An Introduction to Order Theory

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Foreword

To the Instructor

I want to begin by giving credit where credit is due — the exercises presented in these notes got their start with Constantine Tsinakis at Vanderbilt University. I had the good fortune of writing my dissertation under Professor Tsinakis in the late 1980’s, and it was in this setting that I benefitted from his carefully chosen problems. Over the decades that followed, I added many additional problems to suit the particular needs of my students but never improved on the core exercises I first explored under Constantine’s guidance. Readers who also studied under Professor Tsinakis will no doubt see his signature written subtly across these notes, especially in the material on distributive lattices.

I am also indebted to Zack French — a particularly precocious undergraduate — who not only worked through these notes in a guided independent study, but also edited the collection, corrected errors, and provided a number of new exercises. His comments to potential students appear below.

This collection develops basic order theory through a series of problems that are designed to move the reader through the concepts. A few proofs are scattered throughout to provide the reader with some clues regarding the techniques and style of order-theoretic arguments. The exercises presented in these notes have been used as the foundation study for four Masters’ Theses, two graduate courses on order theory, and a number of independent studies.

These exercises may be used in a classroom setting, or as a supervised independent study. However they are used, it is critical that the socratic method play a significant role. Indeed, these exercises were designed with individual reflection and struggle, followed by discussion, debate, and critique as the intended vehicle for moving the reader forward through the study. While it is certainly helpful for the study leader / instructor to have some knowledge of order theory, this is not required. It is important however, that the study leader / instructor be well-versed in the art of formal proof.
If these notes are used in a guided independent study, the instructor should plan on meeting with the student once a week. The student should present her work to the instructor; and the two should discuss the work together and progress toward an acceptable end product. As an independent study, I have found that it will take three to four months to complete all of the exercises.

If these notes are used in a classroom setting, then some variation of the “Moore method” should be used as the pedagogical model. While it is certainly possible to use a “pure” Moore method model and simply “turn students loose” on the exercises and let them compete, I have found it best to assign students exercises in advance and have them present their work to the class. This approach assures that steady progress will be made and encourages lively discussion as students struggle to understand work presented on a problem that they may not have investigated. If undergraduates or beginning graduate students are part of the class, I have also found it helpful to give weekly definitions quizzes to keep the class abreast of the ever-growing list of technical terms and concepts. In a class of experienced graduate students, formal exams are usually not necessary as the reflection-presentation-discussion cycle is sufficient for evaluation of student progress. In the classes where I felt the need to use testing, I have found that one exam at mid-term and one at the end of the semester, coupled with the weekly quizzes and daily presentations, have always provided me with ample data for summative purposes.

Grading student presentations is always a challenge. Here is a grading rubric that I have found works well in this setting.

PRESENTATION GUIDELINES

The goals of homework presentations are

- to emphasize individual mathematics efforts
- to evaluate student understanding of the material
- to develop the ability to communicate mathematical ideas

Presentations are to be made on the assigned date. Postponements will not be accepted except for emergencies or illness. You may work with other students to develop the solution, but you must present the solution. Presentations cannot be copied from others’ work. You should use LaTeX/Beamer, Power Point, MS Word, or some other electronic medium whenever possible to facilitate presentation and to make it easier for others to access your
solutions. You should post your corrected solutions or else email them to me. These presentations individually count 25 points each; together they will count as one exam grade. You will have ten minutes for most presentations. You are expected to use at least five minutes, but not more than fifteen minutes.

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<td>12</td>
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<td>Mathematical notation is used correctly</td>
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<td>Verbal communication is clear</td>
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Of course, pacing and content coverage are a critical concern when any subject is explored in a classroom setting. Depending on the preparedness level and ambition of the class members, it may be necessary to skip some material. I usually skip the material on Zorn’s Lemma after Exercise 29 in Section 1.3. If the class has taken a long time working through some concepts, the material on relatively complemented lattices in Section 2.2 after Exercise 109 can also be skipped, as can the material in Section 2.3 after Exercise 136.

James B. Hart

To the Student

I was introduced to the modified Moore method, as many aspiring mathematicians are, via a course in point-set topology during my undergraduate study. At that time I was still unsure where my future studies might take me. Although I wasn’t completely unfamiliar with proof-based mathematics courses, I was veering toward an applied path that would have had me creating models and algorithms more than exploring the more abstract path of discovering new pure mathematics. Up until that point, I hadn’t quite grasped the art of the proof, and had become somewhat frustrated in my efforts.

That changed the semester I decided to take introductory topology. The course was administered by Dr. James Hart, and his use of the Moore method was transformative for me. By the end of the course, I had a new appreciation for the method of proving statements using precise applications
of definitions and known results. As such, I had the confidence to devote my studies to a more foundational path in mathematics.

The version of the Moore method that Dr. Hart has employed in his courses is exceptionally good for students who, like me, benefit from a “gentle immersion” into true mathematics. Courses that rely on rote memorization of theorems are worthless for showing how results are formed, and courses that merely gloss over major results and provide exercises based on them tend to lose the foundational knowledge necessary to advance the field. On the other hand, if a course is too Spartan in nature, or overly competitive, many potentially great students may fall by the wayside simply because the method is too far outside their particular learning parameters. These notes are meant to be a more balanced approach.

As with traditional Moore method material, the notes have the “Definition, Theorem, Exercise” structure. Many major results have been broken up into exercises so students have the opportunity to develop foundational knowledge for themselves. We have broken with the Moore method tradition, however, and included some proofs. One benefit of this is to provide students the opportunity to compare and adapt methods other than their own, which will be useful later when interpreting other people’s work as part of preliminary research for their own projects. One final addition that is atypical in Moore-style notes are the hints; these have typically been added as a response to consistently recurring snags students encounter with some exercises. Since the time to get through a course using these notes is limited, the hints help to expedite some of these potential hang-ups. My personal opinion is that, when done right, a modified Moore method approach will simulate collaborating on a professional mathematical project (with some of the major details in place), and these notes are a good example of what that might look like.

I would like to take a moment to address anyone who would administer this type of course. As I mentioned before, overly competitive or stripped down versions of the Moore method have the danger of leaving some otherwise good students behind. I would urge instructors to remember the joys of discovery in mathematics, and use this as a guide on how to administer the course. It would be worthwhile to discover which of your students have stronger or weaker backgrounds before beginning the course, and assign exercises to particular students based on these strengths. This engenders a collaborative spirit that will be useful later, and future assignments will tend to progress organically throughout the course.

To carry the metaphor a bit further, the instructor’s role during the course should emulate a referee more than a lecturer. This is an instructor’s primary role aside from assigning exercises; as students present their exercises, the
instructor evaluates their argument on several key points. If instruction is necessary, give it in the form of notes, allow a student adjust course if necessary, and if a problem is particularly troublesome it is worthwhile to encourage class participation.

In conclusion, I encourage everyone involved to recognize that the main goal of these notes is to encourage further study in advanced mathematics. While the strongest students may succeed independent of the methods used, they will likely benefit from this approach, and otherwise capable mathematicians may receive an opportunity to thrive in an otherwise hostile learning environment. I hope working through these notes inspire those who use them as much as they have inspired me during the time Dr. Hart and I put them together, and I wish you all the best of luck.

Zack French
Chapter 1

Posets

1.1 Introduction

In this section, we will introduce some definitions and concepts about posets that will play a key role in the development of the subject in subsequent sections. It stands to reason that we should begin with a few definitions; and that is primarily what this section consists of.

A partially ordered set (or poset for short) is a system $P = (P, \leq)$ consisting of a set $P$ and a binary relation $\leq$ on the set $P$ satisfying the following conditions:

1. For all $x \in P$, we have $x \leq x$ (reflexivity).
2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

The binary relation $\leq$ defined above is called a partial ordering on the set $P$. Note that we are using a convention customary with binary relations — we write $x \leq y$ to mean $(x, y)$ is a member of $\leq$.

The set $\mathbb{S}u(X)$ of all subsets of a set $X$ is partially ordered by set inclusion. That is, the binary relation $\subseteq$ defined by

$$A \subseteq B \iff a \in B \text{ for all } a \in A$$

is a partial ordering on $\mathbb{S}u(X)$. The relation is partial in the sense that not all members of $\mathbb{S}u(X)$ are related under subset inclusion. For example, if $X = \{x, y, z\}$, then the subsets $A = \{x, y\}$ and $B = \{y, z\}$ are not related by subset inclusion.

Two elements $x, y$ of a poset $P = (P, \leq)$ are said to be comparable provided $x \leq y$ or $y \leq x$. We sometimes say that $x$ is below $y$ (or that $y$ is above
x). If this is not the case, we say that x and y are *incomparable* and write \( x \parallel y \). (Note: The notation for incomparability is not universally used.)

In a poset \( P \) we write \( x < y \) to mean that \( x \leq y \) but \( y \not\leq x \). In this case, we say the inequality is *strict*. It is acceptable to write \( x \geq y \) when \( y \leq x \), though we will not often have need of this convention.

A poset \( P \) is said to be a *chain* (or *totally ordered*) provided every element is comparable to every other element in \( P \); that is, for all \( x, y \in P \), we have \( x \leq y \) or \( y \leq x \). The positive integers under their natural ordering is an example of a chain. At the other extreme, we say a poset \( P \) is an *antichain* provided \( x \leq y \) only when \( x = y \). Note that the empty set and singleton sets are the only sets which are both a chain and an antichain under any partial ordering.

**Definition 1.** Let \( P = (P, \leq) \) be any poset. The *order dual* of \( P \) is defined to be the system \( P^{\text{op}} = (P, \leq_{\text{op}}) \) where \( x \leq_{\text{op}} y \iff y \leq x \). We usually denote the order dual of a poset \( P \) by simply writing \( P^{\text{op}} \).

Given any “statement” \( \Phi \) about a poset \( P \), we can obtain its “dual” simply by replacing every occurrence of \( \leq \) in the statement with \( \geq \). This simple fact gives rise to an important feature of order theory known as the *duality principle*:

A statement \( \Phi \) is true of all posets if and only if its dual is also true of all posets.

This simple observation will often be used to shorten proofs, particularly when the conjecture to be proved consists of two parts, one part the dual of the other. In such cases, we will prove one part and state that the other “follows by duality”.

Any subset \( Q \) of a poset \( P \) may be regarded as a poset in its own right under the restriction to \( Q \) of the partial ordering on \( P \). When viewed in this manner, we say the subset \( Q \) is a *subposet* of \( P \). There are two particularly important examples of subposets we will be using:

**Definition 2.** Let \( P = (P, \leq) \) be a poset and let \( L \subseteq P \). We say that \( L \) is a *lowerset* (or *order ideal*) of \( P \) provided, whenever \( x \in P \) is such that \( x \leq y \) for some \( y \in L \), then \( x \in L \). An *upperset* (or *order filter*) of \( P \) is defined to be a lowerset of \( P^{\text{op}} \). We let \( \mathcal{L}(P) \) denote the set of all lowersets for \( P \), partially ordered by set inclusion, and let \( \mathcal{U}(P) \) denote the set of all uppersets of \( P \) partially ordered by *reverse* set inclusion (that is, \( A \leq B \) in \( \mathcal{U}(P) \) if and only if \( B \subseteq A \)).
Definition 3. Let $P = (P, \leq)$ be a poset and let $X \subseteq P$. The set

$$\downarrow X = \{ p \in P : p \leq x \text{ for some } x \in X \}$$

is called the \textit{lower set generated by $X$} in $P$. Likewise, the set

$$\uparrow X = \{ p \in P : x \leq p \text{ for some } x \in X \}$$

is called the \textit{upper set generated by $X$} in $P$.

A lowerset generated by a singleton is called a \textit{principal lowerset}; it is often denoted by $\downarrow x$ instead of $\downarrow \{x\}$.

Definition 4. Let $P = (P, \leq)$ be a poset. We say that $x \in P$ is \textit{minimal} in $P$ provided $\downarrow x = \{x\}$. A \textit{maximal} element in $P$ is a minimal element in $P^\text{op}$.

Definition 5. Let $P = (P, \leq)$ be a poset. We say $P$ has a \textit{least element} provided $P$ has exactly one minimal element. We say that $P$ has a \textit{greatest element} provided $P^\text{op}$ has a least element. We use $\bot$ and $\top$ to denote the least and greatest elements, respectively, of $P$ (when they exist).

A poset which has a least element is said to be \textit{lower-bounded}. A poset which has a greatest element is said to be \textit{upper-bounded}. A \textit{bounded} poset has both a least and a greatest element.

Definition 6. Let $P = (P, \leq)$ be a poset and let $X \subseteq P$. We say that $X$ is \textit{bounded below} (or has a \textit{lower bound} in $P$ provided there exist $y \in P$ such that $y \in \downarrow x$ for all $x \in X$. We say that $X$ is \textit{upper-bounded} in $P$ provided it is lower-bounded in $P^\text{op}$. We let $m(X)$ and $j(X)$ denote the set of all lower-bounds and upper-bounds, respectively, for $X$.

1.2 Some Important Classes of Posets

In this section, we introduce a few of the fundamental classes of posets that will play a role in all of the work to follow.

Definition 7. Let $P$ be a poset and let $X \subseteq P$. We say that $X$ has an \textit{infimum} (or greatest lower-bound) in $P$ provided $m(X)$ has a greatest element. This element is known as the \textit{meet} of $X$ in $P$ and is denoted by $\wedge X$. Likewise, we say that $X$ has a \textit{supremum} (or least upper bound) in $P$ provided $j(X)$ has a least element. This element is known as the \textit{join} of $X$ in $P$ and is denoted by $\vee X$. 
When \( X = \{x_1, \ldots, x_n\} \) has a meet in a poset \( P \), we often denote it by

\[
\bigwedge X = x_1 \land \ldots \land x_n
\]

and likewise denote the join of \( X \) in \( P \) by

\[
\bigvee X = x_1 \lor \ldots \lor x_n
\]

Note the use of the logic operations of conjunction and disjunction to denote the finite meets and joins.

**Exercise 8.** Let \( P = (P, \leq) \) be any poset. Prove that \( P \) is lower-bounded if and only if \( \bigvee \emptyset \) exists in \( P \).

**Definition 9.** A poset \( J \) is called a *join semilattice* provided every pair of elements in \( J \) has a join in \( J \). We say that a poset \( P \) is a *meet semilattice* provided \( P^{\text{op}} \) is a join semilattice.

**Exercise 10.** Let \( \mathbb{N} \) denote the set of nonnegative integers and let \( E(\mathbb{N}) \) denote the collection of all finite subsets of \( \mathbb{N} \) that contain an even number of elements. Consider the poset \( E(\mathbb{N}) = (E(\mathbb{N}), \subseteq) \).

1. Given any \( X, Y \in E(\mathbb{N}) \), show that \( X \cup Y \) serves as the join of \( X \) and \( Y \) in \( E(\mathbb{N}) \). (Hence, \( E(\mathbb{N}) \) is a join semilattice.)

2. Suppose that \( X, Y \in E(\mathbb{N}) \) are such that \( X \land Y \) exists in \( E(\mathbb{N}) \). Prove that \( X \land Y = X \cap Y \).

3. Explain why \( E(\mathbb{N}) \) is not a meet semilattice.

**Exercise 11.** Use mathematical induction to prove that a poset \( P \) is a join semilattice if and only if every nonempty finite subset of \( P \) has a join in \( P \).

**Definition 12.** A poset \( L \) is said to be a *lattice* provided it is both a join and a meet semilattice.

**Exercise 13.** Let \( \mathbb{N} \) denote the set of nonnegative integers. For \( a, b \in \mathbb{N} \), let \( a \leq b \) if and only if there exist \( k \in \mathbb{N} \) such that \( b = ak \). (In other words, \( a \leq b \) if and only if \( a \) is an integer factor of \( b \).) Prove that \( \leq \) is a partial ordering and that \( \mathbb{N} \) is a bounded lattice under this partial ordering.

An *identity* for a lattice is a particular equation which holds true for all elements in a given lattice. In the following exercises, we will look at several identities, some enjoyed by all lattices; other enjoyed only by certain lattices. These identities will prove useful in much of the work to follow.
**Exercise 14.** Let $L$ be a lattice, and let $x,y,z \in L$. Prove that the following identities hold.

1. $x \lor y = y \lor x$
2. $x \land y = y \land x$
3. $(x \lor y) \lor z = x \lor (y \lor z)$
4. $(x \land y) \land z = x \land (y \land z)$
5. $x \lor x = x$
6. $x \land x = x$
7. $x \lor (x \land y) = x$
8. $x \land (x \lor y) = x$

Viewed as binary operations on $L$, Exercise 14 (1) - (4) tell us that $\land$ and $\lor$ are commutative and associative. Exercise 14 (5)-(6) tell us that $\land$ and $\lor$ are idempotent. Identities (7) and (8) in Exercise 14 are called the absorption laws.

**Exercise 15.** Let $L$ be a set equipped with binary operations $m$ and $j$ which are commutative, associative, idempotent, and satisfy the absorption laws. Define binary relations on $L$ as follows:

1. $x \leq y \iff m(x,y) = x$
2. $x \leq j(y) \iff j(x,y) = y$

Prove that $\leq$ is equal to $\leq$; that is, prove $x \leq y \iff x \leq j(y)$. Furthermore, prove that $L$ is a lattice under this partial ordering with $x \lor y = j(x,y)$ and $x \land y = m(x,y)$.

We can think of meets and joins in a lattice $L$ as defining binary operations on the underlying set:

1. Define $\lor : L \times L \to L$ by $\lor(x,y) = x \lor y$
2. Define $\land : L \times L \to L$ by $\land(x,y) = x \land y$

**Exercise 16.** Let $L$ be a set equipped with binary operations $m$ and $j$ which are commutative, associative, idempotent, and satisfy the absorption laws. Define binary relations on $L$ as follows:

1. $x \leq y \iff m(x,y) = x$
2. \( x \sqsupseteq y \iff j(x, y) = y \)

Prove that \( \sqsubseteq = \sqsupseteq \); that is, prove \( x \leq y \iff x \sqsupseteq y \). Furthermore, prove that \( L \) is a lattice under this partial ordering with \( x \lor y = j(x, y) \) and \( x \land y = m(x, y) \).

In light of the previous exercises, we can think of a lattice \( L \) either as a poset in which every pair of elements has a meet and a join, or we may think of a lattice as a triple \( L = (L, \land, \lor) \), where \( L \) is a set, \( \land \) and \( \lor \) are binary operations which are commutative, associative, idempotent, and satisfy the absorption laws. Both viewpoints are useful.

**Definition 17.** Let \( P \) be a poset. A subposet \( D \) of \( P \) is directed provided every finite subset of \( D \) has an upper bound in \( D \). A directed lowerset of \( P \) is called an ideal of \( P \).

Directed sets are sometimes called up-directed, but this is not standard. Note that directed sets are by definition nonempty. A directed set in \( P \) is said to be filtered (or down-directed) in \( P \).

**Definition 18.** A poset \( P \) is said to be directed complete (a DCPO) provided every directed subset of \( P \) has a join in \( P \).

Directed complete posets play a vital role in modern computer science, primarily because directed sets can be given a natural meaning in terms of the information content of computations. We will be using them in quite a different manner, however.

**Exercise 19.** Let \( \mathbb{N} \) denote the set of nonnegative integers. Show that the poset \( \text{Fin}(\mathbb{N}) \) of finite subsets of \( \mathbb{N} \) under subset inclusion is a lattice but is not a DCPO.

**Exercise 20.** Let \( X = (X, \leq) \) be any nonempty poset, and let \( [X \to X] \) denote the set of all functions \( f : X \to X \). For \( f, g \in [X \to X] \), let \( f \preceq g \) if and only if \( f(x) \leq g(x) \) for all \( x \in X \). Prove that \( [X \to X] \) is a poset under \( \preceq \). (This poset is called the function space associated with \( X \), and \( \preceq \) is called the pointwise ordering on \( [X \to X] \).)

**Exercise 21.** Suppose that \( X = (X, \leq) \) is a nonempty DCPO and suppose that \( F \subseteq [X \to X] \) is directed under the pointwise ordering.

For each \( x \in X \), prove that the set \( F(x) = \{ f(x) : f \in F \} \) is directed in \( X \).

Prove that the mapping \( h : X \to X \) defined by \( h(x) = \lor F(x) \) serves as the join of the family \( F \) in \( [X \to X] \). (Hence, the function space of a DCPO is always a DCPO.)
Let \( \mathbb{N} \) denote the set of nonnegative integers under the natural ordering. Let \( \mathbb{N}_0 = \{0\} \times \mathbb{N} \) and let \( \mathbb{N}_1 = \{1\} \times \mathbb{N} \). Suppose \( \omega \not\in \mathbb{N}_0 \cup \mathbb{N}_1 \), and consider the set \( P = \mathbb{N}_0 \cup \mathbb{N}_1 \cup \{\omega\} \). Define a partial ordering \( \preceq \) on \( P \) as follows:

- Let \( u \prec \omega \) for all \( u \in \mathbb{N}_0 \cup \mathbb{N}_1 \).
- For \( j \in \{0, 1\} \), let \( (j, x) \preceq (j, y) \) if and only if \( x \leq y \) for all \( x, y \in \mathbb{N} \).
- Let \( (j, x) \parallel (k, y) \) for all \( x, y \in \mathbb{N} \) if \( j \neq k \).

**Exercise 22.** Consider the poset \( P = (\mathbb{N}, \preceq) \).

1. Prove that \( P = (P, \preceq) \) is a DCPO.
2. Explain why \( P \) is not a meet semilattice.

**Exercise 23.** Let \( P \) be a poset and prove that the following statements are equivalent:

1. Every subset of \( P \) has a meet in \( P \).
2. Every subset of \( P \) has a join in \( P \).
3. The poset \( P \) is both a lower-bounded join semilattice and a DCPO.

**Definition 24.** A poset \( P \) is complete provided every subset of \( P \) has a meet (equivalently, a join) in \( P \). Complete posets are often called complete lattices.

Every finite lattice is necessarily complete. Note that the real numbers under their natural ordering form a chain which is not complete. Also, whenever \( S \) is an infinite set, the collection \( \text{Fin}(S) \) of finite subsets of \( S \) is not a complete lattice under the partial ordering of subset inclusion.

**Definition 25.** We say that a poset \( P \) can be order embedded in another poset \( Q \) provided there exists an isotone injection \( f : P \rightarrow Q \).

**Exercise 26.** Prove that any poset \( P \) can be order-embedded in a complete lattice. Hint: Consider the lattice \( \mathcal{L}(P) \) defined in Definition 2.

**Definition 27.** Let \( P \) be any poset. An ideal of \( P \) is a directed lowerset of \( P \), and a filter of \( P \) is a directed lowerset of \( P^{op} \). Let \( \text{Idl}(P) \) denote the family of all ideals of \( P \), partially ordered by set inclusion. Let \( \text{Fil}(P) \) denote the family of all filters of \( P \), partially ordered by reverse set inclusion.
It should be noted that a filter $F$ in a poset $P$ is an upperset that is down directed. That is, if $A \subseteq F$ is finite, then $A$ has a lower bound in $F$. It should also be noted that $\text{Fil}(P)$ is the order dual of $\text{Idl}(P^\text{op})$. (Thus, we consider $G \leq F$ in $\text{Fil}(P)$ provided $F \subseteq G$.) It will be important to keep this in mind.

**Exercise 28.** For any poset $P$, prove that the union of a directed family of ideals of $P$ is also an ideal of $P$, and conclude that $\text{Idl}(P)$ is a DCPO. Prove that $P$ may be order-embedded in $\text{Idl}(P)$.

Note that $\text{Fil}(P)$ is a dual DCPO for any poset $P$. This means that $\text{Fil}(P)$ is closed under meets of down-directed sets. There is a dual order-embedding of $P$ into $\text{Fil}(P)$.

**Exercise 29.** Let $L = (L, \leq)$ be a join semilattice and let $I \subseteq L$ be nonempty. Prove that $I$ is an ideal of $L$ if and only if $I$ is a lower set of $L$ with the property that $x \vee y \in I$ whenever $x, y \in I$.

Of course, Exercise 29 tells us that if $M = (M, \leq)$ is a meet semilattice and $F \subseteq M$, then $F$ is a filter of $M$ if and only if $F$ is an upperset with the property that $x \land y \in F$ whenever $x, y \in F$.

**Exercise 30.** Let $L$ be a nonempty join semilattice and let $S = \{I_j : j \in J\}$ be a family of ideals of $L$ (indexed by the set $J$).

1. Explain why there exists an ideal of $L$ which contains $\bigcup S$.
2. Prove that $I(S) = \bigcap \{X \in \text{Idl}(L) : \bigcup S \subseteq X\}$ is an ideal of $L$.
3. Explain why $\text{Idl}(L)$ is a complete join semilattice (complete in the sense that every nonempty subset of $\text{Idl}(L)$ has a join in $\text{Idl}(L)$).

Whenever $L$ is a nonempty join semilattice and $X \subseteq L$, then there certainly exist ideals of $L$ that contain $X$. In light of the previous exercise, the intersection of all the ideals that contain $X$ is itself an ideal. It is the smallest ideal of $L$ that contains $X$, and it is said to be generated by the set $X$. The ideal generated by $X$ will be denoted by $(X)$.

**Exercise 31.** Let $L$ be a lower bounded join semilattice.

1. If $I, J \in \text{Idl}(L)$, prove that $I \cap J \in \text{Idl}(L)$.
2. Explain why $\text{Idl}(L)$ is a complete lattice.

It should be noted that similar results hold for the dual DCPO of filters for a nonempty meet semilattice. If we let $M$ be a nonempty meet semilattice,
and we let $T$ denote a family of filters from $M$, then $\bigcap T$ is itself a filter of $M$. Furthermore, $\bigcap T$ serves as the join of $T$ in $\text{Fil}(M)$. Consequently, if $X \subseteq M$, it makes sense to talk about the filter generated by $X$ — it is simply the intersection of all filters that contain $X$. In keeping with our notation for ideals, we will let $[X]$ denote the filter generated by $X$. Consequently, if $M$ is an upper bounded meet semilattice, the $\text{Fil}(M)$ is also a complete lattice. We will need this idea in the next section.

### 1.3 Zorn’s Lemma and the Axiom of Choice

No discussion of basic order theory would be complete without an investigation of Zorn’s Lemma (which is neither a lemma nor attributable solely to Max Zorn, one of its early defenders).

*Let $P$ be a nonempty poset. If every chain in $P$ has an upper bound in $P$, then $P$ has a maximal element.*

The previous, rather innocuous-looking statement is what has come to be known as Zorn’s Lemma. It is generally taken as an axiom for order-theorists and plays a vital role in transfinite induction, as well as many existence proofs. As a quick example, recall that a basis for a vector space $V$ over a field is a maximal, linearly independent subset. We can use Zorn’s Lemma to prove that every nontrivial vector space has a basis.

To see how, let $V$ be any nontrivial vector space over a field $F$. Since the underlying set $V$ is not a singleton by assumption, we may select a vector $\vec{v}$ in $V$ which is not the zero-vector. Clearly this vector is linearly independent when viewed as a singleton; hence, $V$ contains linearly independent subsets. Now, let $P(V)$ denote the set of all linearly independent subsets of $V$, partially ordered by set-inclusion, and let $C \subseteq P(V)$ be any chain. Since every member of $C$ is a linearly independent subset of $V$, it follows that $S = \bigcup C$ is also a linearly independent subset of $V$ (the fact that $C$ is a chain under set-inclusion is critical here). The set $S$ clearly serves as an upper bound for $C$ in $P(V)$; hence we know that $P(V)$ contains a maximal member by Zorn’s Lemma. Any such member is the basis we seek.

**Exercise 32.** Use Zorn’s Lemma to prove that every nonempty poset contains a maximal antichain.

**Exercise 33.** Let $P = (P, \leq)$ be any poset. Use Zorn’s Lemma to prove that there exists a partial ordering $\preceq$ on $P$ which extends $\leq$ such that $(P, \preceq)$ is a chain. To say that $\preceq$ extends $\leq$ means, of course, that $\leq \subseteq \preceq$. Hint: Let $P$ denote the family of all partial orders on $P$, partially ordered by set-inclusion.
Exercise 34. Prove that the following statements are equivalent for any nonempty poset $P$:

1. If every chain in $P$ has an upper bound in $P$, then $P$ has a maximal member.

2. If every chain in $P$ has a least upper bound in $P$, then $P$ has a maximal member.

3. $P$ contains a maximal chain.

The previous exercise gives two axioms equivalent to Zorn’s Lemma. The following exercise gives several more.

Exercise 35. Let $P$ be a nonempty poset. An element $p \in P$ is proper provided $p \neq \top$. If $P$ has no greatest element, then every element of $P$ is proper. Prove that the following statements are equivalent.

1. If every chain in $P$ has an upper bound in $P$, then $P$ has a maximal element.

2. If every chain in $P$ has a proper upper bound in $P$, then every chain is contained in a maximal chain.

3. If every chain in $P$ has a proper upper bound in $P$, then every chain has a maximal upper bound in $P$.

4. If $F$ is a partially ordered family of sets with the property that $\bigcup C \in F$ for every chain $C \subseteq F$, then $F$ has a maximal element.

5. If $F$ is a partially ordered family of sets with the property that a set $U$ is a member of $F$ if and only if every finite subset of $U$ is a member of $F$, then for all $A \in F$, there exists a maximal member of $F$ containing $A$.

Applying Zorn’s Lemma is sometimes whimsically referred to as Zornication.

Definition 36. A poset $P$ is said to be well-ordered provided every nonempty subset of $P$ contains a least element.

Exercise 37. Prove that every well-ordered poset is necessarily a chain.

Definition 38. A poset $P$ satisfies the Descending Chain Condition (DCC) provided every descending chain $x_1 > x_2 > \ldots$ in $P$ contains only finitely many distinct members.
Exercise 39. Prove that a nonempty chain is well-ordered if and only if it satisfies DCC.

Exercise 40. Show by example that there exist posets which satisfy DCC but are not well-ordered.

Exercise 41. Prove that in a well-ordered set, every subset which is bounded above has a least upper bound.

Exercise 42. Let $A$ be a chain, and suppose that $A = B_1 \cup \ldots \cup B_n$, where each $B_i$ is well-ordered. Prove that $A$ is well-ordered.

Definition 43. Let $P$ be a poset, and let $a \in P$. The segment of $P$ generated by $a$ is defined to be the set $S(a) = \downarrow a - \{a\}$.

Exercise 44. Let $C$ be a well-ordered set. Prove that a proper lower set in $C$ must be a segment of $C$.

Exercise 45. Prove that a chain $C$ is well-ordered if and only if every segment of $C$ is well-ordered.

Definition 46. Let $P = (P, \leq)$ and $Q = (Q, \leq)$ be two posets. A mapping $f : P \rightarrow Q$ is called an order homomorphism or an isotone function provided, for all $x, y \in P$,

$$x \leq y \implies f(x) \leq f(y)$$

We say that $P$ and $Q$ are isomorphic as posets provided there exists a bijective order homomorphism $f : P \rightarrow Q$ whose inverse is also an order homomorphism.

Exercise 47. Prove that a well-ordered chain cannot be order-isomorphic to one of its segments.

The following result, which uses Zorn’s Lemma, is of immense historical importance in the development of partially ordered sets. Known as the well-ordering principle, it is remarkable, although somewhat disturbing.

Theorem 48. Every nonempty set can be well-ordered.

Proof. Let $A$ be a nonempty set, and let $L$ denote the family of all well-ordered nonempty subsets of $A$. Note that $L$ is nonempty, since every finite
subset of \( A \) containing \( n \) elements can be well-ordered in \( n! \) ways and will therefore appear in \( L \) exactly \( n! \) times. We are not concerned with the preservation of any existing partial ordering on \( A \).

Define a partial ordering on \( L \) as follows: \( B \leq C \) if and only if \( B \) is a lowerset of \( C \) under the ordering on \( C \). Now, suppose that \( C \) is a chain in \( L \) and consider the set \( B = \bigcup C \). We will first prove that \( B \) is an upper bound for \( C \) in \( L = (L, \leq) \). To this end, we must define a (well) ordering on \( B \) and prove that each member of \( C \) is a lowerset of \( B \) under this ordering.

Of course, the ordering on \( B \) is inherited from the members of the chain \( C \). Indeed, suppose that \( x, y \in B \). It follows that there exist \( C_x, C_y \in C \) such that \( x \in C_x \) and \( y \in C_y \). We know that \( C_x \) is comparable to \( C_y \); for definiteness, assume \( C_x \leq C_y \). Then \( C_x \) is a lowerset of \( C_y \); and it follows that both \( x, y \in C_y \). Since \( C_y \) is a (well-ordered) chain, it follows that \( x \leq y \) or \( y \leq x \). It is a routine matter to prove that the relationship between elements \( B \) thus inherits is indeed a partial ordering. In fact, it is easy to see that \( B \) is a chain.

To see that \( B \) is well-ordered, it suffices to prove that every segment of \( B \) is well-ordered. To this end, let \( S(a) \) be any segment in \( B \) and assume that \( a \in C_a \). It will suffice to prove that \( S(a) \subseteq C_a \), since \( C_a \) is well-ordered. Suppose that \( x \in S(a) \). Then \( x < a \), and there exist \( C_x \in C \) such that \( x \in C_x \). If \( C_x \leq C_a \), then clearly \( x \in C_a \). On the other hand, suppose \( C_a \leq C_x \). In this case, since \( C_a \) is a lowerset in \( C_x \), the fact that \( x < a \) once again implies that \( x \in C_a \).

Clearly, each \( c \in C \) is a lowerset of \( B \). Consequently, \( B \) serves as (the least) upper bound of \( C \) in \( L \); by Zorn’s Lemma, it now follows that \( L \) contains a maximal member. Let \( D \) be such an element. By construction, \( D \) is a well-ordered subset of \( A \). We prove that \( D = A \). Indeed, suppose this is not the case. Then there exist \( u \in A - D \). The set \( D \cup \{u\} \) can clearly be well-ordered — simply set \( d < u \) for all \( d \in D \). Since \( D \) is a lowerset in \( D \cup \{u\} \), we have violated the maximality of \( D \) in \( L \). Thus, we must have \( D = A \), as desired.

\( \square \)

We have just proven that Zorn’s Lemma implies the well-ordering principle. It turns out that the well-ordering principle implies Zorn’s Lemma — that is, as axioms the two ideas are equivalent. To prove this, however, we will need yet another controversial player in the realm of 20th Century mathematics known as the Axiom of Choice:

Let \( P \) be any nonempty set. There exists a function \( f \), defined on the nonempty subsets of \( P \), which assigns every such set one of its members. Such a function is called a choice function for \( P \).
Zorn’s Lemma is unsettling to many because it seems to make a leap from the finite to the infinite (more to the point, from the countable to the uncountable). The previous proof is a beautiful case-in-point. It is intuitively clear that we can well-order finite subsets of a nonempty set \( A \), but it is not at all obvious that we can well-order infinite subsets of \( A \). Nonetheless, Zorn’s Lemma bridges this gap and tells us that a well-ordering of \( A \) exists, although it gives no clue how such an ordering can be constructed. Ernst Zermelo first proposed the Axiom of Choice as an intuitive axiom from which Zorn’s Lemma could be proved, in an attempt to make the conclusions of Zorn’s Lemma less unsettling. Instead, his proposal sparked a controversy which has raged for the better part of a century between those who accept the “transfinite” constructions offered by Zorn’s Lemma and those who do not.

**Exercise 49.** Prove that the well-ordering principle (and hence Zorn’s Lemma) implies the Axiom of Choice. Hint: Well-order the set \( A \), then let the choice function \( f \) assign each subset its least element.

We will now prove that the Axiom of Choice implies Zorn’s Lemma.

**Definition 50.** Let \( P \) be a poset. A map \( g: P \to P \) is said to be **enlarging** provided \( x \leq g(x) \) for all \( x \in P \). A map \( g: P \to P \) is **contracting** if \( g(x) \leq x \) for all \( x \in P \). We say such maps are **strict** provided the inequalities are always strict.

**Exercise 51.** Show by example that an enlarging map need not be order-preserving.

**Definition 52.** Let \( P \) be a lower-bounded poset, and let \( g \) be an enlarging map on \( P \). A subset \( A \) of \( P \) is called a **tower** provided

1. \( \bot \in A \)
2. \( g(A) \) is a subset of \( A \)
3. whenever a chain \( C \subseteq A \) has a least upper bound in \( P \), then this least upper bound is a member of \( A \).

Note that towers exist in any lower-bounded poset \( P \) since \( P \) itself is a tower.

**Definition 53.** Let \( P \) be a poset and let \( x, y \in P \). We say that \( y \) **covers** \( x \) provided \( x < y \) and \( \uparrow x \cap \downarrow y = \{x, y\} \). (Thus, there are no elements “between” \( x \) and \( y \).) We use the symbol \( x \prec y \) to denote this situation.
Exercise 54. Let $P$ be a lower-bounded poset with an enlarging map $g$ and let $T$ denote the intersection of all towers in $P$. Prove that $T$ is discrete as a subposet of $P$ in the sense that, for all $t \in T$, either $t = g(t)$ or $t \prec g(t)$. Hint: Assume there exists $b \in T$ such that $t < b < g(t)$. Let $S = \downarrow b - \downarrow t$ and obtain a contradiction by proving that $T - S$ is a tower.

Definition 55. Let $P$ be a poset. An element $p \in P$ is called a node of $P$ if $p$ is comparable to everything in $P$; that is, $P = \downarrow p \cup \uparrow p$. The set of all nodes for a poset $P$ is called the center of $P$ and will be denoted by $\text{Cen}(P)$.

Exercise 56. The center of a poset is always a chain. Is it always a maximal chain?

Exercise 57. Let $P$ be a lower-bounded poset which admits an enlarging map $g$ and let $T$ denote the intersection of all towers in $T$. Let $b \in \text{Cen}(T)$ and let

$$W = \{ t \in T : t \leq b \text{ or } g(b) \leq t \}$$

1. Prove that $W$ is a tower (hence $T = W$). Hint: Consider cases for Property (2). Most are trivial; one uses Exercise 54.
2. Prove that $\text{Cen}(T)$ is a tower. Conclude that $T$ is a chain in $P$.

Definition 58. Let $X$ be any set. A mapping $f : X \rightarrow X$ has a fixed point provided there exists an element $x \in X$ such that $f(x) = x$.

Exercise 59. Let $P$ be a lower-bounded poset in which every chain has a least upper bound. Without using Zorn’s Lemma or the Axiom of Choice, prove that every enlarging map on $P$ must have a fixed point. Hint: Use Exercise 57. (This is where Property (3) for towers comes into play.)

Exercise 60. Prove that the Axiom of Choice implies Zorn’s Lemma. You will need the last exercise and Exercise 34. How can you get around the problem that the poset in question need not have a lower-bound?
Chapter 2

Lattices

2.1 Modular and Distributive Lattices

In this chapter, we will explore some of the major properties that lattices can satisfy. We begin with what is likely one of the most important properties from an historical perspective. The property takes the form of an identity and is inspired by one of the fundamental properties relating set union and set intersection, as well as a related property enjoyed by all rings.

Definition 61. Let \( L = (L, \wedge, \vee) \) be a lattice. We say that \( L \) is **distributive** provided joins distribute over meets and vice-versa. That is, for all \( x, y, z \in L \), we have

\[
\begin{align*}
\bullet & \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\
\bullet & \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)
\end{align*}
\]

The concept of distributivity is **self-dual**; that is, a lattice is distributive if and only if its order dual is distributive. Interestingly enough, we can say even more than this — the two distributive conditions are actually **equivalent**, as the following result shows.

Lemma 62. Let \( L \) be a lattice and let \( x, y, z \in L \). Then

\[
\begin{align*}
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\
x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z)
\end{align*}
\]

Proof. Assume the first equation, then observe that the absorption laws give us
\[(x \lor y) \land (x \lor z) = [(x \lor y) \land x] \lor [(x \lor y) \land z]\]
\[
= x \lor [(x \lor y) \land z]
\]
\[
= x \lor [(x \land z) \lor (y \land z)]
\]
\[
= [x \lor (x \land z)] \lor (y \land z)
\]
\[
= x \lor (y \land z)
\]

Hence, the first equation implies the second. The fact that the second equation implies the first follows by duality.

\[\square\]

**Exercise 63.** Prove that the inequality \((x \land y) \lor (x \land z) \leq x \land (y \lor z)\) holds in any lattice. Hence, to prove that a lattice is distributive, one need only establish the reverse inequality.

**Exercise 64.** Let \(N_5 = \{\bot, a, b, c, \top\}\) and let \(N_5 = (N_5, \leq)\), where \(a \lor b = b \lor c = \top, a \lor b = b, a \land c = \bot\). Show that \(N_5\) is a lattice that is not distributive.

**Exercise 65.** Let \(M_5 = \{\bot, a, b, c, \top\}\) and let \(M_5 = (M_5, \leq)\), where \(a \lor b = b \lor c = a \land c = \bot\) and \(a \land b = a \land c = b \land c = \bot\). Show that \(M_5\) is a lattice that is not distributive.

The lattice \(M_5\) is called the *nondistributive diamond*; the lattice \(N_5\) is simply called the *pentagon*.

**Definition 66.** Let \(L = (L, \lor, \land)\) be a lattice. A subset \(S\) of \(L\) is said to be a sublattice of \(L\) provided \(S\) is closed under the restrictions of \(\lor\) and \(\land\) to \(S\).

Under their natural ordering, the integers form a sublattice of the lattice of real numbers, as do the rational numbers. Given any set \(S\), the set \(\text{Fin}(S)\) of all finite subsets of \(S\) forms a (distributive) sublattice of \(\text{Su}(S)\).

**Exercise 67.** Let \(\mathbb{N}\) denote the set of nonnegative integers. Define a partial ordering on \(\mathbb{N}\) by \(a \preccurlyeq b\) if and only if \(a\) is a factor of \(b\). (See Exercise 13.) Show that \(\mathbb{N}\) is a distributive lattice under this partial ordering.

**Exercise 68.** Let \(L\) be a nonempty distributive lattice and let \(I\) and \(J\) be ideals of \(L\). Prove that \((I \cup J)\) is the set

\[K = \{x \lor y : x, y \in I \cup J\}\]
It should be noted that a similar result holds for joins in the filter lattice of a distributive lattice $L$. In particular, if $U, V \in \text{Fil}(L)$, then $[U \cup V]$ is the set

$$W = \{x \land y : x, y \in U \cup V\}$$

**Exercise 69.** Let $L$ be a lower bounded, distributive lattice. Prove that $\text{Idl}(L)$ is a distributive lattice. Hint: Let $I, J, K \in \text{Idl}(L)$. Use Exercise 68 to prove that $I \cap (J \cup K) \subseteq (I \cap J) \cup (I \cap K)$, where $\cup$ denotes the join in $\text{Idl}(L)$.

**Exercise 70.** Let $L$ be a lower bounded lattice. Explain why $\text{Idl}(L)$ is distributive if and only if $L$ is distributive. Hint: Exercise 28 is helpful here.

**Definition 71.** A lattice $L$ is said to be modular (or weakly distributive) provided, for all $x, y, z \in L$, $z \leq x$ implies that

$$x \land (y \lor z) = (x \land y) \lor z$$

**Exercise 72.** Prove that every distributive lattice is modular. Show that the nondistributive diamond is modular (but not distributive), and show that the pentagon is not modular.

**Theorem 73.** A lattice is modular if and only if it does not contain the pentagon as a sublattice.

**Proof.** Let $L = (L, \land, \lor)$ be a lattice, and suppose that $J$ is a sublattice of $L$. If $J$ is not modular, then there exist $x, y, z \in J$ such that $z < x$ but $x \land (y \lor z) \neq (x \land y) \lor z$. Since $x, y, z \in L$, it follows that $L$ is not modular. Consequently, if $L$ is modular, we see that $L$ cannot contain the pentagon as a sublattice.

On the other hand, suppose that $L$ is not modular. Then there exist $x, y, z \in L$ such that $z < x$ but $(x \land y) \lor z < x \land (y \lor z)$. We will use this fact to construct a pentagon in $L$. Let $a = z \lor y$, $b = x \land y$, $c = (x \land y) \lor z$ and $d = x \land (y \lor z)$. We will prove that $\{a, b, c, d, y\}$ forms a pentagon.

By assumption, $c < d$. Also, we must have $b < y < a$. To see why, notice first that clearly $b = x \land y \leq y$. If $b = y$, then we have $y \leq x$. However, if this is the case, then

$$(x \land y) \lor z = y \lor z$$

Thus, by assumption, we have $y \lor z < x \land (y \lor z)$ — an impossibility. Thus, we must have $b < y$. To see that $y < a$, again first note that $y \leq y \lor z = a$. If $y = a$, then we must have $z \leq y$. Therefore,
\[ x \land (y \lor z) = x \land y \]

Thus, by assumption, we have \((x \land y) \lor z < x \land y\) — an impossibility. We therefore must have \(y < a\).

Now, suppose \(y \leq c\). By assumption, we know

\[ y < d = x \land (y \lor z) \]

This implies that \(y \leq x\). That means \(c = (x \land y) \lor z = y \lor z = a\), and by assumption \(c = y \lor z < x \land (y \lor z) = d\) — an impossibility.

On the other hand, if \(c = (x \land y) \lor z \leq y\), then \(z \leq y\). It follows that \(d = x \land (y \lor z) = x \land y = b\). This means \(d \leq c\) — contrary to assumption.

From here, it is easy to see that \(c \land y = b\), \(d \lor y = a\), and we have the pentagon.

Exercise 74. Let \(L\) be a lattice which satisfies the following identity:

\((D_1)\) For all \(x, y, z \in L\), \((x \land y) \lor (y \land z) \lor (x \land z) = (x \lor y) \land (y \lor z) \land (x \lor z)\)

Use the absorption laws to prove that \(L\) is modular.

Exercise 75. For a lattice \(L\), show that the following statements are equivalent.

1. The lattice \(L\) is modular.
2. For any \(x, y, z \in L\) we have \((x \land y) \lor (x \land z) = x \land (y \lor (x \land z))\).
3. For any \(x, y, z \in L\) we have \((x \lor y) \land (x \lor z) = x \lor (y \land (x \lor z))\).

Exercise 76. Prove that a lattice \(L\) is distributive if and only if it satisfies Identity \(D_1\) in Exercise 74. Hint: For one implication, join \(x\) to both sides of \(D_1\) and use modularity.

Exercise 77. Let \(L\) be a modular lattice. Let \(x, y, z \in L\) and let

- \(u = (x \lor y) \land (x \lor z) \land (y \lor z)\)
- \(v = (x \land y) \lor (x \land z) \lor (y \land z)\)
- \(a = (y \land z) \lor [x \land (y \lor z)]\)
- \(b = (x \land z) \lor [y \land (x \lor z)]\)
• $c = (x \land y) \lor [z \land (x \lor y)]$

1. Use modularity to prove that $a \lor b = a \lor c = b \lor c = u$.

2. Use modularity to show that $a, b$ and $c$ are equal to the order duals of their definitions.

3. Prove that $a \land b = a \land c = b \land c = v$.

4. Prove that if any two of the above elements are equal, then $u = v$.

5. Use Exercise 76 to prove that if $L$ is not distributive, then $u \neq v$.

**Exercise 78.** Prove that a lattice is distributive if and only if it does not contain the pentagon $N_5$ or the non-distributive diamond $M_5$ as a sublattice.

**Exercise 79.** Prove that a lattice $L$ is distributive if and only if, for all $x, y, z \in L$, $x \lor y = x \lor z$ and $x \land y = x \land z$ together imply that $y = z$. Hint: For one implication, use the fact that $y = (x \lor y) \land y$.

**Definition 80.** Let $L$ be a lattice. A ideal $I \neq L$ of $L$ is said to be prime provided $a \land b \in I$ always implies that $a \in I$ or $b \in I$.

**Exercise 81.** Prove that the following statements are equivalent for a proper ideal $I$ of a lattice $L$:

1. $I$ is a prime ideal;
2. $L-I$ is a filter in $L$;
3. $L-I$ is a prime filter in $L$.

**Exercise 82.** Let $L$ be a lattice and let $2$ denote the two-element chain $\{0, 1\}$ under the natural ordering. Prove that an ideal $I$ of $L$ is prime if and only if there exists a lattice epimorphism $f : L \to 2$ such that $I = f^{-1}(\{0\})$.

**Exercise 83.** Suppose that $L$ and $M$ are lattices and suppose $f : L \to M$ is a lattice epimorphism. (That is, a surjective lattice homomorphism.) Prove the following statements:

1. If $I \in \Id(L)$, then $f(I) \in \Id(M)$.
2. If $J \in \Id(M)$, then $f^{-1}(J) \in \Id(L)$.
3. If $J \in \Id(M)$ is prime in $M$, then $f^{-1}(J)$ is prime in $L$. 

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Exercise 84. A proper ideal of a lattice \( L \) is maximal provided it is a maximal element of \( \text{Idl}(L) - \{L\} \). If \( L \) has a greatest element, use Zorn’s Lemma to prove that every proper ideal of \( L \) is contained in a maximal ideal of \( L \).

Exercise 85. Let \( L \) be a distributive lattice. Prove that every maximal ideal of \( L \) is prime in \( L \).

Exercise 86. Let \( L = (L, \wedge, \vee) \) be a distributive lattice. Suppose that \( I \) is an ideal of \( L \) and \( F \) is a filter of \( L \), and suppose that \( I \) and \( F \) are disjoint. For clarity, if \( U \) and \( V \) are ideals of \( L \), let \( U \sqcup V \) denote the join of these ideals in \( \text{Idl}(L) \).

1. Use Zorn’s Lemma to prove that the set
   \[ X = \{ J \in \text{Idl}(L) : I \subseteq J \text{ and } F \cap J = \emptyset \} \]
   has a maximal member, \( P \).

2. Suppose that there exist \( a, b \in L \) such that \( a \wedge b \in P \) but \( a, b \notin P \). Explain why \( P \sqcup (a) \) and \( P \sqcup (b) \) cannot be disjoint with \( F \).

3. Use Exercise 88 to explain why there exist \( p, q \in P \) such that \( a \vee p \in F \) and \( b \vee q \in F \).

4. Let \( x = (a \vee p) \wedge (b \vee q) \) and prove that
   \[ x = (p \wedge q) \vee (p \wedge b) \vee (a \wedge q) \vee (a \wedge b) \]

5. Explain why \( x \in P \cap F \). Conclude that \( P \) must be a prime ideal.

The last exercise provides proof for one of the most important results in the theory of distributive lattices. Known as the Prime Ideal Theorem, it will play a key role in many later results.

Theorem 87. (Prime Ideal Theorem) Let \( L \) be a distributive lattice. For every ideal \( I \) and filter \( F \) of \( L \) such that \( I \cap F = \emptyset \), there exists a prime ideal \( P \) of \( L \) such that \( I \subseteq P \) and \( F \cap P = \emptyset \).

Exercise 88. Let \( L \) be a distributive lattice and let \( I \) be a proper ideal of \( L \). Prove that, for every \( a \in L - I \), there exists a prime ideal \( P_a \) containing \( I \) which does not contain \( a \).

Exercise 89. Suppose that \( L \) is a lower bounded, distributive lattice and suppose that \( P \) is a prime ideal of \( L \). If \( P \) is contained in two incomparable ideals, use the previous exercise to show \( P \) is contained in two incomparable prime ideals.
Exercise 90. Suppose that $L$ is a lower bounded, distributive lattice and suppose that $P$ is a prime ideal of $L$. If $\uparrow P$ is a chain in $\text{Idl}(L)$, then show that every proper member of $\uparrow P$ is a prime ideal.

Exercise 91. Let $L$ be a distributive lattice, and suppose $a \not\leq b$ in $L$. Prove that there exists a prime ideal containing $b$ which does not contain $a$.

Exercise 92. Suppose that $L$ is a distributive lattice and let $a, b \in L$. Prove that the following statements are true.

1. The set $F(a, b) = \{x \in L : a \leq b \lor x\}$ is a filter of $L$.

2. If $L$ has a least element, and $a \leq b$, then $F(a, b) = L$.

Exercise 93. Suppose that $L$ is a lower-bounded distributive lattice, and suppose $\uparrow P$ is a chain in $\text{Idl}(L)$ for all prime ideals $P$ of $L$. Show that $[F(a, b) \cup F(b, a)] = L$ for all $a, b \in L$. (Hint: Assume the contrary and use the previous exercise.)

Exercise 94. Suppose that $L$ is a lower bounded, distributive lattice, and suppose that $[F(a, b) \cup F(b, a)] = L$ for all $a, b \in L$. If $c \leq a \land b$, show that there exist $p, q \in L$ such that $p \lor a = a \lor b = q \lor b$ and $p \land q = c$. (Use the comments after Exercise 68.)

Exercise 95. Suppose that $L$ is a lower bounded, distributive lattice, and suppose that for all $a, b \in L$, there exist $p, q \in L$ such that $p \lor a = a \lor b = q \lor b$ and $p \land q = \bot$, where $\bot$ is the least element of $L$. If $P$ is any prime ideal of $L$, show that the set $\mathcal{C}_P$ of all prime ideals of $L$ that contain $P$ is a chain under set inclusion.

If we combine Exercises 89 - 94, we have a proof of the following result.

Theorem 96. Let $L$ be a lower bounded, distributive lattice. The following statements are equivalent.

1. If $P$ is any prime ideal of $L$, then $\uparrow P$ is a chain in $\text{Idl}(L)$.

2. If $P$ is any prime ideal of $L$, then the set $\mathcal{C}_P$ of all prime ideals of $L$ that contain $P$ is a chain under subset inclusion.

3. For all $a, b \in L$, we have $[F(a, b) \cup F(b, a)] = L$, where $F(x, y) = \{u \in L : x \leq y \lor u\}$.

4. For all $a, b, c \in L$ such that $c \leq a \land b$, there exist $p, q \in L$ such that $p \lor a = a \lor b = q \lor b$ and $p \land q = c$. (Hint: Assume the contrary and use the previous exercise.)
5. For all \(a, b \in L\), there exist \(p, q \in L\) such that \(p \lor a = a \lor b = q \lor b\) and \(p \land q = \bot\).

Any lower bounded, distributive lattice \(L\) that satisfies the conditions of Theorem 96 is said to be\textit{ relatively normal}. It turns out that relatively normal lattices play an important role in many branches of mathematics. We will return to relatively normal lattices later in these notes.

When working with relatively normal lattices, Criterion (2) and Criterion (5) are the ones most commonly used. Criterion (5) is sometimes called the Monteiro condition, because Antonio Monteiro first established its equivalence to Criterion (2). It is worth noting that Criterion (2) is a “forbidden substructure” condition, because it stipulates that no prime ideal may be contained in a pair of incomparable prime ideals. If we let \(P(L)\) denote the poset of prime ideals for \(L\) ordered by set inclusion, then this means that \(P(L)\) cannot contain “V’s” (that is, subposets of the form \(\{a, b, c\}\) where \(a < b, a < c\) and \(b\) is incomparable to \(c\)).

\textbf{Definition 97.} Two elements \(a\) and \(b\) of a lower bounded meet semilattice \(M\) are\textit{ orthogonal} provided \(a \land b = \bot\). Two elements \(p\) and \(q\) of an upper bounded join semilattice \(J\) are\textit{ comaximal} provided they are orthogonal in \(J^{\text{op}}\).

Orthogonal elements are sometimes called\textit{ disjoint}, but this term is not commonly used. One might expect that orthogonal elements would be called “cominimal” but this term has never gained favor.

\textbf{Lemma 98.} A lower bounded, distributive lattice \(L\) is relatively normal if and only if every pair of incomparable prime filters of \(L\) is comaximal in \(\text{Idl}(L)\).

\textbf{Proof.} Suppose first that \(L\) contains incomparable prime filters \(F\) and \(G\) that are not comaximal. This means that \([F \cup G] \neq L\), so there exist \(x \in L - [F \cup G]\). The Prime Ideal Theorem therefore tells us that there is a prime ideal \(P\) of \(L\) that is disjoint with \([F \cup G]\). Exercise 64 tells us that \(L - F\) and \(L - G\) are prime ideals. These are certainly incomparable and contain \(P\). Consequently, \(L\) fails Criterion (2) and is not relatively normal.

On the other hand, suppose that every pair of incomparable prime filters in \(L\) is comaximal. Let \(P \in P(L)\) and suppose that \(Q, R \in C_P\). If \(Q\) and \(R\) are incomparable, then \(L - Q\) and \(L - R\) are incomparable prime filters of \(L\). By assumption, \([L - Q] \cup [L - R] = L\); consequently, for each \(x \in L\), there exist \(u \in L - Q\) and \(v \in L - R\) such that \(x = u \land v\). In particular, this is true for \(\bot\), the least element of \(L\). However, \(\bot \in P\), which implies that \(u \in P\) or \(v \in P\) — contrary to the fact that \(P\) is disjoint with both \(L - Q\) and \(L - R\).
2.2 Relatively Complemented Lattices

In this section, we will introduce one of the most important families of distributive lattices. Virtually any branch of mathematics that deals with partially ordered objects will at some point deal with structures introduced in this section.

Let $P$ be a poset and let $a, b \in P$. Throughout this section, we will let $[a, b] = \uparrow a \cap \downarrow b$. This subset of $P$ is called an interval in $P$; and, of course, is nonempty if and only if $a \leq b$.

**Definition 99.** Let $L$ be a lattice and let $[a, b] \subseteq L$. An element $x \in [a, b]$ has a relative complement in $[a, b]$ provided there exist $y \in [a, b]$ such that $x \land y = a$ and $x \lor y = b$. We say that $[a, b]$ is relatively complemented provided every element in $[a, b]$ has a relative complement in $[a, b]$. A lattice in which every interval is relatively complemented is called a relatively complemented lattice.

If $L$ is a bounded lattice, then $L = [\bot, \top]$, and relatively complemented elements of $L$ are said to be complemented. A complemented, distributive lattice is called a Boolean lattice in honor of George Boole, a prominent nineteenth century mathematician. (Notice that Boolean lattices are necessarily bounded.) Motivated by this classical definition, relatively complemented, distributive lattices are called generalized Boolean lattices. A generalized Boolean lattice is a Boolean lattice if and only if it is bounded.

It is easy to verify that the nondistributive diamond $M_5$ and pentagon $N_5$ are both complemented lattices. (See Exercises 64 and 65.) Furthermore, there are elements in both lattices that have multiple complements. Consequently, relative complements need not be unique.

**Exercise 100.** Let $L$ be a distributive lattice, and let $a, b \in L$. Prove that an element of $[a, b]$ can have at most one relative complement in $[a, b]$.

Let $L$ be a lower bounded, distributive lattice. If $a, b \in L$ and $a$ has a relative complement in $[\bot, a \lor b]$, then it is unique; and we denote it by $b \setminus a$. Note that $a$ and $b \setminus a$ are orthogonal.
Exercise 101. Let $L$ be a lower bounded, distributive lattice. Prove that the following statements are equivalent.

1. The lattice $L$ is a generalized Boolean lattice.

2. The element $b \setminus a$ exists for all $a, b \in L$. (Hint: For $x \in [a, b]$, consider $y = (b \setminus x) \lor (a \setminus x) \lor a$.)

Exercise 102. If $B$ is a Boolean lattice, use the previous exercise to show that every ideal of $B$ is a lower bounded, generalized Boolean lattice.

Lemma 103. Every prime ideal of a relatively complemented lattice $L$ is maximal.

Proof. Let $P$ be a prime ideal of $L$, let $x \in L - P$, and consider $I = \{P \cup \{x\}\}$. We will prove that $I = L$. To this end, suppose that $y \in L - P$ is distinct from $x$. Let $z \in \downarrow x \cap P$. Since $L$ is relatively complemented, there exist $d \in [z, x \lor y]$ such that $x \land d = z$ and $x \lor d = x \lor y$. Now, since $P$ is a prime ideal and $x \not\in P$, we must conclude that $d \in P$. However, this implies that $x \lor d \in I$; consequently, we may conclude that $y \in I$, as desired. □

Notice that Lemma 103 tells us that the prime ideals of a relatively complemented lattice form an antichain. Consequently, every lower bounded, distributive, relatively complemented lattice is relatively normal. In particular, every Boolean lattice is relatively normal.

Exercise 104. Let $L$ be a distributive lattice and let $a, b \in L$.

1. Prove that $F = \{x \in L : b \leq x \lor c\}$ is a nonempty filter of $L$.

2. Let $G = \uparrow c \cup F$. If $a \in G$, there exist $u \in F$ such that $u \land c \leq a$. Show that $d = a \lor (u \land b)$ is the relative complement of $c$ in $[a, b]$.

Exercise 105. Let $L$ be a distributive lattice that is not a generalized Boolean lattice.

1. There exist $a, b \in L$ and $c \in [a, b]$ such that $c$ has no relative complement. Use the previous exercise to prove that there is a filter of $L$ that contains $c$ but does not contain $a$.

2. Explain why there exists a prime ideal $P$ that contains $a$ but does not contain $c$.

3. Show that $I = \{P \cup \downarrow c\}$ is an ideal of $L$ that does not contain $b$.

4. Prove that there exists a prime ideal $Q$ that properly contains $P$. 

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The previous two exercises, along with Lemma 103, tell us that among distributive lattices, the generalized Boolean lattices are precisely the ones whose prime ideals form antichains.

**Exercise 106.** Let $B$ be a Boolean lattice, and let $I$ be an ideal of $B$. If $M$ is a prime ideal of $I$, show that $M = P \cap I$ for some prime ideal $P$ of $B$.

Let $L$ be a bounded lattice. Whenever $x \in L$ has a unique complement, it is customary to let $x^c$ denote that element. Other common symbols for this element include $\overline{x}$ and $\neg x$.

**Exercise 107.** Let $B$ be a Boolean lattice and let $x, y, z \in B$. Prove the following.

1. $x \land y = \bot \iff x \leq y^c$
2. $x = (x^c)^c$
3. $x \land y \leq z \iff y \leq x^c \lor z$
4. $(x \lor y)^c = x^c \land y^c$
5. $(x \land y)^c = x^c \lor y^c$

The last two properties in the previous exercise are known as De Morgan’s Laws.

**Definition 108.** A lattice $L$ is said to be join continuous if for all $y \in L$ and $X \subseteq L$ such that $\bigvee X$ exists, then $\bigvee \{y \land x : x \in X\}$ exists also; and we have $y \land \bigvee X = \bigvee \{y \land x : x \in X\}$

**Exercise 109.** Prove that every Boolean lattice is join continuous. Hint: Let $x_0 = \bigvee X$ and let $z$ be an upper bound for $\{y \land x : x \in X\}$. Prove that $y \land x_0 \leq z$.

**Definition 110.** Let $P$ be a lower-bounded poset. We say that an element $a \in P$ is an atom of $P$ provided $\bot \prec a$. Likewise, an element $c$ of an upper-bounded poset $P$ is a co-atom of $P$ provided $c$ is an atom of $P^{\text{op}}$. We say that a lower-bounded poset $P$ is atomic provided $\downarrow x$ contains an atom for all $\bot \prec x \in P$.

**Exercise 111.** Let $L$ be a uniquely complemented lattice and let $a \in L$ be an atom. Prove that $a^c$ is a co-atom in $L$ (that is, an atom in $L^{\text{op}}$).
Lattices

Exercise 112. Let **L** be a Boolean lattice. Prove that **L** is atomic if and only if its order dual **L**\textsuperscript{op} is atomic.

Exercise 113. Let **L** be an atomic Boolean lattice. For each \( x < \top \in L \), prove that there exists an atom \( a \in L \) such that \( x < x \lor a \). Hint: Explain why \( \uparrow x \) contains a co-atom \( b \); consider \( b^c \).

Exercise 114. Let **L** be a Boolean lattice. Prove that **L** is atomic if and only if the top element is a join of atoms. Hint: If **L** is atomic, let \( \mathcal{F} = \{ \alpha \in \mathcal{L} : \alpha \text{ is the join of a set of atoms} \} \)

Exercise 115. Let **L** be a complete, atomic Boolean lattice. Prove that every element of **L** is the join of a set of atoms. (Hint: Use join-continuity.)

Exercise 116. Prove that a complete Boolean lattice is atomic if and only if it is order isomorphic to the powerset of some set.

We know that complements, when they exist in a distributive lattice, are necessarily unique. We will conclude this section by looking at a few results probing the connection between distributivity in a lattice and the existence of unique complements.

Definition 117. Let **L** be a uniquely complemented lattice. An element \( \bot < a \in L \) is regular provided \( a \land x = a \land y = \bot \) implies \( a \land (x \lor y) = \bot \). We will say that **L** is regular provided every element exceeds a regular element.

Theorem 118. Every uniquely complemented regular lattice is distributive.

Proof. Suppose by way of contradiction that there exist \( x, y, z \in L \) such that

\[
\begin{align*}
  u &= (x \land y) \lor (x \land z) \neq x \land (y \lor z) = v
\end{align*}
\]

Since \( u \leq v \) in all lattices, it follows that \( u < v \). Since **L** is uniquely complemented, \( t = u^c \land v > \bot \). Also,

\[
\begin{align*}
  t \land y &= (u^c \land v) \land y \\
  &= (u^c \land (x \land y)) \\
  &\leq u^c \land u = \bot
\end{align*}
\]

Similarly, we have \( t \land z = \bot \). Now, suppose that \( b \in \downarrow t \) is regular. Then \( b \land y \leq t \land y = \bot \) and \( b \land z \leq t \land z = \bot \). Hence, we know that \( b \land (y \lor z) = \bot \). But, we also know that
Thus, \( b \land (y \lor z) = \bot \) implies that \( b = \bot \) — contrary to assumption. Thus, \( L \) must be distributive.

**Exercise 119.** Let \( L \) be a uniquely complemented lattice and let \( a \in L \) be an atom. Prove that \( x \land a = \bot \iff x \leq a^c \). Hint: Suppose that \( y = a^c \land x < x \) and prove that \( a^c \) has two distinct complements, \( a \) and \( z = y^c \land x \).

**Exercise 120.** Prove that every uniquely complemented atomic lattice is regular and hence distributive. Hint: Prove that the atoms are regular elements.

**Exercise 121.** Let \( L \) be a uniquely complemented lattice with least element \( \bot \) and greatest element \( \top \). For \( x, y \in L \), prove that the following statements are equivalent.

1. We have \( x < y \).
2. We have \( \bot < x^c \land y \).
3. We have \( x \lor y^c < \top \).

**Definition 122.** Let \( P \) be a poset. We say that \( P \) is weakly atomic provided, for all \( a < b \in P \), there exist \( x, y \in [a, b] \) such that \( y \) covers \( x \).

**Exercise 123.** Prove that every weakly atomic, uniquely complemented lattice is atomic and hence distributive.

**Exercise 124.** Let \( L \) be a complemented modular lattice and suppose that \( a < b \in L \). Prove that \( [a, b] \) is a complemented modular sublattice of \( L \).

**Lemma 125.** Let \( L \) be a complemented modular lattice and suppose that \( a < b \in L \). If \( x \in [a, b] \) and \( s \in [a, b] \) is a complement of \( x \) in \( [a, b] \), then there is a complement \( t \) of \( x \) in \( L \) such that \( s = (a \lor t) \land b \).

**Proof.** Let \( y \) be a complement of \( a \) in \( [\bot, s] \) and let \( z \) be a complement of \( b \) in \( [y, z] \). Now, \( s \in [y, z] \); let \( t \) be a complement of \( s \) in \( [y, z] \). By construction, \( y \leq t \leq z \). Now,

- \( x \lor s = b \) and \( x \land s = a \)
- \( a \land y = \bot \) and \( a \lor y = s \)
\[ b \land z = s \text{ and } b \lor z = \top \]
\[ s \land t = y \text{ and } s \lor t = z \]

Hence, observe that

\[
x \land t = (x \land b) \land (z \land t) \\
= x \land (b \land z) \land t \\
= x \land s \land t \\
= (x \land s) \land (s \land t) \\
= a \land y = \bot
\]

By dualizing the previous string of equalities, we see that \[ x \lor t = \top \] as well. Hence, \( t \) is a complement of \( x \) in \( L \). Finally, observe that

\[ s = z \land b = [a \lor (y \lor t)] \land b = (a \lor t) \land b \]

**Exercise 126.** Prove that every uniquely complemented modular lattice is distributive.

### 2.3 Adjunctions and Heyting Lattices

In this section, we will explore another important class of distributive lattice — the *Heyting* lattices. These lattices are inspired by a branch of mathematics known as *intuitionistic* logic. We begin with a concept which will have far-reaching implications in other sections.

**Definition 127.** Let \( P = (P, \leq) \) and \( Q = (Q, \sqsubseteq) \) be posets, and let \( f : P \rightarrow Q \) and \( g : Q \rightarrow P \) be functions. We say that \( f \) and \( g \) form an *adjunction* provided

\[ f(p) \sqsubseteq q \iff p \leq g(q) \]

for all \( p \in P \) and \( q \in Q \). We will use the symbol \( (f, g) : P \rightleftarrows Q \) to indicate that an adjunction exists between the posets \( P \) and \( Q \) and to label the component functions.

When working with adjunctions, we will usually dispense with distinct symbols for the partial orders on the posets \( P \) and \( Q \) unless special care is needed. Adjunctions are often called *residuations*. If \( (f, g) : P \rightleftarrows Q \), then
we call \( f \) a left adjoint for \( g \) and call \( g \) a right adjoint for \( f \). The functions \( f \) and \( g \) are often called left and right residuals, but we shall stick to the term “adjoint”.

**Exercise 128.** Suppose \((f, g) : P \leftrightarrows Q\) is an adjunction between posets \( P = (P, \leq) \) and \( Q = (Q, \leq)\). Prove the following statements are true:

1. \( f(g(q)) \leq q \) and \( p \leq g(f(p)) \) for all \( p \in P \) and \( q \in Q \).
2. Both \( f \) and \( g \) are order homomorphisms.

**Exercise 129.** Let \((f, g) : P \leftrightarrows Q\) be an adjunction between posets \( P \) and \( Q \). Prove that \( f \) preserves all existing joins in \( P \) and \( g \) preserves all existing meets in \( Q \).

**Exercise 130.** Let \((f, g) : P \leftrightarrows Q\) be an adjunction between posets \( P \) and \( Q \). Prove that \( g \circ f \circ g = g \) and \( f \circ g \circ f = f \).

**Exercise 131.** Let \( P \) and \( Q \) be posets and let \( f : P \rightarrow Q \) and \( g : Q \rightarrow P \) be functions. Prove the following statements are equivalent:

1. The mappings \( f \) and \( g \) form an adjunction.
2. The mapping \( g \) is isotone, and for all \( p \in P \), \( g^{-1}(\uparrow p) = \uparrow f(p) \)
3. The mapping \( f \) is isotone, and for all \( q \in Q \), \( f^{-1}(\downarrow q) = \downarrow g(q) \).

Let \((f, g) : P \leftrightarrows Q\) be an adjunction. Exercise 131 tells us that \( f \) uniquely determines \( g \) and vice-versa. Indeed, we know that, for all \( q \in Q \) and all \( p \in P \),

- \( g(q) = \bigvee_{p \in P} f^{-1}(\downarrow q) \), and
- \( f(p) = \bigwedge_{q \in Q} g^{-1}(\uparrow p) \).

Hence, we are justified in referring to \( g \) as the right adjoint of \( f \) and to \( f \) as the left adjoint of \( g \).

**Exercise 132.** Prove that, whenever \( P \) is a complete lattice and \( Q \) is a poset, then a mapping \( f : P \rightarrow Q \) has a right adjoint if and only if \( f \) preserves arbitrary joins.

Likewise, whenever \( Q \) is a complete lattice and \( P \) is a poset, a mapping \( g : Q \rightarrow P \) has a left adjoint if and only if \( g \) preserves arbitrary meets. Both of these statements require completeness.
Exercise 133. Show by example that preservation of all existing joins is not sufficient to guarantee that \( f \) has a right adjoint if the poset \( P \) is not a complete lattice. Hint: Consider the chain of nonnegative integers and let \( f \) be the constant mapping defined by \( f(x) = 0 \) for all nonnegative integers \( x \).

Exercise 134. Let \( P \) be and \( Q \) be posets and suppose that \( f : P \to Q \) is a bijection. Prove that \( (f, f^{-1}) : P \to Q \) if and only if both \( f \) and \( g \) are order preserving. Explain why this fact provides a “free” proof that order isomorphisms preserve all existing joins and meets.

Definition 135. Let \( L \) be a bounded lattice. We say that \( L \) is a Heyting lattice (or a Brouwerian lattice) if for all \( a, b \in L \), there exists an element \( c \in L \) such that, for all \( x \in L \),

\[
a \land x \leq b \iff x \leq c.
\]

In a Heyting lattice, it is easy to see that the element \( c \) is uniquely determined by \( a \) and \( b \). The element \( c \) is usually denoted by \( a \to b \) and is called the relative pseudocomplement of \( a \) with respect to \( b \). In this context, the arrow is known as the Heyting arrow, or implication.

Exercise 136. Let \( L \) be a Heyting lattice and let \( a, b \in L \). For any \( x \in L \), prove that \( a \to b = \bigvee \{ x \in L : a \land x \leq b \} \).

Heyting lattices are very common structures. For example, all Boolean lattices are Heyting lattices — take \( a \to b = a^c \lor b \), where \( a^c \) denotes the complement of \( a \).

Exercise 137. Let \( L \) be a Heyting lattice and let \( a \in L \) be fixed. Prove that the maps \( m : L \to L \) and \( i : L \to L \) defined by

\[
m(x) = a \land x \quad i(x) = a \to x
\]

form an adjunction.

Exercise 138. Let \( L \) be a Heyting lattice.

1. Prove that \( L \) is join continuous (and in particular is distributive). See Definition 108.

2. Prove that, whenever \( X \subseteq L \) is such that \( \bigwedge X \) exists, then

\[
a \to \bigwedge X = \bigwedge \{ a \to x : x \in X \}.
\]
Definition 139. A complete, join continuous lattice is called a frame.

Note that every complete Heyting lattice is automatically a frame. The following exercise proves that the converse is also true.

Exercise 140. Prove that every frame is a Heyting lattice. Hint: Let \( a \to b = \bigvee \{ c : c \wedge a \leq b \} \).

Exercise 141. Recall that for any set \( X \), a topology is any nonempty collection \( \Omega \subseteq \mathcal{P}(X) \) that satisfies the following conditions:

- For any finite \( \mathcal{F} \subseteq \Omega \), we have \( \bigcap \mathcal{F} \in \Omega \).
- For any collection \( \mathcal{G} \subseteq \Omega \), we have \( \bigcup \mathcal{G} \in \Omega \).

The members of \( \Omega \) are called opens. Prove that \( \Omega \) is a frame under subset inclusion.

Let \( L \) be a Heyting lattice and observe that we have

\[ c \leq x \to b \iff c \wedge x \leq b \iff x \leq c \to b. \]

Define two maps \( \lambda_b : L \to L^\text{op} \) and \( \rho_b : L^\text{op} \to L \) by

\[ \lambda_b(x) = x \to b = \rho_b(x), \]

Exercise 142. Prove that \( (\lambda_b, \rho_b) : L \leftrightarrow L^\text{op} \), where \( \lambda \) and \( \rho \) are defined above. Explain why this tells us that

1. The Heyting arrow is order-reversing in its left argument; that is, if \( x \leq y \), then \( y \to b \leq x \to b \).
2. Whenever \( X \subseteq L \) is such that \( \bigvee X \) exists, then

\[ \bigvee X \to b = \bigwedge \{ x \to b : x \in X \}. \]

In light of the previous exercise, we often say that the Heyting arrow is self-adjoint.

Definition 143. Let \( P \) and \( Q \) be posets. A Galois connection between \( P \) and \( Q \) is an adjunction \( (f, g) : P \leftrightarrow Q^\text{op} \).

Historically speaking, Galois connections (named in honor of Evariste Galois) were the first types of adjunctions to be studied in order theory.

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They arose in the study of automorphism groups for fields. The term “Galois connection” is somewhat archaic; today such entities are sometimes called dual adjunctions. Note that we can rephrase the definition purely in terms of the posets $P$ and $Q$ by using the notion of order reversing (anti-isotone) mappings:

Let $P = (\mathcal{P}, \leq)$ and $Q = (\mathcal{Q}, \leq)$ be posets. A pair $f : P \rightarrow Q$ and $g : Q \rightarrow P$ of anti-isotone maps is a Galois connection provided

\[ p \leq g(q) \iff q \leq f(p) \]

**Exercise 144.** Let $A$ and $B$ be any sets, and let $C \subseteq A \times B$. Define mappings $f : \mathbb{S}(A) \rightarrow \mathbb{S}(B)$ and $g : \mathbb{S}(B) \rightarrow \mathbb{S}(A)$ as follows:

- $f(X) = \{y \in B : (x, y) \in C \text{ for all } x \in X\}$
- $g(Y) = \{x \in A : (x, y) \in C \text{ for all } y \in Y\}$

Prove that the pair $(f, g)$ forms a Galois connection between the powersets of $A$ and $B$.

Let $P$ be any poset. For each $X \subseteq P$, let $u(X) = \{p \in P : x \leq p \text{ for all } x \in X\}$ and let $l(X) = \{q \in P : y \leq x \text{ for all } x \in X\}$. In light of the previous exercise, it is easy to see that the pair $(u, l)$ forms a Galois connection between $P$ and itself (take $C = \{(x, y) \in P \times P : x \leq y\}$).

**Exercise 145.** Let $P$ be any poset and let the mappings $u$ and $l$ be defined as above. Consider the mapping $lu : \mathbb{S}(P) \rightarrow \mathbb{S}(P)$ defined by $lu(X) = l(u(X))$. Prove the following:

1. The map $lu$ is isotone.
2. For all $X \in \mathbb{S}(P)$, $X \subseteq lu(X)$.
3. For all $X \in \mathbb{S}(P)$, $lu(lu(X)) = lu(X)$.
4. For all $p \in P$, $lu(\{p\}) = \downarrow p$.

**Definition 146.** Let $P$ be a poset and let $X \subseteq P$. We say that $X$ is **join-dense** in $P$ provided every element of $x$ is the join (in $P$) of some subset of $P$. We say that $X$ is **meet-dense** provided it is join-dense in $P^{op}$.

**Exercise 147.** Let $P$ be any poset and let the mapping $lu$ be defined as above. Let $\mathcal{D}(P) = lu(\mathbb{S}(P))$, partially ordered by set inclusion.

1. Prove that $\mathcal{D}(P)$ is a complete lattice.
2. Embedding $P$ in $\mathcal{D}(P)$ as principal lowersets, prove that $P$ is join and meet dense in $\mathcal{D}(P)$.

The lattice $\mathcal{D}(P)$ is called the Dedekind MacNeill completion of the poset $P$. It turns out that $\mathcal{D}(P)$ is the smallest complete lattice in which $P$ is both join and meet dense, as the following exercises show.

Exercise 148. Let $P$ be any poset and suppose that $L$ is a complete lattice into which $P$ can be order embedded as a join and meet dense subposet. Prove that $\mathcal{D}(P)$ can also be order embedded in $L$.

2.4 Closure Operators and Compact Generation

In the last section, we briefly introduced adjunctions between posets and looked at one important application — Heyting lattices. In this section, we continue our look at adjunctions by taking advantage of another important property they possess.

Definition 149. Let $P$ be a poset. A function $\varphi : P \rightarrow P$ is called a closure operator on $P$ provided $\varphi$ is isotone, idempotent, and enlarging. That is, provided, for all $x, y \in P$, we have

- $x \leq y \implies \varphi(x) \leq \varphi(y)$ (The mapping is isotone.)
- $\varphi(\varphi(x)) = \varphi(x)$ (The mapping is idempotent.)
- $x \leq \varphi(x)$ (The mapping is enlarging.)

Definition 150. Let $P$ be a poset. A function $\psi : P \rightarrow P$ is called a kernel operator on $P$ provided $\psi$ is isotone, idempotent, and reducing. That is, provided, for all $x, y \in P$, we have

- $x \leq y \implies \psi(x) \leq \psi(y)$
- $\psi(\psi(x)) = \psi(x)$
- $\psi(x) \leq x$

If $P$ is any poset, then the mapping $lu : \text{Su}(P) \rightarrow \text{Su}(P)$ defined in Exercise 145 is a closure operator. The fact that this closure operator arises from an adjunction is no accident, as the following exercises show.

Exercise 151. Let $P$ and $Q$ be posets and let $(f, g) : P \Rightarrow Q$. Prove that $\varphi = g \circ f$ is a closure operator on $P$ and $\psi = f \circ g$ is a kernel operator on $Q$. 

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Exercise 152. Let \( P \) be a poset and let \( \theta : P \to P \) be a function. Let \( Q \) be the image of \( P \) under \( \theta \). Let \( g : Q \to P \) be the inclusion map \( g(q) = q \).

1. If \( \theta \) is a closure operator, prove that \((\theta, g) : P \rightleftharpoons Q\).
2. If \( \theta \) is a kernel operator, prove that \((g, \theta) : Q \rightleftharpoons P\).

Exercise 153. Let \( P \) be a poset and let \( \varphi : P \to P \) be a closure operator. Explain why \( \varphi \) preserves all existing joins in \( P \). In particular, if \( D \subseteq \varphi(P) \) is such that \( \bigvee P D \) exists, then \( \bigvee_{\varphi(P)} D \) exists as well.

Exercises 151 and 152 tell us that there is an intimate connection between adjunctions and closure (or kernel) operators. This connection tells us some important facts about closure and kernel operators. For example, as in the previous exercise, suppose that \( \theta : P \to P \) is a closure operator and let \( Q = \theta(P) \). The fact that the inclusion map \( g : Q \to P \) is the right adjoint to \( \theta \) tells us that whenever \( X \subseteq Q \) is such that \( \bigwedge Q X \) exists, then

\[
g(\bigwedge_Q X) = \bigwedge_P \{g(x) : x \in X\} = \bigwedge_P X
\]

The converse of this statement is also true; that is, whenever \( X \subseteq Q \) is such that \( \bigwedge_P X \) exists, then \( \bigwedge_Q X \) also exists and \( \bigwedge_Q X = \bigwedge_P X \).

To see why this is so, let \( x_0 = \bigwedge_P X \). Suppose \( z \in Q \) is such that \( z \leq \theta(x) \) for all \( x \in X \). (Note that such \( z \) exist by assumption, since \( \theta(x_0) \) has this property.) Then, the fact that \((\theta, g) : P \rightleftharpoons Q\) tells us that

\[
z \leq \theta(x) \iff z = g(z) \leq x(\forall x \in X)
\]

\[
\iff z = g(z) \leq x_0
\]

\[
\iff z \leq \theta(x_0)
\]

Thus, \( \theta(x_0) \) is the greatest lower bound of the set \( \theta(X) \) in \( Q \). However, since \( X \subseteq Q \) by assumption, we know that \( \theta(X) = X \). Consequently, we know

\[
\bigwedge_Q \theta(X) = \bigwedge_Q X = \theta(x_0)
\]

It only remains to prove that \( \theta(x_0) = x_0 \). Since \( \theta \) is a closure operator, we know at once that \( x_0 \leq_P \theta(x_0) \). However, since \( \theta(x_0) \) is itself a lower bound in \( P \) for the set \( X \), it follows that \( \theta(x_0) \leq_P x_0 \) as well.

We can summarize the previous discussion in the following way: If \( P \) is a poset and \( \theta : P \to P \) is a closure operator, then the set \( Q = \theta(P) \) is completely meet faithful in \( P \); that is, for all \( X \subseteq Q \), we have
1. \( \bigwedge_P X \) exists \( \iff \bigwedge_Q X \) exists, and
2. \( \bigwedge_P X = \bigwedge_Q X \)

**Exercise 154.** Let \( P \) be a poset and suppose \( \theta : P \to P \) is a kernel operator. What can be said about the image \( Q = \theta(P) \)?

**Definition 155.** Let \( P \) be a poset and let \( X \subseteq P \) be a subposet of \( P \). We say that \( X \) is a closure retract of \( P \) if, for all \( p \in P \), the set \( U(p) = \{ x \in X : p \leq x \} \) has a least element. We say that \( X \) is a kernel retract of \( P \) if \( X^{\text{op}} \) is a closure retract in \( P^{\text{op}} \). Members of a closure retract are often called closed elements; those of kernel retracts are sometimes called open elements.

**Exercise 156.** Let \( P \) be a poset. Prove that the lattice \( \mathcal{L}(P) \) of lowersets of \( P \) is a closure retract of the powerset of \( P \).

**Exercise 157.** Let \( P \) be a directed poset. Prove that the DCPO \( \text{Idl}(P) \) is a closure retract of the complete join semilattice \( \mathbb{S}u'(P) \) of nonempty subsets of \( P \). What can be said if \( P \) is not directed?

**Exercise 158.** Let \( P \) be a poset and suppose that \( \varphi : P \to P \) is a closure operator of \( P \). Prove that the set \( \varphi(P) \) is a closure retract of \( P \).

**Exercise 159.** Let \( P \) be a poset and suppose that \( X \subseteq P \) is a closure retract. Define a mapping \( \varphi_X : P \to P \) by \( \varphi_X(p) = \bigwedge U(p) \).

1. Prove that \( \varphi_X \) is a closure operator whose range is the set \( X \).
2. Prove that \( \varphi = \varphi_X \).

The previous two exercises tell us that the closure retracts of a poset \( P \) are precisely the images of \( P \) under its closure operators; moreover, the closure retracts of \( P \) uniquely determine the closure operators on \( P \) and vice-versa.

**Exercise 160.** Let \( P \) be a poset and suppose \( D \subseteq P \) is directed. If \( x \in P \), prove that \( D_x = \{ x \land d : d \in D \} \) is directed, provided \( D_x \) is nonempty.

**Definition 161.** Let \( P \) be a DCPO. An element \( c \in P \) is compact provided, whenever \( D \subseteq P \) is directed and such that \( x \leq \bigvee D \), then \( c \leq d \) for some \( d \in D \). Compact elements are sometimes called isolated or finite elements. We will let \( \text{Com}(P) \) represent the subposet of compact elements from \( P \).

**Exercise 162.** Let \( L \) be a complete lattice. Prove that \( \text{Com}(L) \) is a join sub-semilattice of \( L \).
Exercise 163. Let \( L \) be a complete lattice, and let \( c \in L \). Prove that the following statements are equivalent:

1. The element \( c \) is compact in \( L \).
2. Whenever \( X \subseteq L \) is such that \( c \leq \bigvee X \), then there exist finite \( F \subseteq X \) such that \( c \leq \bigvee F \).

Exercise 164. Let \( \mathbb{N} \) denote the chain of positive integers under the natural ordering, and consider the poset \( P = (\mathbb{N} \cup \{\omega, u, v\}, \leq) \), where

1. \( \omega, u, v \not\in \mathbb{N} \)
2. \( \leq \) is the natural order on \( \mathbb{N} \)
3. \( m < \omega \) for all \( m \in \mathbb{N} \)
4. \( \omega < u \) and \( \omega < v \) and \( \omega = u \land v \)

Show that \( u \) and \( v \) are compact in \( P \) but \( \omega \) is not. (Consequently, the meet of two compact elements need not be compact.)

Theorem 165. Let \( P \) be a DCPO. If \( \varphi : P \to P \) is a closure operator, then the following statements are equivalent:

1. Whenever \( D = \{d_i : i \in I\} \subseteq P \) is directed, then \( \varphi(\bigvee D) = \varphi(d_i) \) for some \( i \in I \);
2. Every element of the closure retract \( \varphi(P) \) is compact in \( \varphi(P) \);
3. Every directed subset of \( \varphi(P) \) has a greatest element.

Proof. Note first that, by Exercise 153, every closure retract of a DCPO is itself a DCPO. To prove that Claim (1) implies Claim (2), let \( x \in \varphi(P) \) and suppose that \( D = \{d_i : i \in I\} \subseteq \varphi(P) \) is directed and such that \( x \leq \bigvee_{\varphi(P)} D \).

Since \( D \) is also directed in \( P \), it is easy to prove that \( \varphi(\bigvee_P D) = \bigvee_{\varphi(P)} D \).

Hence, we know that

\[
x \leq \bigvee_{\varphi(P)} D = \varphi(\bigvee_P D) = \varphi(d_i) = d_i
\]

for some \( i \in I \). Hence, \( x \) is compact in \( \varphi(P) \).

To prove that Claim (2) implies Claim (3), suppose that \( D \subseteq \varphi(P) \) is directed. By Claim (2), \( d = \bigvee_{\varphi(P)} D \) is compact in \( \varphi(P) \); hence, we know that \( d = d_i \) for some \( i \in I \). It follows at once that \( d_i \) is the largest element of \( D \).
It remains to prove that Claim (3) implies Claim (1). To this end, let $D = \{d_i : i \in I\} \subseteq P$ be directed. It follows that $\varphi(D)$ is directed in $\varphi(P)$; hence, we know that $\varphi(D)$ has a largest element, say $\varphi(d_i)$. Now, observe that

$$\varphi(\bigvee_D) = \bigvee_{\varphi(P)} \varphi(D) = \varphi(d_i)$$

\[ \square \]

**Definition 166.** A poset $P$ is said to be *compactly generated* provided it is a DCPO and every element of $P$ is the join of a directed family of compact elements in $P$. Compactly generated posets are often called *algebraic* posets.

**Exercise 167.** Let $X$ be any set. Prove that $\text{Su}(X)$ is compactly generated with $\text{Com}(\text{Su}(X)) = \text{Fin}(X)$.

**Exercise 168.** Let $P$ be any poset. Prove that $\mathcal{L}(P)$ is compactly generated with $\text{Com}(\mathcal{L}(P)) = \{\downarrow F : F \in \text{Fin}(P)\}$.

**Exercise 169.** Let $P$ be any poset. Prove that $\text{Idl}(P)$ is compactly generated with $\text{Com}(\text{Idl}(P)) = \{\downarrow p : p \in P\}$.

**Exercise 170.** Suppose that $P$ and $Q$ are isomorphic DCPO’s, and suppose that $f : P \rightarrow Q$ is an isomorphism. Prove that $c$ is compact in $P$ if and only if $f(c)$ is compact in $Q$.

**Exercise 171.** Prove that every lower-bounded, compactly generated distributive lattice is join continuous (and thus a frame).

**Definition 172.** Let $P$ be a DCPO and let $\varphi : P \rightarrow P$ be a closure operator. We say that $\varphi$ is *directed* provided $\bigvee_P D = \bigvee_{\varphi(P)} D$ for all directed $D \subseteq \varphi(P)$.

**Theorem 173.** Let $P$ be a compactly generated poset. If $\varphi : P \rightarrow P$ is a closure operator, then the following are equivalent:

1. The mapping $\varphi$ is a directed closure operator;
2. An element of $\varphi(P)$ is compact if and only if it has a compact preimage;
3. For all $x \in L$, $\varphi(x) = \bigvee_P \varphi(K_x)$. 

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Proof. We first prove Claim (1) implies Claim (2). To this end, we observe that any compact element \( c \in \varphi(P) \) has a compact preimage (regardless of whether or not \( \varphi \) is directed). Suppose instead that \( c \in \varphi(P) \) has a compact preimage \( k \). We show that \( c \) is compact in \( \varphi(P) \). Suppose that \( D = \{ d_i : i \in I \} \subseteq \varphi(P) \) is directed and such that \( c \leq \bigvee_{\varphi(P)} D \). Since \( \varphi \) is directed, this implies that \( c \leq \bigvee_{P} D \). Now,

\[
k \leq \varphi(k) = c \leq \bigvee_{P} D
\]

implies that \( k \leq d_i \) for some \( i \in I \). It follows at once that \( c \leq d_i \) as well. Hence, \( c \) is compact in \( \varphi(P) \).

To prove Claim (2) implies Claim (3), let \( x \in L \). Since \( c \leq x \) for all \( c \in K_x \), we know that \( \bigvee_{P} \varphi(K_x) \leq \varphi(x) \). To obtain the reverse inequality, let \( B = \varphi(K_x) \) and observe that, by Claim (2), \( \varphi(c) \) is compact in \( \varphi(P) \) for all \( c \in B \). Now, since \( \varphi(x) = \bigvee_{\varphi(P)} \varphi(K_x) \) and since \( \varphi(K_x) \) is directed in \( \varphi(P) \), it follows that there exist \( d_c \in \varphi(K_x) \) such that \( c \leq \varphi(c) \leq d_c \). Consequently, we know

\[
\varphi(x) = \bigvee_{P} B \leq \bigvee_{P} \varphi(K_x)
\]

It remains to prove that Claim (3) implies Claim (1). To this end, let \( D = \{ d_i : i \in I \} \subseteq P \) be directed and let \( d = \bigvee_{P} D \). It is clear that

\[
\bigvee_{P} \varphi(D) \leq \varphi(d)
\]

To obtain the reverse inequality, let \( c \in K_d \). It follows that \( c \leq d_i \) for some \( i \in I \); hence, \( \varphi(c) \leq \varphi(d_i) \). By Claim (3), this implies that

\[
\varphi(d) = \bigvee_{P} \varphi(K_d) \leq \bigvee_{P} \varphi(D)
\]

Exercise 174. Let \( P \) be a DCPO and let \( \varphi : P \rightarrow P \) be a closure operator. If \( c \) is compact in \( \varphi(P) \), prove that \( c \) has a compact preimage in \( P \).

Exercise 175. Let \( P \) be a compactly generated poset and suppose that \( \varphi : P \rightarrow P \) is a directed closure operator. Prove that \( \varphi(P) \) is also a compactly generated poset.

Exercise 176. Consider the poset \( Q = (\mathbb{N} \cup \{ \omega \}, \leq) \), where \( \leq \) is the natural ordering on \( \mathbb{N} \) and \( m < \omega \) for all \( m \in \mathbb{N} \). Let \( \mathcal{C} = \{ \downarrow x : x \in \mathbb{N} \cup \{ \omega \} \} \), partially ordered by subset inclusion.
1. Prove that $C$ is a closure retract of $\mathfrak{S}u(\mathbb{N} \cup \{\omega\})$.

2. Prove that $C$ is compactly generated.

3. Prove that the closure operator associated with $C$ is not directed.
   (Hence the converse of Exercise 175 is false.)

2.5 Irreducible Elements in Lattices

Representing a given structure (such as a group, ring, lattice, etc.) in terms of a “canonical” set of elements under a specific operation is a natural problem that arises in the study of algebra. Usually this canonical set consists of those elements which are “irreducible” with regard to the specified operation. An elementary example would be the representation of positive integers as products of primes (which are irreducible with regard to multiplication).

**Definition 177.** Let $L$ be a meet semilattice. An element $p \in L$ is **meet irreducible** if, for all $F \in \text{Fin}(L)$, $p = \bigwedge F$ always implies $p = f$ for some $f \in F$. An element $j$ of a join semilattice $L$ is **join irreducible** provided it is meet irreducible in $L^\text{op}$.

Note that the greatest element of a meet semilattice (if it exists) cannot be meet irreducible. The concepts of **completely meet irreducible** and **completely join irreducible** elements can be defined in a (complete) meet or join semilattice by removing the restriction that the set $F$ be finite.

**Exercise 178.** Let $L$ be a lower-bounded, compactly generated lattice and let $a, b \in L$ be such that $a \not\leq b$.

1. Prove that $K_a - K_b$ is nonempty.

2. Let $c \in K_a - K_b$ and let $X = \{x \in L : b \in \Downarrow x$ and $c \not\in \Downarrow x\}$. Use Zorn’s Lemma to prove that $X$ has a maximal member.

3. Let $q$ be a maximal member of $X$ and prove that $q$ must be completely meet irreducible. (Hint: Assume $q = \bigwedge S$ and prove that some member of $S$ must be in $X$. Use maximality to complete the argument.)

**Exercise 179.** Let $L$ be a complete lattice, let $a \in L$, and let $Q \subseteq L$.

1. Suppose that, for all $b \in L$ such that $a \not\leq b$, there exist $q \in Q$ such that $a \leq q$ and $q \not\leq b$. Prove that $a = \bigwedge Q$.

2. Prove that every element of a lower-bounded, compactly generated lattice is the meet of a set of completely meet irreducible elements.
Claim (2) of the last exercise is a famous result due to Garrett Birkhoff. It will have important implications in much of our later work.

**Definition 180.** Let $L$ be a meet semilattice. An element $p \in L$ is meet prime if, for all $F \in \text{Fin}(L)$, $p \geq \bigwedge F$ always implies $p \geq f$ for some $f \in F$. An element $j$ of a join semilattice $L$ is join prime provided it is meet prime in $L^{\text{op}}$.

**Exercise 181.** Let $L$ be a lattice and let $I$ be a proper ideal of $L$. Prove that $I$ is a prime ideal if and only if $I$ is a meet prime element of $\text{Idl}(L)$.

The concepts of completely meet prime and completely join prime elements can be defined in a (complete) meet or join semilattice by removing the restriction that the set $F$ be finite.

**Exercise 182.** Prove that every meet prime element is meet irreducible.

**Exercise 183.** Show by example that a meet irreducible element in a lattice need not be meet prime and that a meet irreducible element in a complete lattice need not be completely meet irreducible or meet prime.

**Exercise 184.** Prove that in a distributive lattice, every meet irreducible element is meet prime.

**Exercise 185.** Show by example that, even in a complete distributive lattice, a completely meet irreducible element need not be completely meet prime.

**Exercise 186.** Let $L$ be an algebraic (lower-bounded, compactly generated), distributive lattice. Prove that the following statements are equivalent:

1. Every completely meet irreducible element of $L$ is completely meet prime;

2. The lattice $L^{\text{op}}$ is join continuous.

*Hint:* For convenience, recast the join-continuity of $L^{\text{op}}$ as meet continuity in $L$; you will need Birkhoff’s Theorem to prove that Claim (1) implies Claim (2).

**Exercise 187.** Let $L$ be a complete lattice and let $j \in L$ be completely join prime. Prove that $\downarrow j$ is co-atomic; in particular, $j$ is compact in $L$. 
**Exercise 188.** Let $L$ be a lower-bounded, compactly generated, distributive lattice. Prove that the following statements are equivalent:

1. The lattice $L^\text{op}$ is algebraic (lower-bounded and compactly generated);
2. Every completely meet irreducible element of $L$ is completely meet prime;
3. Every element of $L$ is the join of a set of completely join prime elements;
4. The lattice $L$ is order isomorphic to $\mathcal{L}(P)$ for some poset $P$.

We will say that a lattice $L$ is *bicompletely generated* provided both $L$ and $L^\text{op}$ are lower-bounded, compactly generated posets. (In keeping with the fact that compactly generated posets are often called algebraic posets, such lattices are often said to be *bialgebraic*.)

**Exercise 189.** Show by example that there exist frames which are not bialgebraic.

**Exercise 190.** Let $L$ be a lattice in which every element is the join of a finite set of join prime elements. Prove that $L$ is distributive.

**Exercise 191.** Let $L$ be a join semilattice with least element. Prove that $\text{Idl}(L)$ is bialgebraic and distributive if and only if every element of $L$ is the join of a finite set of join prime elements.

The previous exercises show that join prime elements are intimately connected with distributivity. The next few results show that completely meet prime and completely join prime elements are intimately related to one another. This fact also has important structure consequences for lattices.

**Definition 192.** Let $L$ be a lattice and let $a, b \in L$. We say the ordered pair $(a, b)$ splits $L$ provided $\downarrow a \cup \uparrow b = L$ and $\downarrow a \cap \uparrow b = \emptyset$.

**Lemma 193.** Let $L$ be a complete lattice and suppose $a, b \in L$. If $(a, b)$ splits $L$, then $a$ is completely meet prime and $b$ is completely join prime in $L$.

**Proof.** Suppose by way of contradiction that there exists a set $X \subseteq L$ such that $\bigwedge X \leq a$ but $x \not\leq a$ for all $x \in X$. Since $(a, b)$ splits $L$, it follows that $X \subseteq \uparrow b$. Hence; we know $b \leq \bigwedge X$. Thus, $b \leq a$ — an impossibility. Thus, $a$ is completely meet prime. The proof that $b$ is completely join prime is similar.
Exercise 194. Let $L$ be a complete lattice. For all $a \in L$, let
\begin{align*}
  a^+ &= \bigwedge \{x \in L : x \not\leq a\} \\
  a^- &= \bigvee \{y \in L : a \not\leq y\}
\end{align*}
Prove that if $a$ is completely meet prime, then the pair $(a, a^+)$ splits $L$; and if $a$ is completely join prime, then $(a^-, a)$ splits $L$.

Let $L$ be a complete lattice. We will let $	ext{CJP}(L)$ and $	ext{CMP}(L)$ denote the subposets of completely join-prime and completely meet-prime elements of $L$, respectively.

Exercise 195. Let $L$ be a complete lattice. Prove that $\text{CJP}(L)$ is order isomorphic to $\text{CMP}(L)$.

Definition 196. Let $L$ be a lattice, and let $x \in L$. We will say that $p \in L$ is a value of $x$ in $L$ provided $x \not\leq p$; and, for all $y \in L$, if $p < y$, then $x \leq y$. We use the symbol $\text{Val}(x)$ to denote the set of all values of $x$.

Note that the set of values for an element forms an antichain in the lattice. We often summarize the previous definition by saying that an element of a lattice $L$ is a value of $x$ provided it is maximal with respect to not exceeding $x$.

Exercise 197. Let $L$ be an algebraic lattice and let $c \in \text{Com}(L) - \{\bot\}$.
\begin{enumerate}
  \item Prove that $\text{Val}(c)$ is nonempty.
  \item Use Zorn’s Lemma to prove that if $c \not\leq x$ then there exist $p \in \text{Val}(c)$ such that $x \leq p$.
\end{enumerate}

Exercise 198. Let $L$ be an algebraic (lower-bounded, compactly generated) lattice and let $c, d \in \text{Com}(L)$ with $c \leq d$. Prove the following.
\begin{enumerate}
  \item For each $p \in \text{Val}(c)$, there exist $q \in \text{Val}(d)$ such that $p \leq q$.
  \item If $c < d$, then $\text{Val}(c) \neq \text{Val}(d)$.
\end{enumerate}

Exercise 199. Let $L$ be an algebraic lattice and let $c \in \text{Com}(L)$. Prove that $c$ is completely join prime if and only if it has exactly one value.

Exercise 200. Let $L$ be an algebraic lattice and let $x \in L$. Prove that $x$ is completely meet prime if and only if it is a value of a completely join prime element.