Topology Notes

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Dedication

These notes are dedicated to all my topology students: past, present, and future.
To the Instructor

These notes are used for graduate topology at SUNY Potsdam. Most of those enrolled in the course are in our BA/MA program, where the students are capable but have had only a few upper level mathematics courses. In order to serve such students, very little prior knowledge is assumed. Thus, these notes are appropriate for upper level undergraduates as well as graduate students.

The notes are organized into short chapters. We begin with $\mathbb{R}^n$, but only enough about $\mathbb{R}^n$ is covered to establish basic properties of distance, thus motivating the definition of metric in chapter 2. Then metric spaces are the subject of chapters 2 through 4, but the treatment of metric spaces is spare and its primary goal is to establish the equivalence of the $\varepsilon$-$\delta$ definition of continuity with its characterization via open sets. The basic properties of open sets in metric spaces motivate the definition of topological space in chapter 5. The next several chapters address the standard topics of separation properties (Hausdorff, regular, normal), compactness, continuity, and connectedness. The most challenging material is in chapter 10 on compactness. Moving somewhat leisurely, a one semester course ends somewhere in chapter 13 on path connectedness.

The second half of the notes are on quotient spaces, the product topology, Tychonoff’s Theorem, Urysohn’s Lemma, the Tietze Extension Theorem, and the fundamental group. Students coming to topology with a more extensive background may very well be able to cover some of these topics in addition to the earlier ones in a one semester course.

Chapters 1 through 14 are designed to be worked through in order. The following groupings of chapters are independent of each other and may be worked through in any order:

1) Chapters 15, 16, and 17 on product spaces and Tychonoff’s Theorem.
2) Chapters 18, 19, and 20 on Urysohn’s Lemma and the Tietze Extension Theorem.
3) Chapters 21 through 24 on the fundamental group and covering spaces.

The notes are almost self-contained. The fact that the real numbers are complete is needed, and to prove the Tychonoff Theorem in chapter 17, Zorn’s Lemma is necessary. Included at the end of the notes are supplemen-
tary sections on images and inverse images of sets under functions and on series of real numbers, material that our students will have seen in earlier courses and which they are free to use.

At this writing (spring 2015), Topology Notes has been used by several different instructors for a one semester course a total of 10 times for class sizes ranging from 5 to 10 students. The notes have been used for a second semester course 3 times, one of those times for a class as small as 2 students, and have been used a few times for independent study on second semester topology topics. When used for a usual class, the structure I use for the course is as follows. Each student is expected to work through all of the problems and theorems from the notes as we go along. There is a mid-term exam, a final that counts double, a written grade, and an oral grade that counts double. The lowest of these 6 grades is dropped. For the oral grade, theorems and problems are assigned to be presented in class on a regular basis. For the written grade, students are expected to keep a notebook with the solutions to all the problems and theorems in the notes. The notebooks are collected twice a semester and a selection of the problems are graded. The students are welcome to work together or consult the instructor on any assignment other than in-class tests. Consulting other sources is acceptable but not recommended.
Acknowledgements

My purpose is creating these notes has been to present topology in a sequence of steps that students can work out for themselves. I have made no attempt at originality and have freely used ideas and approaches from the following sources:

–my undergraduate and graduate instructors in general topology: W. B. Raymond Lickorish, Kenneth Millet, and especially Martin Scharlemann.

–my late husband David Spellman, who had taken a Moore method topology course from Michael Starbird.

–authors Donald W. Kahn and especially James R. Munkres, the latter of whose treatments of Urysohn’s Lemma, the Tietze Extension Theorem, and the fundamental group I followed rather closely.

My heartfelt gratitude goes to the students who have worked on putting these notes into their current Latex form, especially Kristin Rugg and Ada Morse. Thank you!
Chapter 1

The Space $\mathbb{R}^n$

**Definition 1.1.** As a set, $\mathbb{R}^n$ is the set of all ordered $n$-tuples of real numbers; that is, $\mathbb{R}^n$ is the set of all $(x_1, x_2, \ldots, x_n)$ where $x_i$ is a real number for $i = 1, 2, \ldots, n$. For each $i$, the number $x_i$ is called the $i^{th}$ coordinate of $(x_1, x_2, \ldots, x_n)$. Elements of $\mathbb{R}^n$ are called points.

The set $\mathbb{R}^n$ together with the structure that will be developed is called $n$-space. The set of real numbers is identified with $1$-space, and is called "the real line." Also, $2$-space is called "the plane." An algebraic structure is put on $\mathbb{R}^n$ as follows:

*The addition* of two points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is defined by

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

*The scalar product* of a real number $a$ and a point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is defined by

$$a(x_1, x_2, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n),$$

where for each $i$, $ax_i$ is the product of the real numbers $a$ and $x_i$.

**Proposition 1.2.** Let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, and $Z = (z_1, z_2, \ldots, z_n)$ be any three points in $\mathbb{R}^n$. Let $\theta$ denote the point in $\mathbb{R}^n$ all of whose coordinates are 0; that is $\theta = (0, 0, \ldots, 0)$. ($\theta$ is usually called the origin.) Also, let $a$ and $b$ be any real numbers. Then each of the following hold.

1. $(X + Y) + Z = X + (Y + Z)$.
2. $X + \theta = X$.
3. $X + Y = Y + X$.
4. $X + (-1)X = \theta$.
5. $(a + b)X = aX + bX$.
6. $a(X + Y) = aX + aY$. 

7. \((ab)X = a(bX)\).

8. \(1X = X\).

**Remark 1.3.** Those who have studied linear algebra may notice that Proposition 1.2 shows that \(\mathbb{R}^n\) with addition and scalar multiplication is a vector space over the real numbers.

**Remark 1.4.** In the case of the plane or 3-space, the addition defined above corresponds to the usual geometric definition of addition of vectors as the diagonal of the associated parallelogram.

**Definition 1.5.** If \(X = (x_1, x_2, \ldots, x_n)\) and \(Y = (y_1, y_2, \ldots, y_n)\) are any two elements in \(\mathbb{R}^n\), we define the distance between \(X\) and \(Y\) by

\[
d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.
\]

Notice that the domain of the function \(d\) is \(\mathbb{R}^n \times \mathbb{R}^n\) and its range is in \(\mathbb{R}\).

**Proposition 1.6.** Let \(X\) and \(Y\) be points in \(\mathbb{R}^n\). Then the following hold.

1. \(d(X, Y) \geq 0\).
2. \(d(X, Y) = 0\) if and only if \(X = Y\).
3. \(d(X, Y) = d(Y, X)\).

To establish one further property of the distance function, the notions of inner product and norm will be helpful.

**Definition 1.7.** If \(X = (x_1, x_2, \ldots, x_n)\) and \(Y = (y_1, y_2, \ldots, y_n)\) are any two points in \(\mathbb{R}^n\), we define the inner product of \(X\) and \(Y\) by

\[
\langle X, Y \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n.
\]

**Proposition 1.8.** Let \(X, Y,\) and \(Z\) be points in \(\mathbb{R}^n\), and let \(a\) be any real number. Then each of the following hold.

1. \(\langle X, X \rangle = (d(X, 0))^2\).
2. \(\langle X, Y \rangle = \langle Y, X \rangle\).
3. \(\langle X, Y + Z \rangle = \langle X, Y \rangle + \langle X, Z \rangle\).
4. \(a\langle X, Y \rangle = \langle aX, Y \rangle = \langle X, aY \rangle\).

**Theorem 1.9.** Cauchy-Schwartz Inequality

For any points \(X\) and \(Y\) in \(\mathbb{R}^n\), \(\langle X, Y \rangle \leq \sqrt{\langle X, X \rangle} \sqrt{\langle Y, Y \rangle}\).

[Hint: One way to prove this is to consider \(\langle X + aY, X + aY \rangle\) as a polynomial in the variable \(a\), and use Proposition 1.7].
**Definition 1.10.** The norm $\|X\|$ of a point $X$ in $\mathbb{R}^n$ is defined to be the distance from $X$ to $\theta$, that is, $\|X\| = d(X, \theta)$.

**Proposition 1.11.** For any $X$ and $Y$ in $\mathbb{R}^n$, $\|X\|^2 = \langle X, X \rangle$ and $d(X, Y) = \|X - Y\|$.

**Proposition 1.12.** For any $X$ and $Y$ in $\mathbb{R}^n$, $\|X + Y\| \leq \|X\| + \|Y\|$.

**Corollary 1.13.** For any $X$, $Y$, and $Z$ in $\mathbb{R}^n$, $d(X, Y) \leq d(X, Z) + d(Z, Y)$.

This inequality is called the **Triangle Inequality**.

**Summary of Distance in $\mathbb{R}^n$:**

For any $X$, $Y$, and $Z$ in $\mathbb{R}^n$,

1. $d(X, Y) \geq 0$, and $d(X, Y) = 0$ if and only if $X = Y$.
2. $d(X, Y) = d(Y, X)$.
3. $d(X, Y) \leq d(X, Z) + d(Z, Y)$.

Mathematicians have decided that the above three properties of the distance function are the most important ones when generalizing the distance function.
Chapter 2

Introduction to Metric Spaces

We use the basic properties of distance in $\mathbb{R}^n$ as a model for a more general and abstract notion of distance as follows.

**Definition 2.1.** A **metric** on a set $M$ is a function $d : M \times M \rightarrow \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in M$ and equality holds if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in M$, and
3. **(Triangle Inequality)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

The pair $(M, d)$ is called a **metric space**. If the function $d$ is understood from context, we say “$M$ is a metric space.” The elements of $M$ are called **points**.

**Example 2.2.** The **usual metric** on $\mathbb{R}^n$ is the distance function $d$ that is defined by

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$ 

In Chapter 1, it was proved that $d$ is a metric on $\mathbb{R}^n$.

The verifications that the rest of the examples are metrics are left to the students.

**Example 2.3.** Let $M$ be any set, and define the **discrete metric** $d$ on $M$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$ 

**Example 2.4.** Define the **taxicab metric** $d$ on $\mathbb{R}^n$ by

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|.$$
Example 2.5. Define the square metric $d$ on $\mathbb{R}^n$ by
\[ d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}. \]

Exercise 2.6. Let $(\mathbb{R}^2, d)$ be a metric space. The unit circle is the set of points in $\mathbb{R}^2$ which are distance 1 from $(0,0)$. Draw the different unit circles in $\mathbb{R}^2$ for each of the metrics 2.2 through 2.5. Explain why 2.4 and 2.5 have the names they do.

Example 2.7. Let $(M, d)$ be a metric space and let $a$ be a positive real number. Then the function $d'$ defined by $d'(x, y) = ad(x, y)$ is a metric on $M$.

Proposition 2.8. Let $d$ be the usual metric on $\mathbb{R}^n$, let $d'$ denote the taxicab metric on $\mathbb{R}^n$, and let $d''$ denote the square metric on $\mathbb{R}^n$. For any $X$ and $Y$ in $\mathbb{R}^n$,
\[ \begin{align*}
1. & \quad d''(X, Y) \leq d(X, Y) \leq \sqrt{n} d''(X, Y), \text{ and} \\
2. & \quad d''(X, Y) \leq d'(X, Y) \leq n d''(X, Y).
\end{align*} \]

Example 2.9. Let $(M, d)$ be a metric space. Then the function $d'$ defined by $d'(x, y) = \min\{d(x, y), 1\}$ is a metric on $M$.

Example 2.10. Let $(M, d)$ be a metric space. Then the function $d'$ defined by $d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ is a metric on $M$.

Exercise 2.11. Show that the following do not define metrics on $\mathbb{R}^2$.
\[ \begin{align*}
1. & \quad d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|.
2. & \quad d((x_1, x_2), (y_1, y_2)) = x_1 y_1 - x_2 y_2.
3. & \quad d((x_1, x_2), (y_1, y_2)) = x_1^2 + x_2^2 + y_1^2 + y_2^2.
\end{align*} \]

Example 2.12. Let $(M, d)$ be a metric space, and let $N$ be a nonempty subset of $M$. Then $d'$ is a metric on $N$, where $d'$ is the restriction of $d$ to $N \times N$.

Example 2.13. Let $(M_1, d_1), (M_2, d_2), \ldots, (M_n, d_n)$ be metric spaces, and let $M = M_1 \times M_2 \times \cdots \times M_n$. Define $d : M \times M \to \mathbb{R}$ by
\[ d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n)\}. \]
Then $d$ is a metric on $M$. 

Chapter 3

Open Sets in Metric Spaces

Definition 3.1. Let \((M, d)\) be a metric space. Given \(x \in M\) and a positive real number \(\varepsilon\), the open ball centered at \(x\) with radius \(\varepsilon\) is

\[ B_d(x, \varepsilon) = \{y \in M \mid d(x, y) < \varepsilon\}. \]

We will write \(B(x, \varepsilon)\) for \(B_d(x, \varepsilon)\) when \(d\) is clear from the context. A subset \(U\) of \(M\) is called open if for every \(x \in U\) there exists a positive real number \(\varepsilon\) so that \(B(x, \varepsilon) \subseteq U\).

Proposition 3.2. Open balls are open.

Proposition 3.3. Basic Properties of Open Sets

Let \((M, d)\) be a metric space. Then the following hold.

1. \(\emptyset\) and \(M\) are both open.
2. The union of any collection of open sets is open.
3. The intersection of a finite collection of open sets is open.

Example 3.4. The ray \((a, \infty)\) is open in \(\mathbb{R}\) with the usual metric.

Example 3.5. The set \(\{(x, y) \mid x > 0 \text{ and } y > 0\}\) is open in \(\mathbb{R}^2\) with the usual metric.

Examples 3.6. The following are not open in \(\mathbb{R}^2\) with the usual metric.

1. The x-axis.
2. \(\{(x, y) \mid x^2 + y^2 \leq 1\}\).
3. \(\{(x, y) \mid xy \geq 1\}\).

Proposition 3.7. Let \(U\) be a subset of \(\mathbb{R}^n\). Then the following are equivalent.

1. \(U\) is open in \(\mathbb{R}^n\) with the usual metric.
2. \(U\) is open in \(\mathbb{R}^n\) with the taxicab metric.
3. \( U \) is open in \( \mathbb{R}^n \) with the square metric.

**Example 3.8.** Let \( M \) be any set, and consider the metric space \((M, d)\), where \( d \) is the discrete metric. Then every subset of \( M \) is open.

**Example 3.9.** Let \((M_1, d_1), (M_2, d_2), \ldots, (M_n, d_n)\) be metric spaces, and let \( M = M_1 \times M_2 \times \cdots \times M_n \). Define \( d : M \times M \to \mathbb{R} \) by

\[
d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n)\}.
\]

Then an open ball in \( M \) is a product of open balls in \( M_1, M_2, \ldots, M_n \), respectively.
Chapter 4

Continuous Functions on Metric Spaces

**Definition 4.1.** Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces, let \(f : X_1 \to X_2\) be a function, and let \(a \in X_1\). Then \(f\) is called **continuous at \(a\)** if given \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(x \in X_1\) and \(d_1(a, x) < \delta\), then \(d_2(f(a), f(x)) < \varepsilon\). \(f\) is simply called **continuous** if \(f\) is continuous at all \(a \in X_1\).

**Exercise 4.2.** Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces, and let \(y\) be an element of \(X_2\). Define \(f : X_1 \to X_2\) by \(f(x) = y\) for all \(x \in X_1\). Show that \(f\) is continuous.

**Exercise 4.3.** Let \((M_1, d_1), (M_2, d_2), \ldots, (M_n, d_n)\) be metric spaces and let \(M = M_1 \times M_2 \times \cdots \times M_n\). Define \(d : M \times M \to \mathbb{R}\) by

\[
d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n)\}.
\]

Define, for each \(i = 1, 2, \ldots, n\), \(p_i : M \to M_i\) by \(p_i((x_1, x_2, \ldots, x_n)) = x_i\). These functions are called **projections**. Show that each projection is continuous.

**Exercise 4.4.** Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be defined by \(f(x, y) = x + y\). Let \(\mathbb{R}^2\) have the taxicab metric and let \(\mathbb{R}\) have the usual metric. Show that \(f\) is continuous.

**Exercise 4.5.** Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be defined by \(f(x, y) = xy\). Let \(\mathbb{R}^2\) have the square metric and let \(\mathbb{R}\) have the usual metric. Show that \(f\) is continuous. [Hint: To show \(f\) is continuous at \((x_0, y_0)\), show

\[
|xy - x_0y_0| \leq |x_0||y - y_0| + |y_0||x - x_0| + |x - x_0||y - y_0|,
\]

and for a given positive real number \(\varepsilon\) use \(\delta = \frac{1}{3} \min\left\{\frac{\varepsilon}{|x_0| + |y_0| + 1}, \sqrt{\varepsilon}\right\}\).]

**Exercise 4.6.** Let \((M, d)\) be a metric space and let \(a\) be a positive real number. We have shown that the function \(d'\) defined by \(d'(x, y) = ad(x, y)\) is a metric on \(M\). Let \(M_d\) denote \(M\) with the \(d\) metric, and let \(M_a\) denote \(M\) with the \(d'\) metric.

a) Let \(f : M_d \to M_a\) be defined by \(f(x) = x\). Show that \(f\) is continuous.

b) Let \(g : M_a \to M_d\) be defined by \(g(x) = x\). Show that \(g\) is continuous.
Exercise 4.7. Let \((X_1,d_1), (X_2,d_2),\) and \((X_3,d_3)\) be metric spaces, and let \(f : X_1 \to X_2\) and \(g : X_2 \to X_3\) be continuous. Show that the composition \(g \circ f : X_1 \to X_3\) is continuous.

Proposition 4.8. Let \((X_1,d_1)\) and \((X_2,d_2)\) be metric spaces and let \(f : X_1 \to X_2\) be a function. Then \(f\) is continuous if and only if \(f^{-1}(V)\) is open in \(X_1\) whenever \(V\) is open in \(X_2\).

Exercise 4.9. Let \(\mathbb{R}^2\) and \(\mathbb{R}\) have their usual metrics, and show that addition and multiplication are continuous by using 3.7, 4.4, 4.5, and 4.8.
Chapter 5

The Definition of a Topological Space

**Definition 5.1.** Let $X$ be a set and let $\mathcal{T}$ be a collection of subsets of $X$. The pair $(X, \mathcal{T})$ is called a **topological space** if each of the following hold:

1. $\emptyset$ and $X$ are elements of $\mathcal{T}$.
2. The union of any collection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
3. The intersection of a finite collection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

If $(X, \mathcal{T})$ is a topological space, then $\mathcal{T}$ is called a **topology** on $X$.

**Remarks 5.2. Some Terminology**

1. If $(X, d)$ is a metric space, then consider the collection $\mathcal{T}$ of all open subsets of $X$. Then by 3.3, $(X, \mathcal{T})$ is a topological space. This topology is called the **metric topology** on $X$ induced by $d$.
2. The **usual topology** on $\mathbb{R}^n$ is defined to be the metric topology on $\mathbb{R}^n$ induced by the usual metric on $\mathbb{R}^n$.
3. Let $(X, \mathcal{T})$ be a topological space. If $U \subseteq X$, then $U$ is **open** means “$U \in \mathcal{T}$.” This generalizes the use of the word “open” in the context of metric spaces. Also, an element of $X$ is usually called a **point**.
4. When $\mathcal{T}$ is understood from the context or does not need to be explicitly described, then we will say “$X$ is a topological space” to mean “$(X, \mathcal{T})$ is a topological space.”

**Exercise 5.3.** Let $X$ be any set and let $\mathcal{T} = \{\emptyset, X\}$. Show that $\mathcal{T}$ is a topology on $X$. This topology is called the **indiscrete topology** on $X$.

**Exercise 5.4.** Let $X$ be any set and let $\mathcal{T}$ be the collection of all subsets of $X$. Show that $\mathcal{T}$ is a topology on $X$. This topology is called the **discrete topology** on $X$. Show that the discrete topology on $X$ is the same as the metric topology induced by the discrete metric.
Exercise 5.5. Find all possible topologies on the set $X = \{a, b\}$.

Exercise 5.6. Let $X$ be any set and define

$$\mathcal{T} = \{U \mid U \subseteq X, \text{ and } X - U \text{ is finite or } X\}.$$ 

Show $\mathcal{T}$ is a topology on $X$. 
Chapter 6

Closed Sets

Definition 6.1. Let $X$ be a topological space. A subset $C$ of $X$ is called \textit{closed} if $X - C$ is open.

Theorem 6.2. Let $X$ be a topological space. Then

1. $X$ and $\emptyset$ are closed.
2. The intersection of any collection of closed sets is closed.
3. The union of a finite collection of closed sets is closed.

Definition 6.3. Let $X$ be a topological space, and let $A$ be a subset of $X$.

1. By 6.2, $A$ is a subset of at least one closed set, namely $X$ itself. The \textit{closure} of $A$ is the intersection of all closed sets that contain $A$ as a subset. The closure of $A$ will be denoted $\text{Cl}(A)$.
2. By the definition of topological space, at least one subset of $A$ is open, namely the empty set. The \textit{interior} of $A$ is the union of all open subsets of $A$. The interior of $A$ will be denoted $\text{Int}(A)$.

Proposition 6.4. Let $X$ be a topological space, and let $A$ be the subset of $X$. Then

1. $\text{Int}(A) \subseteq A \subseteq \text{Cl}(A)$,
2. $\text{Cl}(A)$ is closed and $\text{Int}(A)$ is open,
3. $A$ is closed if and only if $A = \text{Cl}(A)$, and
4. $A$ is open if and only if $A = \text{Int}(A)$.

Proposition 6.5. Let $X$ be a topological space, and let $A$ and $B$ be subsets of $X$. Then

1. $\text{Cl}(A) \cup \text{Cl}(B) = \text{Cl}(A \cup B)$, and
2. $\text{Cl}(A) \cap \text{Cl}(B) \supseteq \text{Cl}(A \cap B)$; equality may fail.

**Proposition 6.6.** Let $X$ be a topological space, and let $A$ and $B$ be subsets of $X$. Then

1. $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$, and
2. $\text{Int}(A \cup B) \supseteq \text{Int}(A) \cup \text{Int}(B)$; equality may fail.

**Definition 6.7.** Let $A$ and $B$ be sets. We say $A$ intersects $B$ if $A \cap B \neq \emptyset$. We say $A$ and $B$ are disjoint if $A \cap B = \emptyset$.

**Theorem 6.8.** Let $X$ be a topological space, let $A$ be a subset of $X$, and let $p$ be a point in $X$. Then $p$ is an element of the closure of $A$ if and only if every open set containing $p$ intersects $A$.

**Definition 6.9.** Let $X$ be a topological space, let $A$ be a subset of $X$, and let $p$ be a point in $X$. Then $p$ is called a limit point of $A$ in $X$ if for every open set $U$ containing $p$, $U - \{p\}$ intersects $A$. The set of all limit points of $A$ will be denoted $A'$.

**Corollary 6.10.** (to Theorem 6.8) Let $X$ be a topological space, and let $A$ be a subset of $X$. Then $\text{Cl}(A) = A \cup A'$.

**Definition 6.11.** Let $X$ be a topological space and let $A \subseteq X$. Then boundary of $A$ is $\text{Cl}(A) \cap \text{Cl}(X - A)$. The boundary of $A$ is denoted $\text{Bd}(A)$.

**Exercise 6.12.** Show each of the following.

1. $A \subseteq B$ does not imply $\text{Bd}(A) \subseteq \text{Bd}(B)$.
2. $A' \subseteq \text{Bd}(A)$ may be false.
3. Show that $\text{Bd}(A) \subseteq A'$ may also be false.

For Exercises 6.13 through 6.15, $A$ and $B$ are subsets of a topological space $X$.

**Exercise 6.13.** Prove that $\text{Int}(A)$ and $\text{Bd}(A)$ are disjoint and their union is $\text{Cl}(A)$.

**Exercise 6.14.** Prove $\text{Bd}(A) = \emptyset$ if and only if $A$ is both open and closed.

**Exercise 6.15.** Prove $A$ is open if and only if $\text{Bd}(A) = \text{Cl}(A) - A$. 

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Chapter 7

Bases

Definition 7.1. Let \((X, \mathcal{T})\) be a topological space, and let \(B\) be a collection of subsets of \(X\). Then \(B\) is a basis for \(\mathcal{T}\) if \(B \subseteq \mathcal{T}\) and every nonempty element of \(\mathcal{T}\) is a union of elements of \(B\). The elements of \(B\) are called basic open sets. We say that \(\mathcal{T}\) is generated by \(B\). The plural of “basis” is “bases.”

Theorem 7.2. Let \((X, \mathcal{T})\) be topological space, and let \(B\) be a subset of \(\mathcal{T}\). Then \(B\) is a basis for \(\mathcal{T}\) if and only if for each element \(U\) of \(\mathcal{T}\) and for each point \(p\) in \(U\), there is an element \(B\) of \(B\) such that \(p \in B\) and \(B \subseteq U\).

Note: By the above theorem, if \((X, d)\) is a metric space, then the collection of all open balls is a basis for the metric topology.

Theorem 7.3. Let \(X\) be a set, and let \(B\) be a collection of subsets of \(X\). Then \(B\) is a basis for some topology on \(X\) if and only if the following hold:

1. If \(x \in X\), then there exists \(B \in B\) such that \(x \in B\), and
2. If \(B_1, B_2 \in B\) and \(x \in B_1 \cap B_2\), then there exists \(B_3 \in B\) such that \(x \in B_3\) and \(B_3 \subseteq B_1 \cap B_2\).

Example 7.4. Let \((X, \mathcal{T})\) and \((Y, \mathcal{I})\) be topological spaces. Let \(B\) be the collection of all subsets of \(X \times Y\) of the form \(U \times V\), where \(U \in \mathcal{T}\) and \(V \in \mathcal{I}\).

1. Prove that \(B\) is a basis for some topology on \(X \times Y\). The topology that \(B\) generates is called the product topology.
2. Provide an example to show that \(B\) itself may not be a topology.

Example 7.5. The following are bases for some topology on \(\mathbb{R}\). [For the presentation, show this is true for \(B_2\) and \(B_4\).]

\(B_1 = \{(a, b) \mid a < b\}\).
\(B_2 = \{[a, b) \mid a < b\}\).
\(B_3 = \{(a, b) \mid a < b\}\).
\[ B_4 = \{(a, \infty) \mid a \in \mathbb{R}\} \]
\[ B_5 = \{(-\infty, b) \mid b \in \mathbb{R}\} \]
\[ B_6 = \{B \subseteq \mathbb{R} \mid \mathbb{R} - B \text{ is finite}\} \]

**Note:** The topology generated by \( B_2 \) above is called the **lower-limit topology** on \( \mathbb{R} \) or \( \mathbb{R}_{\text{bad}} \).

**Theorem 7.6.** Let \( X \) be a set, and let \( \mathcal{B} \) be a basis for some topology on \( X \). Let \( \mathcal{T} \) be the topology generated by \( \mathcal{B} \), and let \( A \) be a subset of \( X \).

1. \( x \in \text{Cl}(A) \) if and only if for every \( B \in \mathcal{B} \) that contains \( x \), \( B \) intersects \( A \).
2. \( x \in \text{Int}(A) \) if and only if there exists \( B \in \mathcal{B} \) such that \( x \in B \) and \( B \subseteq A \).
3. \( x \in A' \) if and only if for every \( B \in \mathcal{B} \) that contains \( x \), \( B - \{x\} \) intersects \( A \).

**Exercise 7.7.** For each of the following subsets of \( \mathbb{R} \), determine the closure, interior, set of limit points, and boundary in the topologies generated by each of the bases given in 7.5.

1. \((1,2)\)
2. \(\{1,2\}\)
3. \(\mathbb{R} - \{1,2\}\)
4. \(\mathbb{R} - \{1,\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..\}\)
Chapter 8

Separation Properties

**Definition 8.1.** Let $X$ be a topological space.

1. $X$ is **Hausdorff** if for each pair of distinct points $p$ and $q$ in $X$, there are disjoint open sets $U$ and $V$ such that $p \in U$ and $q \in V$.

2. $X$ is **regular** if for every point $p$ in $X$ and every closed set $A$ with $p \notin A$, there exist disjoint open sets $U$ and $V$ such that $p \in U$ and $A \subseteq V$.

3. $X$ is **normal** if for every pair of disjoint closed sets $A$ and $B$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Exercise 8.2.** Determine which of the six topologies on $\mathbb{R}$ given in Section 7 are Hausdorff.

**Theorem 8.3.** If $X$ and $Y$ are Hausdorff spaces, then $X \times Y$ with the product topology is Hausdorff.

**Theorem 8.4.** If $X$ is Hausdorff, then every singleton subset $\{p\}$ of $X$ is closed.

**Theorem 8.5.** $X$ is regular if and only if for each point $p$ in $X$ and each open set $U$ containing $p$, there is an open set $V$ such that $p \in V$ and $\text{Cl}(V) \subseteq U$.

**Theorem 8.6.** $X$ is normal if and only if for each closed set $A$ in $X$ and open set $U$ containing $A$, there is an open set $V$ such that $A \subseteq V$ and $\text{Cl}(V) \subseteq U$.

**Definition 8.7.** Let $(X,d)$ be a metric space, let $p$ be a point in $X$, and let $\varepsilon > 0$. The **closed ball** about $p$ of radius $\varepsilon$ is $\{x \in X \mid d(p,x) \leq \varepsilon\}$.

**Proposition 8.8.** Closed balls are closed.

**Theorem 8.9.** Metric spaces are regular.

**Definition 8.10.** Let $A$ be a subset of a metric space $(X,d)$, and let $p$ be a point in $X$. The **distance from $p$ to $A$** is denoted $d(p,A)$ and is defined by $d(p,A) = \inf\{d(p,a) \mid a \in A\}$.
**Proposition 8.11.** Let \((X,d)\) be a metric space, let \(A\) be a subset of \(X\), and let \(p\) be a point in \(X\). If \(A\) is closed, then \(d(p,A) = 0\) if and only if \(p\) is an element of \(A\).

**Proposition 8.12.** Let \((X,d)\) be a metric space, and let \(A\) and \(B\) be subsets of \(X\). Then the set of all points \(p\) such that \(d(p,A) < d(p,B)\) is open.

**Theorem 8.13.** Metric spaces are normal.

**Proposition 8.14.** Let \((X,\mathcal{T})\) be a topological space, and let \(Y\) be a subset of \(X\). Let \(\mathcal{S}\) be the collection of all sets \(U \cap Y\) such that \(U \in \mathcal{T}\). Then \(\mathcal{S}\) is a topology on \(Y\).

**Definition 8.15.** \(\mathcal{S}\) as defined above is called the **subspace topology** on \(Y\), and \(Y\) is called a **subspace** of \(X\).

**Theorem 8.16.** Every subspace of a Hausdorff space is Hausdorff.

**Theorem 8.17.** Every subspace of a regular space is regular.

**Theorem 8.18.** Every closed subspace of a normal space is normal.
Chapter 9

Countability Properties

**Definition 9.1.** Let $A$ be a subset of a topological space $X$. We say that $A$ is **dense** in $X$ if $\text{Cl}(A) = X$. A space $X$ is called **separable** if it has a countable dense subset.

**Theorem 9.2.** $\mathbb{R}$ with the usual topology is separable.

**Definition 9.3.** A space $X$ is **2$^\text{nd}$ countable** if $X$ has a countable basis.

**Theorem 9.4.** A 2$^\text{nd}$ countable space is separable.

**Theorem 9.5.** A subspace of a 2$^\text{nd}$ countable space is 2$^\text{nd}$ countable.

**Definition 9.6.** A space $X$ is **1$^\text{st}$ countable** if for every point $p$ in $X$, there is a countable collection $\{U_n\}_{n \in \mathbb{N}}$ of open sets of $X$, each of which contains $p$, such that if $V$ is an open set containing $p$, then there exists $n \in \mathbb{N}$ such that $U_n \subseteq V$.

**Theorem 9.7.** A metric space is 1$^\text{st}$ countable.

**Definition 9.8.** A topological space is **Souslin** if every collection of pairwise disjoint open sets is countable.

**Theorem 9.9.** Separable spaces are Souslin.

**Definition 9.10.** Let $\{p_n\}$ be a sequence of points in a topological space $X$. We say that $\{p_n\}$ **converges** to a point $p$ in $X$ if for every open set $U$ containing $p$, there exists a positive integer $M$ such that $p_n \in U$ for all $n > M$.

**Theorem 9.11.** If $p$ is a limit point of a set $A$ in a 1$^\text{st}$ countable space $X$, then there is a sequence in $A$ that converges to $p$.

**Theorem 9.12.** Every uncountable set in a 2$^\text{nd}$ countable space has a limit point.
Definition 10.1. Let $A$ be a subset of a topological space $X$, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be a collection of subsets of $X$. We say that $\mathcal{U}$ is a cover of $A$ if $A \subseteq \bigcup_{\alpha \in J} U_\alpha$. $\mathcal{U}$ is called an open cover of $A$ if $\mathcal{U}$ is a cover of $A$ and each element of $\mathcal{U}$ is open. If $\mathcal{U}$ is a cover of $A$, then any subset of $\mathcal{U}$ that is also a cover of $A$ is called a subcover of $A$. A space $X$ is compact if every open cover of $X$ has a finite subcover.

Theorem 10.2. Let $X$ be a space whose topology is generated by a basis. Then $X$ is compact if and only if every cover of $X$ by basic open sets has a finite subcover.

Theorem 10.3. Let $(X, \mathcal{T})$ be a topological space, and let $A$ be a subset of $X$. Then $A$ is compact in the subspace topology if and only if every cover of $A$ by elements of $\mathcal{T}$ has a finite subcover.

Theorem 10.4. A compact subset of a Hausdorff space is closed.

Theorem 10.5. A closed subset of a compact space is compact.

Theorem 10.6. A compact Hausdorff space is normal.

Definition 10.7. Let $(X, d)$ be a metric. A subset $A$ of $X$ is bounded if there exists a number $r$ such that $d(p, q) < r$ for all points $p$ and $q$ in $A$.

Theorem 10.8. A compact subset of a metric space is bounded.

Theorem 10.9. If $X$ and $Y$ are compact spaces, then $X \times Y$ is compact in the product topology.

Theorem 10.10. In $\mathbb{R}$ with the usual topology, a closed interval is compact.

Theorem 10.11. In $\mathbb{R}^n$ with the usual topology, a set is compact if and only if it is closed and bounded.

Theorem 10.12. A space $X$ is Lindelöf if every open cover of $X$ has a countable subcover.
Theorem 10.13. Every 2\textsuperscript{nd} countable space is Lindelöf.

Theorem 10.14. Every uncountable subset of a Lindelöf space has a limit point.

Definition 10.15. A space is \textit{countably compact} if every countable open cover has a finite subcover.

Theorem 10.16. A countably compact Lindelöf space is compact.

Theorem 10.17. Let $X$ be a Hausdorff space. Then $X$ is countably compact if and only if every infinite subset of $X$ has a limit point.

Theorem 10.18. Let $X$ be a metric space. Then $X$ is Souslin if and only if every uncountable subset of $X$ has a limit point.

Theorem 10.19. Let $X$ be a metric space. Then the following are equivalent.

1. $X$ is separable.
2. $X$ is 2\textsuperscript{nd} countable.
3. $X$ is Souslin.
4. $X$ is Lindelöf.
5. Every uncountable subset of $X$ has a limit point.

Definition 10.20. A collection of sets is said to have the \textit{finite intersection property} if the intersection of any finite subcollection is nonempty.

Theorem 10.21. A topological space $X$ is compact if and only if every collection $\{C_\alpha\}_{\alpha \in I}$ of closed sets with the finite intersection property satisfies $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$.

Corollary 10.22. Let $X$ be compact. Suppose that $\{F_n\}_{n \in \mathbb{N}}$ is a family of non-empty closed subsets of $X$ such that $F_{n+1} \subseteq F_n$ for all $n$. Then $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$.

Definition 10.23. Let $(X, d)$ be a metric space, and let $A$ be a bounded subset of $X$. The \textit{diameter} of $A$ is denoted $\text{diameter}(A)$ and is defined by $\text{diameter}(A) = \sup \{d(x, y) \mid x, y \in A\}$.

Corollary 10.24. Let $(X, d)$ be a compact metric space. Suppose that $\{F_n\}_{n \in \mathbb{N}}$ is a family of nonempty closed subsets of $X$ such that $F_{n+1} \subseteq F_n$ for all $n$. Suppose also that $\lim_{n \to \infty} \text{diameter}(F_n) = 0$. Then $\bigcap_{n \in \mathbb{N}} F_n$ contains exactly one point.
Chapter 11

Continuity

**Definition 11.1.** Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be a function. Then $f$ is called continuous if $f^{-1}(V)$ is open in $X$ whenever $V$ is open in $Y$.

Note: By 4.3, if $X$ and $Y$ are metric spaces (and hence have metric topologies), then $f : X \to Y$ is continuous in the original "$\varepsilon - \delta$" definition of continuity if and only if $f$ is continuous according to the definition of continuous given above.

**Theorem 11.2.**

1. Constant functions are continuous.
2. The identity function from a space to itself is continuous.
3. If $X$ is a topological space and $A$ is a subspace of $X$, then the inclusion function $A \to X$ is continuous.
4. Compositions of continuous functions are continuous.

**Theorem 11.3.** Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent.

1. $f$ is continuous.
2. $f^{-1}(C)$ is closed in $X$ whenever $C$ is closed in $Y$.
3. For every subset $A$ of $X$, $f(Cl(A)) \subseteq Cl(f(A))$.
4. $f^{-1}(B)$ is open in $X$ whenever $B$ is a basic open set in $Y$.

**Theorem 11.4. Pasting Lemma**

Let $X$ and $Y$ be topological spaces, and let $X = A \cup B$, where $A$ and $B$ closed in $X$. Suppose $f : A \to Y$ and $g : B \to Y$ are continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the function $h : X \to Y$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is continuous.
Theorem 11.5. Let $X$ and $Y$ be topological spaces.

1. The projection functions $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ defined by $p_1(x,y) = x$ and $p_2(x,y) = y$ are continuous.

2. Let $Z$ be a topological space, and let $f_1 : Z \to X, f_2 : Z \to Y$ and $f : Z \to X \times Y$ be functions such that $f(z) = (f_1(z), f_2(z))$ for all $z \in Z$. Then $f$ is continuous if and only if both $f_1$ and $f_2$ are continuous.

Theorem 11.6. If $f : X \to Y$ is continuous and $A$ is compact subset of $X$, then $f(A)$ is compact.

Exercise 11.7. Let $\mathbb{R}_u$ denote the real line with the usual topology, and let $\mathbb{R}_{bad}$ denote the real line with the topology generated by the basis $\mathcal{B}_2 = \{(a,b) \mid a < b\}$. Determine which of the following functions are continuous.

1. $f : \mathbb{R}_{bad} \to \mathbb{R}_u$ defined by $f(x) = x$.
2. $g : \mathbb{R}_u \to \mathbb{R}_{bad}$ defined by $g(x) = x$.
3. $f : \mathbb{R}_{bad} \to \mathbb{R}_{bad}$ defined by $f(x) = 1 - 2x$.
4. $g : \mathbb{R}_{bad} \to \mathbb{R}_{bad}$ defined by $g(x) = 2x - 1$.

Definition 11.8. Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be a function. Then $f$ is called a homeomorphism if $f$ is a bijection, $f$ is continuous, and $f^{-1}$ is continuous. If such a function $f$ exists, then we say $X$ is homeomorphic to $Y$.

Theorem 11.9. For topological spaces $X$ and $Y$, define $X \sim Y$ if and only if $X$ is homeomorphic to $Y$. Then $\sim$ is an equivalence relation.

Exercise 11.10. Let $\mathbb{R}$ have the usual topology. Show that any two open intervals in $\mathbb{R}$ are homeomorphic. Show that $\mathbb{R}$ is homeomorphic to an open interval.

Theorem 11.11. If $X$ is compact, $Y$ is Hausdorff, and $f : X \to Y$ is a continuous bijection, then $f$ is a homeomorphism.
Chapter 12

Connectedness

Throughout this section, let $X$ be a topological space.

**Definition 12.1.** $X$ is called **disconnected** if there exist nonempty disjoint open sets $U$ and $V$ such that $X = U \cup V$. Such a pair $(U, V)$ is called a **separation** of $X$. If $X$ is not disconnected, then $X$ is called **connected**.

**Theorem 12.2.** The following are equivalent.

1. $X$ is disconnected.

2. There exists a nonempty proper subset of $X$ that is both open and closed.

3. There exists a continuous function from $X$ onto a two-point set with the discrete topology.

**Theorem 12.3.** The continuous image of a connected space is connected. Also, if $X$ and $Y$ are homeomorphic, then $X$ is connected if and only if $Y$ is connected.

**Lemma 12.4.** If $(U, V)$ is a separation of $X$ and if $Y$ is a connected subset of $X$, then $Y \subseteq U$ or $Y \subseteq V$.

**Theorem 12.5.** Suppose $\{A_\alpha\}_{\alpha \in I}$ is a collection of connected subsets of $X$, each of which intersects a connected set $B$. Then $B \cup (\bigcup_{\alpha \in I} A_\alpha)$ is connected.

**Theorem 12.6.** If $A$ is a connected subset of $X$ and if $A \subseteq B \subseteq \text{Cl}(A)$, then $B$ is connected.

**Theorem 12.7.** If $X$ and $Y$ are connected, then $X \times Y$ is connected.

**Definition 12.8.** Let $S$ be a subset of $\mathbb{R}$. Then $S$ is an **interval** if $a \in S$, $b \in S$, and $a < c < b$ imply $c \in S$. 
Theorem 12.9.

1. Every interval in $\mathbb{R}$ (with the usual topology) is connected.
2. $\mathbb{R}_{bad}$ is disconnected.

**Theorem 12.10. Intermediate Value Theorem**

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous, and let $k$ be a number between $f(a)$ and $f(b)$. Then there exists a number $c$ between $a$ and $b$ so that $f(c) = k$.

**Theorem 12.11. A Fixed Point Theorem**

Let $f : [0, 1] \rightarrow [0, 1]$ be continuous. Then there exists $x \in [0, 1]$ so that $f(x) = x$.

**Definition 12.12.** Let $a \in X$. The **component** of $a$ in $X$ is the union of all connected subsets of $X$ that contain $a$.

**Proposition 12.13.**

1. Each component in $X$ is connected.
2. The distinct components in $X$ form a partition of $X$.

**Definition 12.14.** A topological space is called **locally connected** if it has a basis of connected sets.

**Proposition 12.15.** $X$ is locally connected if and only if every component of each open set is open in $X$.

**Exercise 12.16.** Give an example of a locally connected subset of $\mathbb{R}^2$ which is not connected. Give an example of a connected subset of $\mathbb{R}^2$ which is not locally connected.
Chapter 13

Path Connectedness

Throughout this section, let $X$ be a topological space.

**Definition 13.1.** A path in $X$ is continuous function $\alpha: [a, b] \to X$ (where $[a, b]$ denotes a closed interval in $\mathbb{R}$ with the usual topology). If $\alpha$ is a path and $\alpha(a) = x$ and $\alpha(b) = y$, we say that $\alpha$ is a path from $x$ to $y$, and we call $x$ the initial or starting point of $\alpha$ and we call $y$ the terminal or ending point of $\alpha$.

**Proposition 13.2.** Define a relation $\approx$ on $X$ by $x \approx y$ if and only if there is a path from $x$ to $y$. Then $\approx$ is an equivalence relation.

**Definition 13.3.** The equivalence classes in $X$ under the relation $\approx$ are called the path components of $X$. If $x \approx y$ for all $x$ and $y$ in $X$, then $X$ is called path connected.

**Proposition 13.4.**

1. The continuous image of a path connected space is path connected.

2. If $A$ and $B$ are path connected subsets of $X$ and $A \cap B \neq \emptyset$, then $A \cup B$ is path connected.

**Exercise 13.5.** Let $d$ denote the usual metric in $\mathbb{R}^n$, and let $\theta$ denote the origin. The unit ball in $\mathbb{R}^n$ is $\{x \mid d(x, \theta) \leq 1\}$, and the unit sphere in $\mathbb{R}^n$ is $\{x \mid d(x, \theta) = 1\}$.

1. Show that the unit ball is path connected.

2. Show that the unit sphere is path connected if and only if $n > 1$.

**Proposition 13.6.** If $X$ is path connected, then $X$ is connected.

**Exercise 13.7.** Give an example of a space that is connected but not path connected.

**Exercise 13.8.** Show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ when $n > 1$. 

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Definition 13.9. Let $x$ be a point in $X$. Then $X$ is called **locally path connected** at $x$ if for every open set $U$ containing $x$, there is a path connected open set $V$ with $x \in V$ and $V \subseteq U$. If $X$ is locally path connected at every point, then $X$ is called **locally path connected**.

Theorem 13.10. The following are equivalent.

1. $X$ is locally path connected.
2. If $U$ is open in $X$, then every path component of $U$ is open in $X$.

Theorem 13.11. If $X$ is connected and locally path connected, then $X$ is path connected.

Exercise 13.12. Give an example of a locally path connected space $X$ and a continuous function $f : X \to Y$ with $f(X)$ not locally path connected.
Chapter 14

Quotient Spaces

**Definition 14.1.** Let $X$ and $Y$ be topological spaces, and let $p : X \to Y$ be an onto function. A subset $U$ of $X$ is called *saturated* (with respect to $p$) if $p^{-1}(p(U)) = U$. The function $p$ is called a *quotient map* if $p$ is continuous and $p(U)$ is open whenever $U$ is an open saturated set.

**Exercises 14.2.**

1. Let $p : X \to Y$ be continuous. Show that if there exists a continuous function $f : Y \to X$ so that $p(f(y)) = y$ for all $y \in Y$, then $p$ is a quotient map.

2. Let $A \subseteq X$. A *retraction* from $X$ to $A$ is a continuous function $r : X \to A$ so that $r(a) = a$ for all $a \in A$. Show that a retraction is a quotient map.

**Exercise 14.3.** Let $X$ be a topological space, let $A$ be any set, and let $p : X \to A$ be an onto function. Show that there is exactly one topology on $A$ that makes $p$ a quotient map.

**Definition 14.4.** The topology on $A$ given by 14.3 is called the *quotient topology* on $A$, relative to $p$.

**Exercise 14.5.** Let $\mathbb{R}$ have the usual topology, and let $p : X \to \{a, b, c\}$ be the function defined by $p(x) = a$ if $x > 0$, $p(x) = b$ if $x < 0$, and $p(0) = c$. Find the quotient topology on $\{a, b, c\}$, relative to $p$.

**Definition 14.6.** Let $X$ be a topological space on which an equivalence relation is defined, and let $X^*$ denote the set of equivalence classes. Let $p : X \to X^*$ be the function that assigns each element of $X$ to its equivalence class. Then $X^*$ is called a *quotient space* when it has the quotient topology, relative to $p$.

**Definition 14.7.** Let $X = [0, 1] \times [0, 1]$, where $X$ has the subspace topology as a subset of $\mathbb{R}^2$ with the usual topology. Define an equivalence relation on $X$ as follows:
1. When \(0 < x < 1\) and \(0 < y < 1\), the point \((x,y)\) is equivalent only to itself.

2. When \(0 < x < 1\), then \((x,0)\) and \((x,1)\) are equivalent.

3. When \(0 < y < 1\), then \((0,y)\) and \((1,y)\) are equivalent.

4. The following four points are equivalent: \((0,0), (0,1), (1,0), (1,1)\).

The resulting quotient space \(X^*\) is a torus.

**Exercise 14.8.** Similarly to the previous definition, define a Mobius band and a Klein bottle as quotient spaces. For extra fun, define a two-hole torus.
Chapter 15

Subbases

Proposition 15.1. Let $X$ be a set, and let $\mathcal{I}$ be a collection of subsets of $X$ such that the union of all the elements of $\mathcal{I}$ is $X$. Let $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{I}$. Then $\mathcal{B}$ is a basis for some topology on $X$.

Definition 15.2. Let $X$ be a set. A subbasis for a topology on $X$ is a collection $\mathcal{S}$ of subsets of $X$ whose union is $X$. The basis $\mathcal{B}$ given by Proposition 15.1 is called the basis generated by the subbasis $\mathcal{I}$, and the topology on $X$ given by Proposition 15.1 is called the topology generated by the subbasis $\mathcal{I}$.

Exercise 15.3. Consider the following subsets of $\mathbb{R}$:

$$\mathcal{I}_1 = \{(-\infty, a) \mid a \in \mathbb{R}\} \text{ and } \mathcal{I}_2 = \{(a, \infty) \mid a \in \mathbb{R}\}.$$  

1. Show that $\mathcal{I}_1 \cup \mathcal{I}_2$ is a subbasis.

2. Show that the topology generated by $\mathcal{I}_1 \cup \mathcal{I}_2$ is the usual topology on $\mathbb{R}$.

Proposition 15.4. Let $X$ have the topology generated by a subbasis $\mathcal{I}$, and let $Y$ be a topological space. A function $f : Y \to X$ is continuous if and only if $f^{-1}(S)$ is open for every $S \in \mathcal{I}$. 
Chapter 16

Product Topology

Definition 16.1. Let \( \{X_\alpha\}_{\alpha \in J} \) be an indexed family of sets. The Cartesian product \( \prod_{\alpha \in J} X_\alpha \) is the set of all functions \( x : J \to \bigcup_{\alpha \in J} X_\alpha \) such that \( x(\alpha) \in X_\alpha \) for each \( \alpha \in J \). Frequently \( x_\alpha \) is used to denote \( x(\alpha) \); \( x_\alpha \) is called the \( \alpha \)-th coordinate of \( x \). Also, \((x_\alpha)_{\alpha \in J}\) is frequently used to denote \( x \). If each of the sets \( X_\alpha \) are the same set \( X \), then \( \prod_{\alpha \in J} X_\alpha \) is denoted \( X^J \), and the elements of \( X^J \) are called \( J \)-tuples of elements of \( X \).

Exercise 16.2. Let \( J \) be a nonempty set, and let \( \{X_\alpha\}_{\alpha \in J} \) and \( \{Y_\alpha\}_{\alpha \in J} \) be families of sets.

1. Prove that if \( Y_\alpha \subseteq X_\alpha \) for all \( \alpha \in J \), then \( \prod_{\alpha \in J} Y_\alpha \subseteq \prod_{\alpha \in J} X_\alpha \).
2. Find condition(s) under which the converse to (1) is true.
3. Show \( \left( \prod_{\alpha \in J} X_\alpha \right) \cap \left( \prod_{\alpha \in J} Y_\alpha \right) = \prod_{\alpha \in J} (X_\alpha \cap Y_\alpha) \).
4. Show \( \left( \prod_{\alpha \in J} X_\alpha \right) \cup \left( \prod_{\alpha \in J} Y_\alpha \right) \subseteq \prod_{\alpha \in J} (X_\alpha \cup Y_\alpha) \).

Definition 16.3. Let \( \{X_\alpha\}_{\alpha \in J} \) be an indexed family of topological spaces. For each \( \beta \in J \), define the \( \beta \)-th projection function \( p_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta \) by \( p_\beta(x) = x_\beta \). The product topology on \( \prod_{\alpha \in J} X_\alpha \) is the topology generated by the subbasis \( \mathcal{S} = \{p_\beta^{-1}(U) \mid \beta \in J \text{ and } U \text{ is open in } X_\beta \} \). Unless explicitly stated otherwise, we will always assume that the topology on a product of topological spaces is the product topology.

Exercise 16.4. Let \( J = \{1, 2\} \), and let \( X_1 \) and \( X_2 \) be topological spaces. Show that \( \prod_{\alpha \in J} X_\alpha \) is homeomorphic to \( X_1 \times X_2 \).

For Theorems 16.5 through 16.12, let \( \{X_\alpha\}_{\alpha \in J} \) be a family of topological spaces.
Proposition 16.5.

1. The projection functions are continuous.

2. The product topology is the smallest topology on $\prod_{\alpha \in J} X_\alpha$ for which the projection functions are continuous.

Theorem 16.6. Let $Y$ be a topological space, and let $f_\alpha : Y \to X_\alpha$ be a function for each $\alpha \in J$. Define $f : Y \to \prod_{\alpha \in J} X_\alpha$ by $f(y) = (f_\alpha(y))_{\alpha \in J}$. Then $f_\alpha$ is continuous for all $\alpha \in J$ if and only if $f$ is continuous.

Theorem 16.7. If $X_\alpha$ is Hausdorff for all $\alpha \in J$, then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff.

Theorem 16.8. If $A_\alpha$ is a closed subset of $X_\alpha$ for all $\alpha \in J$, then $\prod_{\alpha \in J} A_\alpha$ is a closed subset of $\prod_{\alpha \in J} X_\alpha$.

Theorem 16.9. If $A_\alpha \subseteq X_\alpha$ for all $\alpha \in J$, then $\text{Cl}\left(\prod_{\alpha \in J} A_\alpha\right) = \prod_{\alpha \in J} \text{Cl}(A_\alpha)$.

Exercise 16.10. Let $\mathscr{B} = \{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha\}$.

1. Show that $\mathscr{B}$ is a basis for some topology on $\prod_{\alpha \in J} X_\alpha$.

2. Show that the topology generated by $\mathscr{B}$ is not necessarily the product topology.

Definition 16.11. The topology generated by $\mathscr{B}$ in Exercise 16.10 is called the box topology.

Exercise 16.12. Show that the statement of Theorem 16.6 may not be true if it is changed by giving $\prod_{\alpha \in J} X_\alpha$ the box topology instead of the product topology.
Chapter 17

The Tychonoff Theorem

Definition 17.1. Let $X$ be a topological space. A collection $U$ of open sets is **finitely inadequate** if $U$ covers $X$ but $U$ has no finite subcover.

Theorem 17.2. Let $X$ be a topological space given by a basis, and let $\Psi$ be the collection of all finitely inadequate basic open covers of $X$. Then every chain by inclusion of elements of $\Psi$ has an upper bound.

Theorem 17.3. Let $X$ be a topological space with basis $\mathcal{B}$. If $X$ is not compact, then there exists a finitely inadequate open cover $\mathcal{U}$ by elements of $\mathcal{B}$ such that if $B \in \mathcal{B} - \mathcal{U}$ then $\mathcal{U} \cup \{B\}$ has a finite subcover.

Lemma 17.4. Let $X$ be a topological space with a basis $\mathcal{B}$ generated by a subbasis $\mathcal{S}$. Suppose that every open cover by subbasic open sets has a finite subcover. Further suppose that there exists a finitely inadequate basic open cover $\mathcal{U}$. Then there exists $B \in \mathcal{U}$ and finitely many elements $S_1, S_2, \cdots, S_n$ of $\mathcal{S} - \mathcal{U}$ such that $B = S_1 \cap S_2 \cap \cdots \cap S_n$.

Theorem 17.5. **Alexander Subbasis Lemma**

Let $X$ be a topological space generated by a subbasis $\mathcal{S}$. If every open cover by elements of $\mathcal{S}$ has a finite subcover, then $X$ is compact.

Theorem 17.6. **Tychonoff Theorem**

Let $J$ be an indexing set, and let $X_\alpha$ be a topological space for each $\alpha \in J$. If $X_\alpha$ is compact for each $\alpha \in J$, then $\Pi_{\alpha \in J}X_\alpha$ is compact.
Chapter 18

Urysohn’s Lemma

This section is a sequence of exercises which together prove the following.

**Theorem 18.1. Urysohn’s Lemma**

Let $X$ be a normal space, and let $A$ and $B$ be disjoint closed subsets of $X$. Given any closed interval $[a, b]$, there exists a continuous function $f : X \to [a, b]$ such that $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$.

Throughout this section, let $A$ and $B$ be disjoint closed subsets of a normal space $X$.

**Exercise 18.2.** Show that it suffices to show that there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

**Exercise 18.3.** Let $J$ be the set of all rational numbers in the interval $[0, 1]$. Show that there exists a family $\{U_j\}_{j \in J}$ of open sets such that $A \subseteq U_0$, $X - B = U_1$, and $\text{Cl}(U_i) \subseteq U_j$ whenever $i$ and $j$ are elements of $J$ with $i < j$.

**Exercise 18.4.** Extend the family of open sets in 18.3 by defining $U_j = \emptyset$ for rational $j$ less than $0$, and $U_j = X$ for rational $j$ greater than $1$. We now have an open set $U_j$ for each rational number $j$. For each $x \in X$, define $f(x) = \inf\{j \mid x \in U_j\}$.

1. Show that for each $x \in X$, the set $\{j \mid x \in U_j\}$ is nonempty and bounded below. [This establishes the fact that $f(x)$ exists for all $x \in X$.]

2. Show that $f(x) \in [0, 1]$ for all $x \in X$.

3. Show that $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

It remains to show $f$ is continuous.
Exercise 18.5. Show the following.

1. If $x \in \text{Cl}(U_j)$, then $f(x) \leq j$.
2. If $x \notin U_j$, then $f(x) \geq j$.

Exercise 18.6. Show that given $x \in X$ and given an open interval $(a,b)$ with $f(x) \in (a,b)$ there exists an open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq (a,b)$.

[Hint: if $i$ and $j$ are rational numbers with $a < i < f(x) < j < b$, then $U = U_j - \text{Cl}(U_i)$ will work.]

Exercise 18.7. Show that $f$ is continuous.
Chapter 19

Uniform Convergence

Definition 19.1. Let $X$ be a set, and let $Y$ be a metric space with metric $d$. A sequence $\{f_n\}$ of functions $f_n : X \to Y$ is said to converge uniformly to $f : X \to Y$ if for every $\varepsilon > 0$ there exists a positive integer $N$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and all $x \in X$.

Theorem 19.2. Let $X$ be a topological space and let $Y$ be a metric space. If $\{f_n\}$ is a sequence of continuous functions $f_n : X \to Y$ that converge uniformly to $f : X \to Y$, then $f$ is continuous.

Exercise 19.3. Let $X$ be a topological space and let $Y$ be a metric space. Suppose $\{f_n\}$ is a sequence of continuous functions $f_n : X \to Y$, and suppose $\{x_n\}$ is a sequence of points in $X$ that converge to $x \in X$. Show that if $\{f_n\}$ converges uniformly to $f : X \to Y$, then $\{f_n(x_n)\}$ converges to $f(x)$.

Theorem 19.4. Weierstrauss M-Test

Let $X$ be a set, and let $\{f_i\}$ be a sequence of functions $f_i : X \to \mathbb{R}$. For each positive integer $n$, let $s_n : X \to \mathbb{R}$ be defined by $s_n(x) = \sum_{i=1}^{n} f_i(x)$. If there exists a sequence $\{M_i\}$ of real numbers such that $|f_i(x)| \leq M_i$ for all $x \in X$ and all positive integers $i$, and if $\sum_{i=1}^{\infty} M_i$ converges, then $\{s_n\}$ converges uniformly to a function $s : X \to \mathbb{R}$.
Chapter 20

Tietze Extension Theorem

The bulk of this section is a sequence of exercises which together prove the following theorem. At the end of the section are a couple of corollaries to Urysohn’s Lemma and the theorem.

**Theorem 20.1. Tietze Extension Theorem**

Let $X$ be a normal space and let $A$ be a closed subset of $X$.

1. Any continuous function on $A$ to a closed interval $[a, b]$ extends to a continuous function on all of $X$ to $[a, b]$.

2. Any continuous function on $A$ to the set of real numbers extends to a continuous function on all of $X$ to the set of real numbers.

Throughout this section let $A$ be a closed subset of a normal space $X$.

**Exercises 20.2.**

1. Show that to prove statement 1 of the Tietze Extension Theorem, it suffices to prove it in the case $[a, b] = [-1, 1]$.

2. Show that to prove statement 2 of the Tietze Extension Theorem, it suffices to prove that any continuous function from $A$ to $(-1, 1)$ may be extended to a continuous function from $X$ to $(-1, 1)$.

**Exercise 20.3.** Let $r$ be a positive real number and suppose $f : A \rightarrow [-r, r]$ is continuous. Show that there exists a continuous function $g : X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$ so that $|g(x) - f(x)| \leq \frac{2}{3}r$ for all $x \in A$. [Hint: Use Urysohn’s Lemma with the sets $f^{-1}([-r, -\frac{r}{3}])$ and $f^{-1}([\frac{r}{3}, r])$.]

**Exercise 20.4.** Let $f : A \rightarrow [-1, 1]$ be a continuous function. Show that there exists a sequence $\{g_i\}$ of continuous functions on $X$ such that the image of $g_i$ lies in $[-\frac{2^{i-1}}{3^i}, \frac{2^{i-1}}{3^i}]$ for each positive integer $i$ and such that

$$
|f(x) - \sum_{i=1}^{n} g_i(x)| \leq \left(\frac{2}{3}\right)^n.
$$
Exercises 20.5. Let \( f \) and the sequence \( \{g_i\} \) be as in 20.4, and let \( g \) be the function on \( X \) defined by \( g(x) = \sum_{i=1}^{\infty} g_i(x) \).

1. Show that for each \( x \in X \), \( \sum_{i=1}^{\infty} g_i(x) \) does in fact converge.

2. Show that \( g(x) \in [-1,1] \) for all \( x \in X \).

3. Show that \( g(x) = f(x) \) for all \( x \in A \).

4. Show that \( g \) is continuous.

Thus part (1) of the Tietze Extension Theorem has been proved.

Exercise 20.6. Let \( f : A \to (-1,1) \) be a continuous function. By what has been proven, there exists a continuous function \( g : X \to [-1,1] \) such that \( g(x) = f(x) \) for all \( x \in A \).

1. Show that there exists a continuous function \( k : X \to [0,1] \) such that \( k(x) = 1 \) for all \( x \in A \) and \( k(x) = 0 \) for all \( x \in g^{-1}([-1,1]) \).

2. Let \( h \) be the function on \( X \) defined by \( h(x) = k(x)g(x) \). Show that \( h \) is a continuous extension of \( f \) and the range of \( h \) lies in \((-1,1)\).

Corollary 20.7. If a connected normal space has the property that singleton sets are closed and if the space has more than one point, then it is uncountable.

Definition 20.8. A space \( Y \) has the universal extension property if whenever \( X \) is a normal space, \( A \) is a closed subset of \( X \), and \( f : A \to Y \) is a continuous function, then there exists a continuous extension of \( f \) on \( X \).

Corollary 20.9. For any set \( J \), \( \mathbb{R}^J \) has the universal extension property.
Chapter 21

The Definition of the Fundamental Group

Let $X$ be a topological space.

**Definition 21.1.** Let $x$ and $y$ be points in $X$, and let $\alpha : [0, 1] \to X$ and $\beta : [0, 1] \to X$ both be paths from $x$ to $y$. A **path homotopy** between $\alpha$ and $\beta$ is a continuous function $F : [0, 1] \times [0, 1] \to X$ that satisfies each of the following criteria:

1. $F(0, t) = x$ for all $t \in [0, 1]$.
2. $F(1, t) = y$ for all $t \in [0, 1]$.
3. $F(s, 0) = \alpha(s)$ for all $s \in [0, 1]$.
4. $F(s, 1) = \beta(s)$ for all $s \in [0, 1]$.

If there exists a path homotopy between $\alpha$ and $\beta$, then we say $\alpha$ and $\beta$ are **(path) homotopic**.

**Proposition 21.2.** Given points $x$ and $y$ in $X$, homotopy defines an equivalence relation on the set of all paths in $X$ from $x$ to $y$.

**Definition 21.3.** Let $x, y,$ and $z$ be points in $X$. If $\alpha : [0, 1] \to X$ is a path from $x$ to $y$, and if $\beta : [0, 1] \to X$ is a path from $y$ to $z$, then the **product** (or **concatenation**) of $\alpha$ and $\beta$ is the function $\alpha \beta : [0, 1] \to X$ defined by

$$
\alpha \beta(s) = \begin{cases} 
\alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\
\beta(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 
\end{cases}
$$

**Proposition 21.4.** Let $x, y,$ and $z$ be points in $X$.

1. If $\alpha : [0, 1] \to X$ is a path from $x$ to $y$ and $\beta : [0, 1] \to X$ is a path from $y$ to $z$ then $\alpha \beta$ is a path from $x$ to $z$.

2. If $\alpha_1$ and $\alpha_2$ are homotopic paths from $x$ to $y$ and if $\beta_1$ and $\beta_2$ are homotopic paths from $y$ to $z$, then $\alpha_1 \beta_1$ is homotopic to $\alpha_2 \beta_2$. 

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Definition 21.5. 1. For any point \( x \in X \) the constant path at \( x \) is the function \( e_x : [0, 1] \to X \) defined by \( e_x(s) = x \).

2. For any path \( \alpha : [0, 1] \to X \) the inverse (or reverse) of \( \alpha \) is the function \( \alpha^{-1} : [0, 1] \to X \) defined by \( \alpha^{-1}(s) = \alpha(1 - s) \).

Proposition 21.6. Let \( x \) and \( y \) be points in \( X \) and let \( \alpha : [0, 1] \to X \) be a path from \( x \) to \( y \):

1. The paths \( \alpha e_y \) and \( e_x \alpha \) are both homotopic to \( \alpha \).

2. The inverse of \( \alpha \) is a path from \( y \) to \( x \).

3. The path \( \alpha \alpha^{-1} \) is homotopic to \( e_x \), and the path \( \alpha^{-1} \alpha \) is homotopic to \( e_y \).

Proposition 21.7. Let \( x, y, z, \) and \( w \) be points in \( X \). If \( \alpha : [0, 1] \to X \) is a path from \( x \) to \( y \), \( \beta : [0, 1] \to X \) is a path from \( y \) to \( z \), and \( \gamma : [0, 1] \to X \) is a path from \( z \) to \( w \), then \( \alpha(\beta \gamma) \) is homotopic to \( (\alpha \beta) \gamma \).

Definition 21.8. Fix a point \( a \) in \( X \). A loop based at \( a \) is a path in \( X \) from \( a \) to \( a \). Via 21.2 we have an equivalence relation on the set of all loops based at \( a \). For any loop \( \alpha \) based at \( a \), let \( \langle \alpha \rangle \) denote the equivalence class of \( \alpha \). The set of all equivalence classes is denoted \( \pi_1(X, a) \); that is, \( \pi_1(X, a) = \{ \langle \alpha \rangle | \alpha \text{ is a loop based at } a \} \). On \( \pi_1(X, a) \) define an operation by \( \langle \alpha \rangle \langle \beta \rangle = \langle \alpha \beta \rangle \).

Theorem 21.9. The operation defined on \( \pi_1(X, a) \) in 21.8 is well-defined, and \( \pi_1(X, a) \) under this operation is a group.

Definition 21.10. The set \( \pi_1(X, a) \) together with the operation defined in 21.8 is called the fundamental group of \( X \) based at \( a \).

Theorem 21.11. If \( X \) is path-connected and \( a \) and \( b \) are any two points in \( X \), then \( \pi_1(X, a) \) is isomorphic to \( \pi_1(X, b) \).

Theorem 21.12. Let \( X \) and \( Y \) be topological spaces, and let \( f : X \to Y \) be continuous. Fix a point \( a \) in \( X \) and define \( f_* : \pi_1(X, a) \to \pi_1(Y, f(a)) \) by \( f_*\langle \alpha \rangle = \langle f \circ \alpha \rangle \). Then \( f_* \) is well-defined and is a group homomorphism.

Definition 21.13. Given a continuous function \( f : X \to Y \) between topological spaces \( X \) and \( Y \) and given some fixed point \( a \) in \( X \), the function \( f_* \) defined in 21.12 is called the induced homomorphism.

Theorem 21.14. Induced homomorphisms satisfy the following categorical properties:

1. If \( f : X \to Y \) and \( g : Y \to Z \) are continuous functions on topological spaces, then \( (g \circ f)_* = g_* \circ f_* \).
2. \((\text{Id}_X)_* = \text{Id}_{\pi_1(X,a)}\), where \(\text{Id}_X\) and \(\text{Id}_{\pi_1(X,a)}\) denote the identity functions on \(X\) and \(\pi_1(X,a)\) respectively.

**Corollary 21.15.** If \(X\) and \(Y\) are path-connected topological spaces and if \(X\) is homeomorphic to \(Y\), then the fundamental groups of \(X\) and \(Y\) (regardless of where they are based) are isomorphic.
Chapter 22

Covering Spaces

Definition 22.1. Let $X$ and $\tilde{X}$ be topological spaces. Let $p : X \to \tilde{X}$ be continuous and onto. A subset $U$ of $X$ is said to be even\textit{ly} covered by $p$ if there exists a family of open sets $\{V_\alpha\}_{\alpha \in J}$ such that

1. $p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$,
2. $V_\alpha$ and $V_\beta$ are disjoint whenever $\alpha \neq \beta$, and
3. for each $\alpha \in J$ the restriction $p |_{V_\alpha}$ of $p$ to $V_\alpha$ is a homeomorphism from $V_\alpha$ onto $U$.

Each such set $V_\alpha$ is called a \textit{sheet} or \textit{slice} of $p^{-1}(U)$. If every point of $X$ is contained in an evenly covered open set, then $p$ is called a \textit{covering map} and $\tilde{X}$ is called a \textit{covering space} or \textit{cover} of $X$.

Exercise 22.2. The unit circle is $S^1 = \{(x, y) \in \mathbb{R} | x^2 + y^2 = 1\}$. Show that the function $p : S^1 \to S^1$ defined by $p(x, y) = (x^2 - y^2, 2xy)$ is a covering map. (Include a verification that $p(x, y) \in S^1$ for all $(x, y) \in S^1$.) Draw a picture illustrating how $f$ maps $S^1$ to $S^1$.

Exercise 22.3. Show that the function $f : \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map. (You may use without proof that the cosine and sine functions are continuous.) Draw a picture illustrating how $f$ maps $\mathbb{R}$ to $S^1$.

Proposition 22.4. If $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ are covering maps then $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$ is a covering map.

Proposition 22.5. If $p : \tilde{X} \to X$ is a covering map and $a$ is a point in $X$, then $p^{-1}\{a\}$ has the discrete topology.

Proposition 22.6. Let $\tilde{X}$ be connected, and let $p : \tilde{X} \to X$ be a covering map. If $a$ and $b$ are points in $X$, then the cardinality of $p^{-1}\{a\}$ and $p^{-1}\{b\}$ are equal.
Remark 22.7. In the situation of Proposition 22.6, if \( p^{-1}(\{a\}) \) is finite with \( k \) elements, we say \( \tilde{X} \) is a \( k \)-fold cover.

**Theorem 22.8. Lesbegue Number Lemma**

Let \( X \) be a compact metric space and let \( \mathcal{U} \) be an open cover of \( X \). Then there exists a positive real number \( \delta \) such that if \( A \) is a subset of \( X \) with \( \text{diameter}(A) < \delta \), then there exists an element \( U \) of \( \mathcal{U} \) such that \( A \subseteq U \).

**Definition 22.9.** Given a compact metric space \( X \) and an open cover \( \mathcal{U} \) of \( X \), such a number \( \delta \) as guaranteed by 22.8 is called a Lesbegue number for the cover \( \mathcal{U} \).

**Definition 22.10.** Let \( X, Y, \) and \( Z \) be topological spaces, and let \( f : X \to Y \) and \( g : Z \to Y \) be continuous functions. A lift of \( g \) is a continuous function \( \tilde{g} : Z \to X \) such that \( f \circ \tilde{g} = g \). If \( \tilde{g} \) is a lift of \( g \), we say, “\( g \) lifts to \( \tilde{g} \).”

**Theorem 22.11. Path Lifting Lemma**

Let \( p : \tilde{X} \to X \) be a covering map. Let \( a \) be a point in \( X \) and let \( \tilde{a} \in p^{-1}(a) \). Then every path in \( X \) beginning at \( a \) has a unique lift to a path beginning at \( \tilde{a} \).

**Theorem 22.12. Homotopy Lifting Lemma**

Let \( p : \tilde{X} \to X \) be a covering map. Let \( a \) be a point in \( X \) and let \( \tilde{a} \in p^{-1}(a) \). Suppose \( F : [0, 1] \times [0, 1] \to X \) is continuous and \( F(0, 0) = a \). Then \( F \) lifts to a map \( \tilde{F} : [0, 1] \times [0, 1] \to \tilde{X} \) with \( \tilde{F}(0, 0) = \tilde{a} \). Moreover, if \( F \) is a path homotopy, \( \tilde{F} \) is a path homotopy.

**Theorem 22.13.** Let \( p : \tilde{X} \to X \) be a covering map. Suppose \( a \) and \( b \) are points in \( X \), and suppose \( \alpha \) and \( \beta \) are homotopic paths from \( a \) to \( b \). Let \( \tilde{a} \in p^{-1}(a) \), and let \( \tilde{\alpha} \) and \( \tilde{\beta} \) denote the respective lifts of \( \alpha \) and \( \beta \) beginning at \( \tilde{a} \). Then \( \tilde{\alpha} \) and \( \tilde{\beta} \) end at the same point and are homotopic.

**Lemma 22.14.** Let \( p : \mathbb{R} \to S^1 \) be the covering map from 22.3 defined by \( p(x) = (\cos 2\pi x, \sin 2\pi x) \). Let \( \mathbb{Z} \) be the group of integers under addition. Define \( \psi : \pi_1(S^1, (1, 0)) \to \mathbb{Z} \) by \( \psi([\alpha]) = \bar{\alpha}(1) \), where \( \bar{\alpha} \) is the unique lift of \( \alpha \) beginning at 0. Then \( \psi \) is a well-defined homomorphism.

**Theorem 22.15.** \( \pi_1(S^1, (1, 0)) \) is isomorphic to \( \mathbb{Z} \), the group of integers under addition.

**Theorem 22.16.** Let \( p : \tilde{X} \to X \) be a covering map. Let \( a \) be a point in \( X \), and let \( \tilde{a} \in p^{-1}(a) \). Then \( p_* : \pi_1(\tilde{X}, \tilde{a}) \to \pi_1(X, a) \) is an injection.

**Theorem 22.17.** Let \( p : \tilde{X} \to X \) be a covering map. Let \( a \) be a point in \( X \) and let \( \tilde{a}, \tilde{b} \in p^{-1}(a) \). If \( \tilde{X} \) is path connected, then the two subgroups \( p_* (\pi_1(\tilde{X}, \tilde{a})) \) and \( p_* (\pi_1(\tilde{X}, \tilde{b})) \) are conjugate.

**Definition 22.18.** A space \( X \) is called simply connected if \( X \) is path connected and has a trivial fundamental group.
Exercise 22.19. Show that any convex subset of $\mathbb{R}^n$ is simply connected. [A subset $A$ of $\mathbb{R}^n$ is called convex if for any points $x$ and $y$ in $A$, the set $\{tx + (1-t)y | t \in [0,1]\}$ is a subset of $A$.]

Theorem 22.20. Let $p : \tilde{X} \to X$ be a covering map. Let $a$ be a point in $X$. If $\tilde{X}$ is path connected, there exists a surjection $\psi : \pi_1(X,a) \to p^{-1}(a)$. If $\tilde{X}$ is simply connected, then $\psi$ is a bijection.

Definition 22.21. A covering space $\tilde{X}$ of $X$ is called a universal cover of $X$ if $\tilde{X}$ is simply connected.

Exercise 22.22.

1. Show that $S^1 \times S^1$ is homeomorphic to the torus as defined as a quotient space of the square in section 14.

2. Find a universal cover of the torus.
Chapter 23

Computation of Some Fundamental Groups

Theorem 23.1. Simple Version of Van Kampen’s Theorem

Suppose $X$ is a topological space such that $X = U \cup V$ where $U$ and $V$ are open and $U \cap V \neq \emptyset$. Let $x \in U \cap V$. If $U \cap V$ is path connected and $\pi_1(U, x)$ and $\pi_1(V, x)$ are trivial, then $\pi_1(X, x)$ is trivial.

Definition 23.2. The $n$-sphere is

$$S^n = \{ (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1 \}.$$ 

On $S^n$ define an equivalence relation by $x \sim y$ if and only if $x = y$ or $x = -y$. The quotient space of $S^n$ under this equivalence relation is called projective $n$-space, and is denoted $P^n$. The space $P^2$ is called the projective plane.

Exercise 23.3. Show that the fundamental group of $S^n$ is trivial for $n \geq 2$.

Exercise 23.4.

1. Check that the relation defined in 23.2 really is an equivalence relation.
2. Show that $S^n$ is a two-fold cover of $P^n$.
3. Compute the fundamental group of $P^n$ for $n \geq 1$.

Theorem 23.5. If $X$ and $Y$ are topological spaces and $a$ and $b$ are points in $X$ and $Y$ respectively, then $\pi_1(X \times Y, (a, b))$ is isomorphic to $\pi_1(X, a) \times \pi_1(Y, b)$.

Exercise 23.6. Compute the fundamental group of the torus.

Definition 23.7. Let $X$ be a topological space, and let $A$ be a subset of $X$. We say that $A$ is a strong deformation retract of $X$ if there exists a continuous function $F : X \times [0, 1] \to X$ such that

1. $F(x, 0) = x$ for all $x \in X$,
2. $F(x, 1) \in A$ for all $x \in X$, and
3. \( F(a, t) = a \) for all \( a \in A \) and \( t \in [0, 1] \).

Such a map \( F \) is called a strong deformation retraction.

**Theorem 23.8.** Let \( A \) be a strong deformation retract of \( X \). Then the inclusion function induces an isomorphism of fundamental groups.

**Exercise 23.9.**

1. Show that \( \mathbb{R}^n \) minus a singleton set is simply connected for \( n \geq 3 \).

2. Show that the fundamental group of \( \mathbb{R}^2 \) minus a singleton set is isomorphic to the group of integers under addition.

**Exercise 23.10.** An annulus is a space homeomorphic to \( S^1 \times [0, 1] \). Compute the fundamental groups of an annulus and the Möbius band.

**Definition 23.11.** Let \( X = \bigcup_{i=1}^{k} (S^1 \times \{i\}) \) and define an equivalence relation on \( X \) as follows: a point \( (a, i) \) is only equivalent to itself when \( a \neq (1, 0) \), and for all \( i \) and \( j \), \( ((1, 0), i) \) is equivalent to \( ((1, 0), j) \). The resulting quotient space is called the bouquet of \( k \) circles. A bouquet of two circles is called the figure eight.

**Exercise 23.12.**

1. Draw a universal cover of the figure eight.

2. Use the drawing from (1) to give a geometric argument that the fundamental group of the figure eight is not abelian.

**Exercise 23.13.** Show that no two of the spaces in the following list are homeomorphic: \( [0, 1], \mathbb{R}^1, S^1, \mathbb{R}^2, S^2, P^2, \mathbb{R}^2 \) minus a singleton set, the torus, the figure eight, \( \mathbb{R}^3 \).
Chapter 24

Further Results on Covering Maps

Throughout this section, suppose all topological spaces are path connected and locally path connected.

**Theorem 24.1. General Lifting Lemma**

Let \( p : \tilde{X} \to X \) be a covering map, and let \( \tilde{a} \in p^{-1}(a) \). Let \( f : Y \to X \) be a continuous function with \( f(b) = a \). Then \( f \) has a lift \( \tilde{f} : Y \to \tilde{X} \) such that \( \tilde{f}(b) = \tilde{a} \) if and only if \( f_* (\pi_1(Y, b)) \subseteq p_* (\pi_1(\tilde{X}, \tilde{a})) \). Moreover if \( f \) has a lift \( \tilde{f} \) such that \( \tilde{f}(b) = \tilde{a} \), then \( \tilde{f} \) is unique.

**Definition 24.2.** Two covering maps \( p : \tilde{X} \to X \) and \( q : \tilde{X}' \to X \) are equivalent if there exists a homeomorphism \( h : \tilde{X} \to \tilde{X}' \) such that \( q \circ h = p \).

**Exercise 24.3.** Show that given a space \( X \), the relation defined in 24.2 is in fact an equivalence relation.

**Theorem 24.4.** Let \( p : \tilde{X} \to X \) and \( q : \tilde{X}' \to X \) be covering maps with \( p(\tilde{a}) = q(\tilde{a}') = a \). Then \( p \) and \( q \) are equivalent if and only if \( p_* (\pi_1(\tilde{X}, \tilde{a})) \) and \( q_* (\pi_1(\tilde{X}', \tilde{a}')) \) are conjugate subgroups of \( \pi_1(X, a) \).

**Corollary 24.5.** Given a space \( X \), its universal cover is unique up to homeomorphism.

**Exercise 24.6.** Classify all (equivalence classes of) covering maps of \( S^1 \).

**Definition 24.7.** Let \( p : \tilde{X} \to X \) be a covering map. A deck transformation is a homeomorphism \( h : \tilde{X} \to \tilde{X} \) such that \( p \circ h = p \).

**Theorem 24.8.** Given a covering map \( p : \tilde{X} \to X \), the set of all deck transformations forms a group under the operation of composition.

**Exercise 24.9.** Describe explicitly the group of deck transformations for the universal cover of the torus given in 22.22.

**Theorem 24.10.** Let \( p : \tilde{X} \to X \) be a covering map and let \( \tilde{a}, \tilde{b} \in p^{-1}(a) \). Then there exists a deck transformation \( h : \tilde{X} \to X \) such that \( h(\tilde{a}) = \tilde{b} \) if and only if \( p_* (\pi_1(\tilde{X}, \tilde{a})) = p_* (\pi_1(\tilde{X}, \tilde{b})) \). If such an \( h \) exists, it is unique.
Definition 24.11. Let $p : \tilde{X} \to X$ be a covering map, and let $\tilde{a} \in p^{-1}(a)$. If $p_* (\pi_1(\tilde{X}, \tilde{a}))$ is normal in $\pi_1(X, a)$, then the covering map is said to be regular.

Exercise 24.12. Give an example of a covering map that is not regular.

Theorem 24.13. Let $p : \tilde{X} \to X$ be a regular covering map, and let $\tilde{a} \in p^{-1}(a)$. Then the group of deck transformations is isomorphic to the quotient group $\pi_1(X, a)/p_* (\pi_1(\tilde{X}, \tilde{a})).$
Chapter 25

Supplementary Section A: Images and Inverse Images of Functions

Definition 25.1. Let $f : A \rightarrow B$ be any function between any two sets $A$ and $B$. If $A_0$ is a subset of $A$, we denote by $f(A_0)$ the subset of $B$ that consists of all images of elements of $A_0$ under the function $f$; this set is called the image of $A_0$ under $f$. Formally, $f(A_0) = \{ b \in B \mid b = f(a) \text{ for some } a \in A_0 \}$.

On the other hand, if $B_0$ is a subset of $B$ we denote by $f^{-1}(B_0)$ the set of all elements of $A$ whose images under $f$ lie in $B_0$; it is called the inverse image of $B_0$ under $f$. Formally, $f^{-1}(B_0) = \{ a \in A \mid f(a) \in B_0 \}$.

Exercises 25.2.

1. Let $f : A \rightarrow B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$.
   (a) Show $f^{-1}(f(A_0)) \supseteq A_0$. Show $f^{-1}(f(A_1)) = A_1$ for all $A_1 \subseteq A$ if and only if $f$ is injective.
   (b) Show $f(f^{-1}(B_0)) \subseteq B_0$. Show $f(f^{-1}(B)) = B$ if and only if $f$ is surjective.

2. Let $f : A \rightarrow B$. Let $B_0 \subseteq B$ and $B_1 \subseteq B$. Show the following.
   (a) If $B_0 \subseteq B_1$, then $f^{-1}(B_0) \subseteq f^{-1}(B_1)$.
   (b) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

3. Let $f : A \rightarrow B$. Let $B_\alpha \subseteq B$ for all $\alpha$ in some index set $J$. Show the following.
   (a) $f^{-1}(\bigcup_{\alpha \in J} B_\alpha) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$.
   (b) $f^{-1}(\bigcap_{\alpha \in J} B_\alpha) = \bigcap_{\alpha \in J} f^{-1}(B_\alpha)$.

4. Let $f : A \rightarrow B$. Let $A_0 \subseteq A$ and $A_1 \subseteq A$. Show the following.
(a) If $A_0 \subseteq A_1$, then $f(A_0) \subseteq f(A_1)$.

(b) $f(A_0 - A_1) \supseteq f(A_0) - f(A_1)$; give an example where equality fails.

5. Let $f : A \to B$. Let $A_\alpha \subseteq A$ for all $\alpha$ in some index set $J$. Show the following.

(a) $f(\bigcup_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in J} f(A_\alpha)$.

(b) $f(\bigcap_{\alpha \in J} A_\alpha) \subseteq \bigcap_{\alpha \in J} f(A_\alpha)$; give an example where equality fails.
Chapter 26

Supplementary Section B: Series of Real Numbers

**Definition 26.1.** Let \( \{a_i\} \) be a sequence of real numbers. For each positive integer \( n \) the \( n \)-th partial sum \( s_n \) is defined by \( s_n = \sum_{i=1}^{n} a_i \). If the sequence \( \{s_n\} \) of partial sums converges to \( s \), we say that the series \( \sum_{i=1}^{\infty} a_i \) **converges** and its **sum** is \( s \). If the series \( \sum_{i=1}^{\infty} |a_i| \) converges we say that \( \sum_{i=1}^{\infty} a_i \) **converges absolutely**.

**Exercises 26.2.** Let \( \{a_i\} \) and \( \{b_i\} \) be sequences of real numbers.

1. (a) If \( \sum_{i=1}^{\infty} a_i \) converges and has sum \( s \), and if \( \sum_{i=1}^{\infty} b_i \) converges and has sum \( t \), then \( \sum_{i=1}^{\infty} (a_i + b_i) \) converges and has sum \( s + t \).

   (b) If \( \sum_{i=1}^{n} a_i \) converges and has sum \( s \), and if \( c \) is a constant, then \( \sum_{i=1}^{\infty} ca_i \) converges and has sum \( cs \).

2. If \( \sum_{i=1}^{\infty} a_i \) converges absolutely, then \( \sum_{i=1}^{\infty} a_i \) converges.

3. If \( a_i \) is nonnegative for all \( i \), then \( \sum_{i=1}^{n} a_i \) converges if and only if the sequence of partial sums is bounded.

4. If the series \( \sum_{i=1}^{\infty} a_i \) converges, then the sequence \( \{a_i\} \) converges to 0.