, we will use the limit definition of the derivative to verify our conjecture about the derivative of \( \cos(x) \).

**Problem 6.17.** Let \( f(x) = \cos(x) \).

(a) Write down the limit definition for \( f'(x) \).

(b) Use the fact that \( \cos(x + y) = \cos(x)\cos(y) - \sin(y)\sin(x) \) to rewrite the limit definition.

(c) Use a process similar to what you did before to evaluate \( f'(x) \). Does this match what you found in Problem 6.16?

**Problem 6.18.** Let \( f(x) = \sin(x) - \cos(x) + 3 \). Compute \( f'(x) \).
In order to compute more complicated derivatives, we want to be able to compute the derivatives of products and quotients, not just sums and differences. We start by exploring the derivative of the product of two functions.

**Problem 6.19.** Let \( f(x) = x^3 \), \( g(x) = x^4 \), and \( p(x) = f(x)g(x) \).

(a) Compute \( p(x) \) and simplify.

(b) Compute \( p'(x) \).

(c) Now compute \( f'(x) \) and \( g'(x) \).

(d) Is \( p'(x) \) equal to \( f'(x)g'(x) \)?

Since we now know that \( (fg)' \neq f'g' \), let’s try to determine what \( (fg)' \) really should be.

For the next 3 problems, let \( f \) and \( g \) be differentiable functions, and let \( p(x) = f(x)g(x) \).

**Problem 6.20.** Use the limit definition of the derivative to write down the limit definition of \( p'(x) \) in terms of \( f \) and \( g \). Do not simplify.

In order to help simplify the expression from the previous problem, a geometric construction may prove helpful. Suppose \( f(x) \) represents the length of an elongating horizontal line segment at time \( x \), and \( g(x) \) represents the length of an elongating vertical line segment. Then \( f(x)g(x) \) is the area of an enlarging rectangle. \( f(x+h) \) is the length of the horizontal line segment at a later time \( x+h \), and \( g(x+h) \) is the length of the vertical line segment at that later time, so \( f(x+h)g(x+h) \) is the area of a larger rectangle. See Figure 6.1 below.

**Problem 6.21.** As mentioned above, \( p(x+h) = f(x+h)g(x+h) \) represents the total area of the large rectangle in Figure 6.1. Rewrite this area as a sum of the areas of several other rectangles in the figure. Do not simplify your expression.

**Problem 6.22.** Rewrite your expression for \( p'(x) \) by replacing \( f(x+h)g(x+h) \) with the expression you found in the previous problem. Now find a formula for \( p'(x) \) in terms of \( f(x) \), \( f'(x) \), \( g(x) \), and \( g'(x) \). (Hint: You will need to recall the limit definitions of the derivatives of \( f(x) \) and \( g(x) \).)

We can now summarize this result in the following theorem, known as the Product Rule.

**Theorem 6.23. (The Product Rule).** Let \( f \) and \( g \) be differentiable functions. Then \( (fg)' = \)???

**Problem 6.24.** Let \( f(x) = x^3 \), \( g(x) = x^4 \), and \( p(x) = f(x)g(x) \), as in Problem 6.19.
Trigonometric Functions, The Product Rule, And The Quotient Rule

We will now return to trigonometric functions. Recall that the tangent, cosecant, secant, and cotangent functions are defined as follows:

\[
\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \csc(x) = \frac{1}{\sin(x)} \quad \sec(x) = \frac{1}{\cos(x)} \quad \cot(x) = \frac{1}{\tan(x)}
\]

Note that the domain of each of these functions consists of those values of \(x\) for which the denominators are nonzero.

In order to calculate the derivative of each of these new functions, we will need to know how to find the derivative of the quotient of two functions. But first, we will examine a very particular quotient.

Instructor Note: Students may have difficulty with the fractions involved in the next two problems, and may need some encouragement.

**Problem 6.28.** Let \(f(x) = \frac{1}{x}\). Use the limit definition of the derivative to calculate \(f'(x)\). Note that you will need to recall how to work with common denominators and how to simplify fractions divided by another number.
Problem 6.30 generalizes the result of the previous one. It is similar in that it involves fractions, but you will not be cancel anything, but there will be a moment when you can replace one expression with a $g'(x)$.

**Problem 6.29.** This problem is meant to help you with one particular step in Problem 6.30. Fill in the blanks below to make a true statement.

\[ A - B = - (\quad - \quad) \]

**Problem 6.30.** Let $g$ be a function and suppose $g(x) \neq 0$. Now, let $r(x) = \frac{1}{g(x)}$. Use the limit definition of the derivative to compute $r'(x)$ in terms of $g(x)$ and $g'(x)$.

We can now use this fact to differentiate some of the trigonometric functions defined earlier.

**Problem 6.31.** Use the previous problem to differentiate $\sec(x)$.

**Problem 6.32.** Compute the derivative of $\csc(x)$.

**Problem 6.33.** Let $f$ and $g$ be functions, and suppose $g(x) \neq 0$. Also, let $q(x) = \frac{f(x)}{g(x)}$. Use the Product Rule and Problem 6.30 to find $q'(x)$ in terms of $f(x), f'(x), g(x),$ and $g'(x)$.

**Theorem 6.34. (The Quotient Rule).** For any functions $f$ and $g$, the derivative of the quotient \( \frac{f(x)}{g(x)} \) is ???

**Problem 6.35.** Use the quotient rule to find the derivative of $\tan(x)$.

**Problem 6.36.** Compute the derivative of $\cot(x)$.

**Problem 6.37.** Let $r(x) = \frac{x^4}{x^2 + 1}$. Compute $\frac{dr}{dx}$.

**Problem 6.38.** Let $s(x) = \frac{x^2}{\sin(x)}$. Compute $s'(x)$.

**Problem 6.39.** Compute the derivative of $x^2\tan(x)$.

**Problem 6.40.** Compute the derivative of $\frac{\sec(x) + 5}{3x^2}$.

In an earlier chapter we developed the Power Rule to find the derivative of $x^n$, but the method that we used to prove this rule only applied when $n$ was a positive integer. We now have the ability to extend the power rule more values of $n$. We will start with a few examples.

**Problem 6.41.** Rewrite $x^{-2}$ as a fraction and then compute its derivative.
Problem 6.42. Compute the derivative of $x^{-3}$.

Problem 6.43. Compute the derivative of $x^{-4}$.

Problem 6.44. Compute the derivative of $x^{-100}$.

Problem 6.45. Let $n$ be a positive integer, and let $f(x) = \frac{1}{x^n}$. Use the Quotient Rule to compute $f'(x)$. Simplify your answer completely.

We can now write the Power Rule for all integers:

Theorem 6.46. (The Power Rule for integer powers). Let $n$ be any integer. The derivative of $f(x) = x^n$ is ???

Problem 6.47. Compute the first and second derivative of the function $f(x) = x^4 - 2x^3 + 8x + \frac{1}{x} - \frac{3}{x^2} + \frac{7}{x^3}$.

Problem 6.48. Compute the derivative of the function $g(x) = \frac{x^7 + 1}{x^5}$.

Instructor Note: A lot of material was covered in this chapter. The instructor may find it useful to assign a number of computationally focused practice problems.
Chapter 7

Exponential Functions And The Chain Rule

In this chapter, we examine exponential functions. In addition, we see how to find the derivative of the composition of two (or more) functions.

Problem 7.1. On the same set of axes, sketch the graphs of the functions \( p(x) = 2^x \) and \( q(x) = 3^x \). Where do each of these functions cross the \( y \)-axis? Which of these two functions is bigger when \( x < 0 \)? Which is bigger when \( x > 0 \)?

Problem 7.2. Let \( b \) be a positive number, and let \( f(x) = b^x \). Write down the limit definition for \( f'(x) \), and show that

\[
f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}.
\]

Instructor Note: The limit below, of course, turns out to be the natural logarithm of 2. Since we do not define logarithms until the next chapter, we will need to be content with using such numerical approximations for now.

Problem 7.3. Use technology to estimate the value of

\[
\lim_{h \to 0} \frac{2^h - 1}{h}
\]

to three decimal places. Specifically, this means you should zoom in far enough on a graph or use numbers close enough to zero so that the first three digits do not change.

Problem 7.4. Let \( p(x) = 2^x \), as before. Use your estimate from the previous problem to compute \( \frac{dp}{dx} \).

Instructor Note: In the following problems students may find both \( p(0) \) and \( p'(0) \). This provides a good opportunity for reinforcing the difference between those values.
Problem 7.5. Once again, let \( p(x) = 2^x \). Find the slope of the tangent line to \( p \) at \( x = 0 \).

Problem 7.6. Use technology to estimate the value of \( \lim_{h \to 0} \frac{3^h - 1}{h} \) to three decimal places.

Problem 7.7. Let \( q(x) = 3^x \), as before. Use your estimate from the previous problem to compute \( \frac{dq}{dx} \).

Problem 7.8. Once again, let \( q(x) = 3^x \). Find the slope of the tangent line to \( q \) at \( x = 0 \).

Definition 7.9. We define the number \( e \) to be the number for which

\[
\lim_{h \to 0} \frac{e^h - 1}{h} = 1.
\]

Problem 7.10. Use information from previous problems to explain why \( e \) must be between 2 and 3.

Problem 7.11. Use information from previous problems to decide whether \( e \) must be between 2 and 2.5, or between 2.5 and 3.

Instructor Note: The following problem can be modified to make students better approximate \( e \), if that is what the instructor desires.

Problem 7.12. It turns out that \( e \approx 2.72 \). Explain how you could use a trial-and-error technique to make this approximation.

The previous few problems were intended to give you a sense of the value of the number \( e \) which, like the number \( \pi \), shows up in many places in mathematics. We will usually use the letter \( e \) in equations, but if you ever need an approximate value, it is typical to use 2.72 or 2.718 (in the same way that it is common to use 3.14 or 3.14159 as approximate values of \( \pi \)). Some calculators also have both an \( e \) button and a \( \pi \) button.

Problem 7.13. Sketch the graph of the function \( f(x) = e^x \).

Problem 7.14. Compute the derivative of the function \( f(x) = e^x \).

Problem 7.15. Compute the equation of the tangent line to \( f(x) = e^x \) at \( x = 0 \).

The next few problems are intended to review some of the concepts you have recently learned: the derivative of \( e^x \), the derivatives of the trigonometric functions, and the product and quotient rules.

Problem 7.16. Let \( y = xe^x \). Compute \( y' \).
Problem 7.17. Let \( y = \frac{x}{e^x} \). Compute \( \frac{dy}{dx} \).

Problem 7.18. Let \( y = x^2 e^x \). Compute \( y' \).

Problem 7.19. Let \( y = e^x \sin(x) \). Compute \( \frac{dy}{dx} \).

Problem 7.20. Let \( y = \frac{\tan(x)}{e^x} \). Compute \( \frac{dy}{dx} \).

We are now going to take a departure from exponential functions in preparation for a new technique for differentiation. This technique is related to composition of functions, so we start with a quick reminder of function composition.

Instructor Note: In our experience, many students have difficulty recognizing certain functions (e.g. \((x^2 + 1)^8\)) as compositions and thus have difficulty applying the chain rule. These problems are meant to gently remind students about function composition to make the transition to the chain rule a little easier.

Problem 7.21. Compute \( f(g(x)) \) for the following functions.

(a) \( f(x) = x^2 \) and \( g(x) = \sin(x) \)
(b) \( f(x) = \sin(x) \) and \( g(x) = x^2 \)
(c) \( f(x) = e^x \) and \( g(x) = \cos(x) \)
(d) \( f(x) = \cos(x) \) and \( g(x) = e^x \)
(e) \( f(x) = e^x \) and \( g(x) = 3x + 1 \)

Problem 7.22. The functions below can be written as \( y = f(g(x)) \). Find the appropriate \( f \) and \( g \) in each case. It may be easier to find \( g(x) \) first.

(a) \( y = \sin(2x) \)
(b) \( y = (3x^5 - x^{27})^{100} \)
(c) \( y = [\cos(x)]^2 \). This is often written as \( y = \cos^2(x) \).

We shall now compute some derivatives of composite functions.

Instructor Note: The following problems are meant to foreshadow the Chain Rule.

Problem 7.23. Use the Product Rule to compute the derivative of \( y = \sin^2(x) \). (Recall that \( \sin^2(x) = [\sin(x)]^2 \).) In what way is this derivative similar to the derivative of \( x^2 \)?
Problem 7.24. Use the Product Rule (and the derivative above) to compute the derivative of \( y = \sin^3(x) \). In what way is this derivative similar to the derivative of \( x^3 \)?

Problem 7.25. Using the pattern above, make a conjecture as to the derivative of \( y = \sin^4(x) \). Then use the product rule (and derivatives you already know) to determine if your conjecture was correct.

Problem 7.26. The function \( y = \sin^4(x) \) above can be thought of as \( y = f(g(x)) \). What is \( g(x) \)? What is \( f(x) \)?

Problem 7.27. We’ll switch to a different function now. Use the Product Rule to compute the derivative of \( y = (x^4 + x^2 + 1)^2 \). (Combine like terms, but do not multiply out.) In what way is this derivative similar to the derivative of \( x^2 \)?

Problem 7.28. Use the Product Rule to compute the derivative of \( y = (x^4 + x^2 + 1)^3 \). (As before, combine like terms, but do not multiply out.) In what way is this derivative similar to the derivative of \( x^3 \)?

Problem 7.29. The function \( y = (x^4 + x^2 + 1)^3 \) can be thought of as \( y = f(g(x)) \). What is \( g(x) \)? What is \( f(x) \)?

Problem 7.30. Let \( g \) be a function. Use the Product Rule to compute \( [g(x)^2]' \). (Your answer will include both \( g \) and \( g' \).)

Problem 7.31. Let \( g \) be a function. Use the Product Rule to compute \( [g(x)^3]' \). (Your answer will include both \( g \) and \( g' \).)

Problem 7.32. Let \( g \) be a function.

(a) Make an educated guess as to what \( [g(x)^{113}]' \) will be.

(b) Use this guess to predict what \( [\sin^{113}(x)]' \) will be.

(c) Now generalize your guess to \( [(g(x))^n]' \).

The previous problems involve finding the derivative of a composition of functions. The following theorem tells us how to find the derivative of the composition of two functions in full generality. The proof is beyond the scope of this class, and will have to wait until Real Analysis.

**Theorem 7.33. The Chain Rule.** Given two functions \( f \) and \( g \), the derivative of \( f(g(x)) \) is defined by the product

\[
\left( f(g(x)) \right)' = f'(g(x)) \cdot g'(x).
\]

That is, the derivative of the composition of two functions is equal to the derivative of the “outside” function evaluated at the “inside” function, times the derivative of the “inside” function.
This formula is sometimes written in the following way:

\[
\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}
\]

**Problem 7.34.** In Problem 7.26 you wrote \( y = \sin^4(x) \) as \( y = f(g(x)) \).

(a) Use this breakdown, together with the Chain Rule as described in Theorem 7.33, to compute the derivative.

(b) Is your answer the same as what you found in Problem 7.25?

**Problem 7.35.** In Problem 7.29 you wrote \( y = (x^4 + x^2 + 1)^3 \) as \( y = f(g(x)) \).

(a) Use this breakdown, together with the Chain Rule as described in Theorem 7.33, to compute the derivative.

(b) Is your answer the same as what you found in Problem 7.28?

**Problem 7.36.** Compute the derivative of \( y = (x^4 + x^2 + 1)^{4321} \).

**Problem 7.37.** Compute the derivative of \( y = \cos^{20}(x) \). (Recall that \( \cos^{20}(x) = [\cos(x)]^{20} \).)

**Problem 7.38.** Compute the derivative of \( (e^x - 5)^{10} \).

**Problem 7.39.** Let \( g \) be a function. Use the Chain Rule to find a formula for \( \left[ \sin \left( g(x) \right) \right]' \).

**Problem 7.40.** Compute the derivative of \( y = \sin(x^2) \).

**Problem 7.41.** Compute the derivative of \( y = \sin(2x) \).

**Problem 7.42.** Let \( g \) be a function. Use the Chain Rule to find a formula for \( \left[ \cos \left( g(x) \right) \right]' \).

**Problem 7.43.** Compute the derivative of \( y = \cos(x^2) \).

**Problem 7.44.** Compute the derivative of \( y = \cos(\pi x) \).

**Problem 7.45.** Compute the derivative of \( y = \tan(x^3) \). You will need to recall the derivative of \( \tan(x) \).

**Problem 7.46.** Let \( g \) be a function. Find a formula for the derivative of \( y = e^{g(x)} \).

**Problem 7.47.** Compute the derivative of \( y = e^{x^3} \).

**Problem 7.48.** Compute the derivative of \( y = e^{5x} \).

**Problem 7.49.** Compute the derivative of \( y = e^{\cos(x)} \).

**Problem 7.50.** Compute the derivative of \( y = \sin^{10}(3x) \).

**Problem 7.51.** Compute the derivative of \( y = x^2 \cos(11x) \).
Chapter 8

Logarithms And Implicit Differentiation

We now turn our attention to inverse functions and, in particular, logarithms.

**Problem 8.1.** Let \( f(x) = x^2 \) on the interval \([0, \infty)\).

(a) What familiar function would “undo” what the function \( f \) does to \( x \)? Label this function as \( f^{-1} \).

(b) Compute \( f^{-1}(f(x)) \).

(c) Compute \( f(f^{-1}(x)) \).

(d) For this problem, why do we need to restrict the domain of \( x^2 \) to the interval \([0, \infty)\)? (Hint: Is \( \sqrt{x^2} \) equal to \( x \) when \( x = -3 \)?)

**Instructor Note:** The defining equation for inverse functions below is very important in what follows and should be emphasized by the instructor.

**Definition 8.2.** Suppose there are two functions \( f \) and \( f^{-1} \) such that \( f(f^{-1}(x)) = x = f^{-1}(f(x)) \). Then we say \( f \) and \( f^{-1} \) are inverse functions. Note that not every function has an inverse. Also note that \( f^{-1} \) does not denote the reciprocal of \( f \).

**Definition 8.3.** Let \( b \) be a positive real number. The inverse of the exponential function \( f(x) = b^x \) is called the **logarithmic function in base** \( b \) and is written \( f(x) = \log_b(x) \). That is, if \( y = \log_b(x) \), then \( y \) is the power that you raise the number \( b \) to to get \( x \) (i.e. \( b^y = x \)). When using the base \( e \), we often use \( \ln(x) \) to denote \( \log_e(x) \), which is called the **natural logarithm**.

**Problem 8.4.** Before we compute the derivatives of logarithmic functions, we review their meaning. Evaluate the following logarithms without using technology.

(a) \( \log_{10}(1000) \)
(b) \( \log_{10}(1,000,000,000) \)
(c) \( \log_2(0.5) \)
(d) \( \log_7(7) \)
(e) \( \log_{83}(1) \)

(f) Still without using your technology, what two integers is \( y = \log_{10}(3000) \) in between?

**Problem 8.5.** Sketch a graph of each of the functions below, labeling 3–4 key points. Also state what the domain and range of each function is.

(a) \( y = 10^x \)
(b) \( y = e^x \)
(c) \( y = \log_{10}(x) \)
(d) \( y = \ln(x) \)

**Problem 8.6.** Without using technology, simplify the following:

(a) \( \log_{10}(10^5) \)
(b) \( \log_3(3^{\frac{1}{2}}) \)
(c) \( \ln(e^{100}) \)
(d) \( \ln(e^{x}) \)

**Problem 8.7.** Without using technology, simplify the following:

(a) \( 10^{\log_{10}(10000)} \)
(b) \( 10^{\log_{10}(0.1)} \)
(c) \( 5^{\log_5(125)} \)
(d) \( 5^{\log_5(10)} \)
(e) \( e^{\ln(17)} \)
(f) \( e^{\ln(x)} \)

Now that we have re-familiarized ourselves with logarithms, it is time to sort out how to differentiate them.

**Problem 8.8.** We know that \( e^{\ln(x)} = x \). Differentiate both sides (using the Chain Rule on the left side). Your derivative of the left side should involve \([\ln(x)]'\). Rearrange and solve for \([\ln(x)]'\), thus giving us the derivative of \(\ln(x)\)!
Problem 8.9. Find the equation of the tangent line to \( y = \ln(x) \) at \( x = 1 \). Sketch a graph of both the curve \( y = \ln(x) \) and the tangent line you found above.

Problem 8.10. Let \( g \) be a function. Find a formula for the derivative of \( y = \ln[g(x)] \).

Problem 8.11. Let \( f(x) = \ln(x^2 + 1) \). Compute \( f'(x) \).

Problem 8.12. Let \( f(x) = e^{3x} \ln(4x) \). Compute \( f'(x) \).

Problem 8.13. Let \( f(x) = \frac{\ln(\sin(x))}{x} \). Compute \( f'(x) \).

The following theorem should be familiar to you from high school.

Theorem 8.14. The following three properties of logarithms are true for any base \( b \).

\( (a) \) For any positive real numbers \( x \) and \( y \), \( \log_b(xy) = \log_b(x) + \log_b(y) \).

\( (b) \) For any positive real numbers \( x \) and \( y \), \( \log_b \left( \frac{x}{y} \right) = \log_b(x) - \log_b(y) \).

\( (c) \) For any positive real number \( x \), and any real number \( r \), \( \log_b(x^r) = r \log_b(x) \).

Problem 8.15. Let \( f(x) = \ln \left[ \frac{(x^2 + 1)(x+2)(x-3)}{e^{x^2}(x+5)(x^3 - 1)} \right] \).

(\(a\)) Which differentiation rules would be required to differentiate \( f' \)? How long would it take you?

(\(b\)) Use Theorem 8.14 to simplify \( f \) as far as possible, then compute \( f' \).

Often we encounter equations involving two variables \( x \) and \( y \) in which

- we know \( y \) depends on \( x \) somehow, but
- we cannot solve the equation for \( y \) to see this dependence explicitly.

If we still want to see how the variable \( x \) affects the variable \( y \), however, we need to calculate the derivative of \( y \). We can do this using an application of the Chain Rule, a process called **implicit differentiation**.

Problem 8.16. Consider the unit circle, defined by the equation \( x^2 + y^2 = 1 \).

(\(a\)) Sketch a graph of the unit circle, and the tangent lines to the circle at the points \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) and \( \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \).
(b) Treat \( y \) as simply some unknown function \( y(x) \) and use the Chain Rule (particularly on the \( y^2 \) term) to differentiate both sides of the equation. Your answer will include \( x \), \( y \) and/or \( y' \).

(c) Solve for \( y' \). (This process is implicit differentiation.)

(d) What is the slope of the tangent line at \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \)?

(e) What is the slope of the tangent line at \( \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \)?

(f) Do these slopes match your sketch from part (a)?

**Problem 8.17.** Consider the graph of the implicit relation \( 2x^2 - y^4 = 2 \) below.

![Graph of implicit relation](image)

(a) Draw the tangent line to the graph at the point \((3, -2)\), and determine an approximate value for the slope.

(b) Use implicit differentiation to find the exact value of this slope.

We can now use the combination of logarithms and implicit differentiation to find a few new derivatives that would be hard or even impossible to find otherwise.

**Problem 8.18.** Let \( y = 2^x \).

(a) Take the natural logarithm of both sides, and simplify.

(b) Use implicit differentiation on your simplified expression to find \( y' \).
   (Your answer may initially have both \( x \) and \( y \) in it; if so, at the very end replace \( y \) with \( 2^x \).)

(c) Compare your answer to the derivative of \( 2^x \) you found in the previous chapter.
Problem 8.19. Let \( b \) represent any positive real number, and let \( y = b^x \).

(a) Take the natural logarithm of both sides, and simplify.

(b) Use implicit differentiation on your simplified expression to find \( y' \).

(As before, your answer may initially have both \( x \) and \( y \) in it, but you can replace \( y \) with \( b^x \).)

The procedure used in the last problem is called logarithmic differentiation, in which the natural logarithm is applied to both sides of an equation, the result is simplified using the properties of logarithms, and implicit differentiation is used to compute the derivative.

Theorem 8.20. Let \( b \) represent a positive real number, and let \( f(x) = b^x \). Then \( f'(x) = ???. \)

Problem 8.21. Recall that in Problem 7.2, using the limit definition of the derivative, we determined that if \( f(x) = b^x \), then

\[
f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}.
\]

Take this formula for the derivative of \( b^x \) and set it equal to the formula for the derivative of \( b^x \) you just found in Theorem 8.20. Based on this, what must \( \lim_{h \to 0} \frac{b^h - 1}{h} \) be equal to?

Problem 8.22. As practice with this new short-cut, what is the derivative of \( y = 10^x \)?

Problem 8.23. What is \( f'(x) \) if \( f(x) = 10^{g(x)} \)? What is \( f'(x) \) if \( f(x) = 10^{\tan(x)} \)?

Problem 8.24. We’ll now apply this same technique to a new (but not really) function. Let \( r \) be a real number (that is, a constant but not necessarily an integer), and let \( y = x^r \). Use logarithmic differentiation to find \( y' \). Be sure to simplify your answer completely.

Finally, we are able to state the Power Rule in its full generality!

Theorem 8.25. The Power Rule. Let \( r \) be a real number and let \( f(x) = x^r \), then \( f'(x) = ???. \)

Problem 8.26. Use the Power Rule to compute the derivatives of each of the functions below. You may need to rewrite a few of them before you can use the power rule.

(a) \( x^{3/2} \)

(b) \( \sqrt{x} \)
Problem 8.27. Suppose the height of a hot air balloon (in meters) is given by the function \( h(t) = \sqrt{t} \) after \( t \) seconds.

(a) Recall that the derivative of the distance function is (instantaneous) velocity. Find the velocity function \( v(t) = h'(t) \), including units.

(b) What is the velocity of the balloon when \( t = 4 \) seconds? How does your answer compare to what you found in Problem 2.28 (c)?

(c) The derivative of the velocity function is acceleration. Find the acceleration function \( a(t) = v'(t) = h''(t) \).

(d) What is the acceleration of the balloon when \( t = 4 \)? Is the balloon speeding up or slowing down at that point?

In the following problem we return to logarithmic differentiation as a simpler way to approach products and quotients.

Problem 8.28. Let \( y = \frac{(x^2 + 1)(x + 2)(x - 3)}{e^{x^2}(x + 5)(x^3 - 1)} \). Use logarithmic differentiation to find \( y' \).

We will now complete this chapter with a quick look at inverse trigonometric functions.

Problem 8.29. The notation \( \sin^{-1} \) or arcsin represents the inverse sine function. In particular, \( y = \arcsin(x) \) means that \( \sin(y) = x \); that is, \( y \) is the angle whose sine is \( x \).

(a) What is \( \arcsin(0) \)? How is this related to the sine function?

(b) What is \( \arcsin(1) \)? How is this related to the sine function?

(c) What is \( \arcsin(-1) \)?

(d) What is \( \arcsin(2) \)? Is this a trick question?*

(e) Graph \( y = \arcsin(x) \). You may use technology to help with this, but pay particular attention to the domain and range.

(f) Why might some people (possibly your teacher) think that \( \arcsin \) is a less confusing notation than \( \sin^{-1} \)?

Problem 8.30. Let \( y = \arcsin(x) \).

*Yes it is.
(a) Take the sine of both sides, and simplify.

(b) Use implicit differentiation on your simplified expression to find \( y' \). At this point, it is fine if your answer has \( y \) in it.

(c) With implicit differentiation, it is normally fine if the expression for \( y' \) has both \( x \) and \( y \) in it; in this case, however, we can rewrite the expression so that it only contains \( x \). Use the information in part (a) and the fact that \( \sin^2(y) + \cos^2(y) = 1 \) (so, what must \( \cos(y) \) be?) to rewrite your expression for \( y' \) so that it is only in terms of \( x \).

**Problem 8.31.** We’ll now look at the inverse tangent function, written as \( \tan^{-1} \) or \( y = \arctan \). As above, \( y = \arctan(x) \) means that \( \tan(y) = x \); that is, \( y \) is the angle whose tangent is \( x \).

(a) What is \( \arctan(0) \)? How is this related to the tangent function?

(b) What is \( \arctan(1) \)? How is this related to the tangent function?

(c) What is \( \arctan(-1) \)?

(d) What is \( \arctan(2) \)? How is this similar to or different from \( \arcsin(2) \)?

(e) Graph \( y = \arctan(x) \). You may use technology to help with this, but pay particular attention to the domain and range.

**Problem 8.32.** Let \( y = \arctan(x) \).

(a) Take the tangent of both sides, and simplify.

(b) Use implicit differentiation on your simplified expression to find \( y' \). At this point, it is fine if your answer has \( y \) in it.

(c) As above, we can rewrite the expression so that it only contains \( x \). Use the information in part (a) and the fact that \( \sin^2(y) + \cos^2(y) = 1 \) (possibly divided by \( \cos^2(y) \) so that it is an equation about \( \tan^2(y) \) and \( \sec^2(y) \)) to rewrite your expression for \( y' \) so that it is only in terms of \( x \).
Chapter 9

Antiderivatives

In previous chapters, we learned how to find the derivative of a function $f$. In the remaining chapters we will focus on finding a function whose derivative is $f$.

**Problem 9.1.** (a) Find a function whose derivative is $2x$.

(b) Find a function whose derivative is $3x^2$.

(c) Find a function whose derivative is $25x^{24}$.

(d) Find a function whose derivative is $-3x^{-4}$.

**Definition 9.2.** Let $f$ be a function. We say the function $F$ is an antiderivative for $f$ if $F'(x) = f(x)$ for all $x$.

**Problem 9.3.** Find an antiderivative for each of the following functions:

(a) $f(x) = 4x^3$

(b) $f(x) = 40x^3$

(c) $f(x) = 2x^3$

(d) $f(x) = x^3$

**Problem 9.4.** Find an antiderivative for each of the following functions:

(a) $f(x) = 10x^9$

(b) $f(x) = 50x^9$

(c) $f(x) = x^9$

(d) $f(x) = x^{20}$

(e) $f(x) = -4x^{-5}$

(f) $f(x) = -40x^{-5}$
(g) \( f(x) = x^{-5} \)

**Problem 9.5.** Suppose \( f(x) = 3x^{-10} \).

(a) Which number do you look at (3 or -10) to determine the exponent of an antiderivative of \( f(x) \)?

(b) What will that exponent be?

(c) Find an antiderivative of \( f \).

**Problem 9.6.** Find an antiderivative for each of the following functions:

(a) \( f(x) = x^{1/2} \)

(b) \( f(x) = 10x^{1/2} \)

(c) \( f(x) = 500\sqrt{x} \)

(d) \( f(x) = \sqrt{x} \)

(e) \( f(x) = \frac{1}{\sqrt{x}} \)

(f) \( f(x) = \frac{17}{\sqrt{x}} \)

**Problem 9.7.** Find an antiderivative for \( g(x) = x^3 + \frac{1}{x^3} - 3 \).

**Problem 9.8.** Let \( f(x) = x^n \) where \( n \) is a number.

(a) Find a formula for an antiderivative of \( f \).

(b) There is one value of \( n \) which will not work with your formula from part (a). Which value of \( n \) is it?

(c) Find an antiderivative of \( x^n \) for this problematic value of \( n \) from part (b).

It turns out that a function can have many different antiderivatives, as the next few problems illustrate.

**Problem 9.9.** Of the functions you are familiar with, which have a derivative of zero? (While these are the only functions having a derivative of 0, the proof of this fact will be reserved for Real Analysis.)

**Problem 9.10.** Find three different antiderivatives for \( f(x) = 0 \).

**Problem 9.11.** Find three different antiderivatives for \( g(x) = 2x \).

**Problem 9.12.** Find three different antiderivatives for \( h(x) = e^x \).
Problem 9.13. Find an expression that represents *ALL* antiderivatives of \( h(x) = e^x \).

As you have seen above, if a function has an antiderivative it actually has an infinite family of antiderivatives, which leads to the following definition.

**Definition 9.14.** If a function \( f \) has an antiderivative, we call the collection of all such antiderivative functions the **general antiderivative** for \( f \), and denote this family of functions by

\[
\int f(x)
\]

Another name for this collection of functions is the **indefinite integral** of \( f \).

**Instructor Note:** Part (a) of the problem below may cause a few problems with the \( n \), and with the exceptional case of \( n = -1 \).

Problem 9.15. Compute the following indefinite integrals:

(a) \( \int x^n \, dx \)

(b) \( \int \cos(x) \, dx \)

(c) \( \int \sin(x) \, dx \)

(d) \( \int e^x \, dx \)

Problem 9.16. We have seen all of the functions in the integrals below as the derivatives of common functions. With that in mind, evaluate the following integrals:

(a) \( \int \sec^2(x) \, dx \)

(b) \( \int \sec(x) \tan(x) \, dx \)

(c) \( \int \frac{1}{1+x^2} \, dx \)

(d) \( \int \frac{1}{\sqrt{1-x^2}} \, dx \)

**Instructor Note:** Some students may find part (c) below trivial, but this problem still has value in different ways for students at different levels of understanding.
Problem 9.17. In this problem we will turn our attention to various exponential functions.

(a) Compute the derivative of $10^x$.

(b) Evaluate $\int 10^x \, dx$. (Your answer to part (a) may be helpful.)

(c) Evaluate $\int 2^x \, dx$.

Problem 9.18. Compute the following indefinite integrals:

(a) $\int \left( t^4 + 3t - \frac{1}{t^2} \right) \, dt$

(b) $\int \sqrt{u} \, du$

Problem 9.19. Complete the following.

(a) Find $f(x)$ if $f'(x) = 2x$ and $f(1) = 2$.

(b) Find $g(t)$ if $g'(t) = 2t$ and $g(0) = 2$.

(c) Find $h(u)$ if $h'(u) = \sin(u)$ and $h(0) = 5$

Instructor Note: We have found this to be a difficult topic for many students. Often students fall back on memorized formulas from physics class. It is important to emphasize the relationship between acceleration, velocity, and position.

Before we learn a method for finding some slightly more complicated antiderivatives, we will turn our attention to a few applications from physics. Recall that the derivative of the position function is the velocity function, and the derivative of the velocity function is the acceleration function.

Problem 9.20. Suppose $d(t)$ gives the position, in meters, of an object at any given time $t$ (where the time is measured in seconds). Given the relationship between position, velocity and acceleration described above, what are the units of the velocity function, $v(t)$, for the object? What are the units of the acceleration function, $a(t)$, for the object?

Problem 9.21. Given the relationship between position, velocity and acceleration described above, what is the antiderivative of the acceleration function? What is the antiderivative of the velocity function? (The antiderivative of the position function has a physical interpretation, as well, but we will need to wait until the next chapter to see it).
Instructor Note: Many students have difficulty finding the constants of integration in the problems below. The instructor should keep this in mind, and may want to discuss this issue in greater detail with the whole class.

Problem 9.22. Suppose the velocity of an object is \( v(t) = \frac{1}{10}t^2 + 2 \) feet/second.

(a) Find the position function \( d(t) \) for the object. What are the units of this function? At this point it is okay if your position function has an unknown constant in it.

(b) If the position of the object is 10 feet at \( t = 3 \) seconds, solve for the unknown constant in part (a), and re-write the position function with this new information.

(c) Suppose, instead, that the position of the object is 2 feet at \( t = 1 \) second. Solve for the unknown constant in part (a), and re-write the position function with this new information.

Problem 9.23. Suppose the acceleration of a car from a stop is given by \( a(t) = \sqrt{t} \) feet/\( s^2 \).

(a) Find the velocity function, \( v(t) \), for the car. What are the units of this function? At this point it is okay if your velocity function has an unknown constant in it.

(b) The fact that the car starts off at a stop means that when \( t = 0 \), the velocity is zero; that is, \( v(0) = 0 \). Use this to figure out what the unknown constant is in your formula for \( v(t) \).

(c) Find the velocity of the car after 4 seconds. Be sure to include units.

Problem 9.24. The acceleration due to the Earth’s gravity is approximately \( 9.8 \text{m/s}^2 \) (down). For this and all problems, use techniques from this class rather than physics formulas you may have seen elsewhere.

(a) The acceleration function is often written as \( a(t) = -9.8 \text{m/s}^2 \). In this form, why is there a negative sign?

(b) If an object is thrown upwards with an initial velocity of 10 \( \text{m/s} \), find a function which gives the velocity of the object after \( t \) seconds. Note that the word initial is used to mean at time \( t = 0 \).

(c) If this same object is thrown from the top of a 150 \( \text{m} \) building, that is, \( d(0) = 150 \), find a function which gives the height of the object after \( t \) seconds.

Problem 9.25. Suppose Galileo drops a cannonball off the top of the Leaning Tower of Pisa (60 meters off the ground).
(a) Write down the acceleration function, \( a(t) \), for the cannonball.

(b) Find a function which gives the velocity of the cannonball after \( t \) seconds. Note that the word “dropped” means that the initial velocity of the cannonball is 0 m/s.

(c) Find a function which gives the height of the cannonball (off the ground) after \( t \) seconds.

(d) How long does it take for the cannonball to hit the ground?

(e) How fast is the cannonball falling when it hits the ground?

**Problem 9.26.** One of the lunar astronauts dropped a wrench from the top of the space module (12 meters above the surface of the moon). How long did it take for the wrench to hit the ground? (The acceleration due to gravity on the moon is 1.622 m/s\(^2\) down.)

**Problem 9.27.** In general, suppose that an object is traveling with a constant acceleration \( a \), with an initial velocity of \( v_0 \), and an initial position of \( d_0 \).

(a) Write down the acceleration function, \( a(t) \), for the object.

(b) Find the velocity function, \( v(t) \), for the object.

(c) Find the position function, \( d(t) \), for the object.

We will now turn our attention to some slightly more complicated antiderivatives.

**Problem 9.28.** (a) Compute the derivative of \( e^{x^2} \).

(b) Evaluate \( \int 2x e^{x^2} \, dx \).

(c) Evaluate \( \int 200x e^{x^2} \, dx \).

(d) Evaluate \( \int x e^{x^2} \, dx \).

(e) Finding the antiderivatives above involves recognizing that the derivative used the chain rule. In using the chain rule for the derivative, what function played the role of \( g(x) \)? What is \( g'(x) \) in that case?

**Problem 9.29.** (a) Compute the derivative of \( \cos(e^x) \).

(b) Evaluate \( \int e^x \sin(e^x) \, dx \).

(c) Evaluate \( \int 45e^x \sin(e^x) \, dx \).
(d) As before, finding the antiderivative involves recognizing that the derivative used the chain rule. What function played the role of \( g(x) \) with the chain rule? What is \( g'(x) \) in that case?

We call this particular technique the Method of Substitution: we locate a candidate for the “inside” function \( g(x) \) (or, more commonly, \( u \)), compute \( g'(x) \) (or \( \frac{du}{dx} \)), and then replace \( x \) and \( dx \) with expressions involving \( u \) and \( du \), making it more clear exactly how to “undo” the Chain Rule.

**Problem 9.30.** Consider \( \int \cos(5x) \, dx \).

(a) What is playing the role of \( g(x) \)? What is \( g'(x) \)?

(b) Evaluate the integral.

**Problem 9.31.** Consider \( \int x^9 \cos(3x^{10} + 4) \, dx \).

(a) What is playing the role of \( g(x) \)? What is \( g'(x) \)?

(b) Evaluate the integral.

With these two problems, we have only scratched the surface of the Method of Substitution. A more detailed treatment will take place in Calculus II.
Chapter 10

Area and The Fundamental Theorem of Calculus

In this chapter we will explore how to calculate the area of some not-straightforward shapes, and end with one of the most fascinating applications of the antiderivative: the Fundamental Theorem of Calculus.

**Problem 10.1.** We will start by estimating some areas.

(a) Draw the function $y = 3x$. Shade in the triangular region under the curve on the interval $0 \leq x \leq 2$. (The term “under the curve” means the region beneath the curve, but still above the $x$-axis.) Estimate (or, if possible, find exactly) the area of this shaded region.

(b) Draw the function $y = 5x + 1$. Shade in the region beneath the curve on the interval $0 \leq x \leq 1$. Estimate (or, if possible, find exactly) the area of this shaded region.

(c) Draw the function $y = x^2$ and shade in the region under the curve on the interval $0 \leq x \leq 2$. Estimate (or, if possible, find exactly) the area of this shaded region.

(d) Which of the above areas were you able to find exactly, and which did you need to estimate?

**Definition 10.2.** Suppose $f$ is a function that is positive for every $x$ on the interval $[a, b]$. The area of that region in the $xy$-plane bounded on the sides by the vertical lines $x = a$ and $x = b$, above by the graph of $f$, and below by the $x$-axis is written as

$$\int_{a}^{b} f(x) \, dx$$

We call this the **definite integral of $f$ from $a$ to $b$.**

**Problem 10.3.** (a) Consider the three shaded regions from Problem 10.1. Which one has an area that can be represented by the definite integral

$$\int_{0}^{1} (5x + 1) \, dx$$


Problem 10.4. In the next few problems we’re just going to focus on one function: \( f(t) = 3t \). We’re using the variable \( t \) instead of \( x \) because we want to use \( x \) for something else later.

(a) Compute \( \int_0^1 3t \, dt \) by drawing the curve \( y = 3t \), shading in the region under the curve on the interval \( 0 \leq t \leq 1 \), and computing its area.

(b) Likewise, compute \( \int_0^2 3t \, dt \) by drawing the relevant picture, shading in the appropriate region, and finding the right area.

(c) Do the same for \( \int_0^3 3t \, dt \).

Problem 10.5. Still looking at \( f(t) = 3t \), we will define an area function
\[
A(x) = \int_0^x 3t \, dt.
\]

(a) Notice that \( A(1) = \int_0^1 3t \, dt \). You just found this in the previous problem. What value (number) is \( A(1) \) equal to? What are \( A(2) \) and \( A(3) \) equal to?

(b) Find a general formula for \( A(x) \) by drawing the curve \( f(t) = 3t \), shading in the triangular region under the curve on the interval \( 0 \leq t \leq x \), and computing the area as you did before. Simplify your answer as much as possible. Your formula will still have an \( x \) in it.

(c) Check that your formula for \( A(x) \) is accurate by using this formula to compute \( A(1) \), \( A(2) \), and \( A(3) \). Did your answers match what you got in part (a) of this problem?

Problem 10.6. As above, let \( f(t) = 3t \) and consider the simplified formula that you found for \( A(x) = \int_0^x 3t \, dt \). Use this formula to find \( A'(x) \). Does this look familiar?

Problem 10.7. We’re now going to repeat the same processes as above with a new function: \( f(t) = 5t + 1 \). As before, we’ll define \( A(x) \) to be an area function: \( A(x) = \int_0^x (5t + 1) \, dt \).

(a) Notice that \( A(1) = \int_0^1 (5t + 1) \, dt \). Compute \( A(1) \) exactly. (You may have found this in a previous problem.)

(b) Compute \( A(2) \) by drawing the relevant picture, shading in the appropriate region, and computing the area.
(c) Likewise, compute $A(3)$.

(d) What do you think $A(0)$ is equal to?

(e) Find a general formula for $A(x)$ by drawing the curve $f(t) = 5t + 1$, shading in the triangular region under the curve on the interval $0 \leq t \leq x$, and computing the area as you did before. Simplify your answer as much as possible. Your formula will still have an $x$ in it.

(f) Check that your formula for $A(x)$ is accurate by using this formula to compute $A(0)$, $A(1)$, $A(2)$, and $A(3)$. Did your answers match what you got in the earlier parts of this problem?

**Problem 10.8.** As above, let $f(t) = 5t + 1$ and consider the simplified formula that you found for $A(x) = \int_0^x (5t + 1) \, dt$. Find $A'(x)$. Does this look familiar?

**Problem 10.9.** Let’s return to considering the function $f(t) = 3t$. This time we’ll define $A(x)$ to be an area function: $A(x) = \int_2^x 3t \, dt$. Notice that in this case the interval starts at $t = 2$ rather than at $t = 0$.

(a) Compute $A(3)$ by drawing the relevant picture, shading in the appropriate region, and computing the area.

(b) Likewise, compute $A(4)$.

(c) What do you think $A(2)$ is equal to?

(d) Find a general formula for $A(x)$ by drawing the curve $f(t) = 3t$, shading in the region under the curve on the interval $2 \leq t \leq x$, and computing the area as you did before. Simplify your answer as much as possible. Your formula will still have an $x$ in it.

(e) Check that your formula for $A(x)$ is accurate by using this formula to compute $A(2)$, $A(3)$, and $A(4)$. Did your answers match what you got in the earlier parts of this problem?

**Problem 10.10.** As above, let $f(t) = 3t$ and consider the simplified formula that you found for $A(x) = \int_2^x 3t \, dt$. Find $A'(x)$. Does this look familiar?

**Problem 10.11.** Consider the function $A(x) = \int_0^x 3t \, dt$ from Problem 10.5 and the function $A(x) = \int_2^x 3t \, dt$ from Problem 10.9. You found simplified version of both functions, and also found both derivatives.

(a) Compare the two simplified functions for $A(x)$. 
(b) Compare the two derivatives.

Thus far we have been computing $A(x)$ directly using geometry, and using the formula we get for $A(x)$ to determine $A'(x)$. This method only works in certain situations. Our goal is to develop the patterns we’ve noticed for $A'(x)$, and then use $A'$ to find $A$.

**Problem 10.12.** Suppose that $A(x) = \int_0^x t^2 \, dt$. Make a conjecture as to what $A'(x)$ is. (In the next few problems, we’ll do the formal work required to check this conjecture.)

**Problem 10.13.** Consider $f(t) = t^2$ and, as before, let $A(x)$ represent the area under the curve on the interval $0 \leq t \leq x$.

(a) Shade in the region under $f(t) = t^2$ on the interval $0 \leq t \leq 1$ and find an approximate value for $A(1)$.

(b) Shade in the region under $f(t) = t^2$ on the interval $0 \leq t \leq 2$ and find an approximate value for $A(2)$.

(c) Shade in the region under $f(t) = t^2$ on the interval $0 \leq t \leq 3$ and find an approximate value for $A(3)$.

**Instructor Note: In the next few problems, students begin approximating areas with rectangles. The instructor can moderate a class discussion about the choices of rectangles used, and the impact those choices make on final answers.**

**Problem 10.14.** Let $f(t) = t^2$ and $A(x) = \int_0^x t^2 \, dt$ as above.

(a) Sketch the the curve $f(t) = t^2$ and shade in the region under the curve on the interval $2 \leq t \leq 3$.

(b) Explain why the area of this region is equal to $A(3) - A(2)$.

(c) There are different ways to approximate the area of this region; we are going to use a single rectangle that roughly overlaps the shaded region. The base of this rectangle should be on the interval $2 \leq t \leq 3$. Draw a rectangle overlapping with the shaded region.

(d) What is the length of the base of this rectangle? What is the height of this rectangle? (There are several reasonable answers for what the height of the rectangle should be.)

**Problem 10.15.** Let $f(t) = t^2$ and $A(x) = \int_0^x t^2 \, dt$ as above. We’re going to do the same procedure as in the problem above, but in a slightly more abstract way.
(a) Shade in the region under the curve on the interval $x \leq t \leq x + h$.

(b) Explain why the area of the shaded region is equal to $A(x + h) - A(x)$.

(c) As with the previous problem, we are going to use a single rectangle that roughly overlaps the shaded region. The base of this rectangle should be on the interval $x \leq t \leq x + h$. Draw a rectangle overlapping with the shaded region.

(d) What is the length of the base of this rectangle?

(e) What is the height of this rectangle? (Again, there are several reasonable answers for what the height of the rectangle should be. Your answer will involve $x$ and may or may not involve $h$, as well.)

(f) Write a formula for the area of your rectangle.

Instructor Note: In the following problem students take a limit using the approximate areas found using rectangles. In this case it does not matter which rectangle students use (within reason), and a good discussion can be had around this point.

Problem 10.16. As above, let $A(x) = \int_0^x t^2 \, dt$. We are now ready to compute $A'(x)$.

(a) Write down the limit definition of $A'(x)$.

(b) Note that while the rectangle you drew in the previous problem does not have exactly the same area as $A(x + h) - A(x)$, as $h$ approaches 0, the two areas get closer and closer to each other. As a result, replace $A(x + h) - A(x)$ in the definition of $A'(x)$ with the area of your rectangle.

(c) Simplify this new expression to evaluate $A'(x)$.
(d) Compare the answer you got above to what your conjecture in Problem 10.12.

**Problem 10.17.** Now, suppose \( A(x) = \int_1^x t^3 \, dt \). What do you think \( A'(x) \) will be?

**Problem 10.18.** Finally, let \( f \) be a function and \( A(x) = \int_0^x f(t) \, dt \). What do you think \( A'(x) \) will be?

**Theorem 10.19.** *(The Fundamental Theorem of Calculus, Part I.)* For any continuous function \( f \) and any number \( a \) in its domain, let \( A(x) = \int_a^x f(t) \, dt \). Then \( A'(x) = \)???

**Note:** The Fundamental Theorem of Calculus used the word “continuous”. In simple terms, a continuous function is one which has no jumps or holes. More precisely, \( f \) is a **continuous function** if \( \lim_{x \to a} f(x) = f(a) \) for every real number \( a \) in the domain.

Now that we have a way to find \( A'(x) \) we will develop a way to determine \( A(x) \).

**Problem 10.20.** As earlier, let \( A(x) = \int_0^x t^2 \, dt \). Without coming up with a formula for \( A(x) \) we were able to determine a formula for \( A'(x) \). Now that we know what \( A'(x) \) is, find \( A(x) \). (Your answer will involve an unknown constant.)

**Problem 10.21.** As above, let \( A(x) = \int_0^x t^2 \, dt \).

(a) What do you think \( A(0) \) will be?

(b) Consider the formula for \( A(x) \) that you found in Problem 10.20. Use the knowledge of \( A(0) \) to determine what the unknown constant is.

(c) Use this formula to compute \( A(1) \), \( A(2) \), and \( A(3) \). Are these answers similar to what you found in Problem 10.13?

**Problem 10.22.** Now assume \( A(x) = \int_1^x t^3 \, dt \).

(a) According to the Fundamental Theorem of Calculus, Part I, what is \( A'(x) \)?

(b) Now that we know what \( A'(x) \) is, find \( A(x) \). (Your answer will involve an unknown constant.)

(c) What do you think \( A(1) \) is equal to?
(d) Use the value of $A(1)$ to find a formula for $A(x)$. 

The following problem is intended to summarize the relationship between $A$ and $f$.

**Problem 10.23.** Suppose that $f$ is a continuous function, that $a$ is a number in its domain, and that $A(x) = \int_a^x f(t) \, dt$. What is the relationship between $f$ and $A$? Describe this relationship in two different ways. (Think “How is the function $f$ related to the function $A$?” and “How is the function $A$ related to the function $f$?”)

The next batch of problems will lead us to developing a more general formula for computing area.

**Problem 10.24.** Let $A(x) = \int_0^x t^2 \, dt$ and consider the simplified the formula you found in 10.21.

![Graph of $y = t^2$](image)

Note that in almost all the questions below, the answers will have $a$ and/or $b$ in them.

(a) Compute $A(a)$ and shade in the region represented by $A(a)$.

(b) Compute $A(b)$ and shade in the region represented by $A(b)$.

(c) Compute $A(b) - A(a)$ and shade in the region represented by $A(b) - A(a)$.

(d) Evaluate $\int_a^b t^2 \, dt$. Explain how this is related to your answer above.

(e) Evaluate $\int_a^d t^2 \, dt$ and shade in the related region.
Problem 10.25. As above, let \( A(x) = \int_{0}^{x} t^2 \, dt \).

(a) Note that \( F(t) = \frac{1}{3} t^3 - 11 \) is another antiderivative of \( t^2 \). Compute \( F(b) - F(a) \).

(b) Suppose that \( G(t) = \frac{1}{3} t^3 + C \). Compute \( G(b) - G(a) \).

(c) What do you notice about \( F(b) - F(a) \), \( G(b) - G(a) \), \( A(b) - A(a) \), and \( \int_{a}^{b} t^2 \, dt \)?

Problem 10.26. Now assume \( A(x) = \int_{1}^{x} t^3 \, dt \), and recall the simplified formula you found in 10.22.

![Diagram](attachment:image.png)

(a) Compute \( A(a) \) and shade in the region represented by \( A(a) \).

(b) Compute \( A(b) \) and shade in the region represented by \( A(b) \).

(c) Compute \( A(b) - A(a) \) and shade in the region represented by \( A(b) - A(a) \).

(d) Evaluate \( \int_{a}^{b} t^3 \, dt \).

Problem 10.27. As above, let \( A(x) = \int_{1}^{x} t^3 \, dt \).

(a) Pick any antiderivative \( F \) of \( t^3 \). (There are many possible choices for \( F \)).

(b) Compute \( F(b) - F(a) \).
(c) What do you notice about $F(b) - F(a)$, $A(b) - A(a)$, and $\int_a^b t^3 \, dt$?

The following problem is intended to tie together one of the observations you may have made about the antiderivatives.

**Problem 10.28.** Suppose that $f$ is a function defined on the interval $a \leq t \leq b$, and that $F$ and $G$ are two antiderivatives of $f$.

(a) Compare $F$ and $G$.  (How are they the same?  How might they be different?)

(b) Compare $F(b) - F(a)$ and $G(b) - G(a)$.

**Problem 10.29.** We’ll now consider a new function: $f(t) = \sin(t)$.

(a) Pick any antiderivative $F$ of $f$.

(b) Compute $F(b) - F(a)$.  Your answer will have $a$ and $b$ in it.

(c) Make a conjecture about what $\int_a^b \sin(t) \, dt$ will be.

(d) Shade in the region represented by $\int_0^\pi \sin(t) \, dt$ and use your conjecture above to determine the area of that shaded region.

This brings us to the formal connection between derivatives, antiderivatives, and definite integrals.

**Theorem 10.30.** (*The Fundamental Theorem of Calculus, Part II.*) If $F$ is any antiderivative for $f$, then $\int_a^b f(x) \, dx =$ ???

We can now use The Fundamental Theorem of Calculus (Part II) to evaluate definite integrals.

**Problem 10.31.** Evaluate $\int_{-2}^2 x^2 \, dx$.

**Problem 10.32.** Evaluate $\int_0^4 \sqrt{x} \, dx$.

**Problem 10.33.** Write an integral to represent the area under the curve $y = \frac{1}{x}$ on the interval $1 \leq x \leq e$.  Evaluate the integral.

**Problem 10.34.** Evaluate $\int_1^3 u^3 \, du$.

**Problem 10.35.** Evaluate $\int_0^1 e^w \, dw$. 
Problem 10.36. Evaluate $\int_0^2 10^x \, dx$.

Problem 10.37. Evaluate $\int_0^1 \frac{1}{1+x^2} \, dx$.

In the next few problems we’ll examine the relationship between the definite integral and area if a function has negative values.

Problem 10.38. Consider the function $f(t) = \cos(t)$.

(a) Draw a graph of the function on the interval $0 \leq t \leq \pi$.

(b) Compute $\int_0^{\pi/2} \cos(t) \, dt$, and shade in the region associated with that area.

(c) Shade in the region between $y = \cos(t)$ and the $t$-axis on the interval $\frac{\pi}{2} \leq t \leq \pi$.

(d) Evaluate $\int_{\pi/2}^{\pi} \cos(t) \, dt$. What is the relationship between the shaded area and the value of the definite integral?

Problem 10.39. Suppose that $f(t)$ is a function such that $f(t) \leq 0$ on the interval $a \leq t \leq b$. Make a conjecture about the relationship between the integral $\int_a^b f(t) \, dt$ and the area between the curve $y = f(t)$ and the $t$-axis.

Problem 10.40. Evaluate $\int_0^\pi \cos(t) \, dt$. Use a picture of the graph of $y = \cos(t)$ on the interval $0 \leq t \leq \pi$ to explain why you got the answer you did.

We’ll end the chapter by going back to the Fundamental Theorem of Calculus, Part I, and giving a general proof. This is very similar to Problems 10.15–10.16, but using a general function $f(t)$ instead of $t^2$.

Problem 10.41. Let $f(t)$ be a function such that $f(t) \geq 0$ on a domain including $a$, and assume that $A(x) = \int_a^x f(t) \, dt$. Use the figure below as an example of the graph of such a function.
(a) Shade in the region under the curve corresponding to $A(x)$.

(b) Shade in the region under the curve corresponding to $A(x + h)$.

(c) Shade in the region under the curve on the interval $x \leq t \leq x + h$.

(d) Explain why the area of the shaded region on the interval $x \leq t \leq x + h$ is equal to $A(x + h) - A(x)$.

(e) As before, draw a single rectangle that roughly overlaps the shaded region. The base of the rectangle should be on the interval $x \leq t \leq x + h$.

(f) What is the length of the base of this rectangle?

(g) What is the height of this rectangle? (There are several reasonable answers.)

(h) Write a formula for the area of your rectangle.

Instructor Note: Once again, in the following problem, students may make different choices of an approximating rectangle, which can lead to a rich discussion.

Problem 10.42. As in the previous problem, let $A(x) = \int_a^x f(t)\, dt$. We are now ready to compute $A'(x)$.

(a) Write down the limit definition of $A'(x)$.

(b) Note that while the rectangle you drew in the previous problem does not have exactly the same area as $A(x + h) - A(x)$, as $h$ approaches 0, the two areas get closer and closer to each other. As a result, replace $A(x + h) - A(x)$ in the definition of $A'(x)$ with the area of your rectangle.

(c) Simplify this new expression to evaluate $A'(x)$.

With this we conclude our course on Single Variable Calculus. Congratulations!