Advanced Calculus with Generalizations: Second Semester

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0.1 Introduction - Advanced Calculus

Advanced Calculus (or Mathematical Analysis) is one of the most important classes one takes as a math student. Unlike a class in, say, abstract algebra, where the main concepts (rings, groups, fields) are new, in this class we will look at topics you should be familiar with from your earlier calculus sequence (limits, differentiation, integration, and sequences & series). That is the good news. Since you are familiar with all these topics, what this class then is about is an increase in rigor. This is not about determining the integral of $f$, but proving what type of $f$ is an integrable function. We will also, as is expected in higher-level mathematics, look at some generalizations of these ideas. This serves as an introduction to what many mathematicians do when researching a topic.

Now, a word about prerequisites. Most courses of this type require a C or better in a course on proof techniques. Let us be blunt here: A grade of C or better does not mean that you have some vague recollection of a course that had a lot of proofs in it. It means that you learned the material and are capable of both creating your own proofs and following proofs of others. The types of proof techniques you should be familiar with include, but are not limited to, induction, direct proof, proof by contradiction, and proof by exhaustion.

This is an extremely important course as it develops the ability to apply logic, which is what makes the math major so desirable to employers (see www.maa.org/careers/ for more details). It takes hard work, but it is worth it.

Now, all that said, let me add that mathematics is one of the most creative disciplines out there. There is very little difference between the imagination, hard work, and focus needed to paint a still life and to determine a proof. Note that I did not say paint the Mona Lisa and prove Fermat’s Last Theorem. Just as you don’t have to paint masterpieces in order to enjoy painting, you do not have to solve the greatest open problems in mathematics to enjoy the thrill and pride associated with developing a proof. Learning to find proofs and write proofs is a skill that comes with practice; a lot of prac-
tice. If you start to feel discouraged and like you are flailing and lost, I suggest you think about the man who did solve Fermat’s Last Theorem and his thoughts on doing math.

I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it’s completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it’s all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they’re momentary, sometimes over a period of a day or two, they are the culmination of, and couldn’t exist without, the many months of stumbling around in the dark that proceed them.

Andrew Wiles, Princeton University

Get ready and get excited to take a journey deeper into analysis than you have taken before. And enjoy the trip!
0.2 R.L. Moore and his Method

That student is taught the best who is told the least.

R.L. Moore, 1882-1974

[Dr. Moore] told us early on that he had no use for the university guidelines stating that we should expect three hours of outside class work for each hour in the classroom. He said he wanted us to think about his class all day, every day, to go to bed thinking about it, to wake up in the night thinking about it, to get up the next morning thinking about it, to think about it walking to class, to think about it while we were eating. If we weren’t prepared to do that, he didn’t want us in his class. It was also quickly evident that he meant what he said....

John Green, PhD, University of Texas, 1968, under R. L. Moore

The core of any course using the Moore Method (a type of Inquiry-Based Learning) is the understanding that people learn best by doing the work, not by being told the results. Moore developed his method in 1911 while teaching at the University of Pennsylvania and then took it with him to the University of Texas where he worked from 1920 until his retirement in 1969. The mathematics building at UT is named after him.

To begin with, a true Moore Method course has no book. Instead, the “book” for the course is written by the students as the semester/year goes on. That is, at the end of this course the collection of definitions, axioms, and results you have will be enough for a text. Each and every student is expected to do his/her own work. The use of outside sources (including, but not limited to, books, the internet, tutors, friends, classmates, and professors who are not me) is strictly forbidden.

You will be given handouts which contain Definitions, Axioms, Problems, Exercises, and a few Theorems. You are to provide proofs for anything labeled Theorem (which by virtue of being called a theorem is true). Any problem will start with the directive “Prove or Disprove.” It is up to the student to determine the truthfulness of the statements and, if it is true, give a proof; or if it is false, give a counterexample with explanation. Exercises can be presented at the board for a grade, but not turned in. The Exercises are to help illustrate definitions and topics. Similarly, there are Remarks throughout to explain concepts and introduce new discussions.

In addition to submitting written work, students will be presenting at the board. There will be little, if any, lecture in this class. On class days, you will be called on to present solutions. You may choose to show your work on any problem not yet presented in class. If you are not the first person chosen on a given day, you will not have your choice of all assigned problems. This
means that you may not get to present your first choice problem, so you should be prepared with solutions to more than one problem.

During student presentations, the rest of the class is encouraged (expected) to ask questions, and to think critically about the solution presented by their classmates. The students in the class have the job of determining the validity of arguments presented, and the instructor will occasionally allow incorrect solutions to stand in class. These incorrectly presented problems may appear on tests. While “Should that two be a three?” is a question, you should also ask “Can you explain how you went from Step 2 to Step 3”. Don’t worry, if you have that question, so do others and the presenter should be able to answer that question and others. Every once in awhile there are questions someone doesn’t know how to answer. It happens to everyone, so the presenter should not feel bad or embarrassed. The response of “I’m not sure, I’ll have to get back to you,” is absolutely fine, but the speaker is then responsible for finding out the answer and showing it to the class. It will not take you long to figure out how to be the speaker and how to be an active audience member. Then it’s just a matter of enjoying the learning.
A quote from Paul Halmos¹:

“Don’t just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?”

and another

“Mathematics is not a deductive science – that’s a cliché. When you try to prove a theorem, you don’t just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.”

and a final one

“It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.”

¹If you do not know who Paul Halmos was, you may find it worthwhile to look him up.
Note: As with any written work, this is not a solitary effort, but a collaboration between the author and references, colleagues, & the past. These notes were inspired by the notes posted on the JIBL site from Scott Beaver of Western Oregon State University. They were also influenced by Jacqueline Jensen-Vallin and Ron Taylor and their efforts to help me convert to Inquiry-Based Learning. That said, all typos and other manners of errors are mine and mine alone. If the reader finds one (or many) mistake(s), please bring it to my attention, especially if the mistake causes a problem to be impossible. In addition, if you have questions on how to use this material, please do not hesitate to contact me at robert.vallin@sru.edu
Chapter 1

Differentiation

Definition 1. Suppose \( f \) is a function defined on the interval \( I = [a, b] \). We say \( f \) is differentiable at \( x_0 \in (a, b) \) if

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]

exists and is finite. If such a value exists, it is called the derivative of \( f \) at \( x_0 \) and is denoted by \( f'(x_0) \). If \( f \) is differentiable at each \( x_0 \in (a, b) \), then we say \( f \) is differentiable on \( (a, b) \).

Remark 2. 1. The expression

\[
\frac{f(x) - f(x_0)}{x - x_0}
\]

is called a difference quotient.

2. Another difference quotient and limit used in the definition of derivative is

\[
\frac{f(x_0 + h) - f(x_0)}{h}
\]

In this case, the derivative could be defined by

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]

3. We are making the assumption that any given \( x_0 \) is an interior point of an interval \([a, b] \subset \mathcal{D}(f)\). This is to avoid needing one-sided limits.

4. The reader may also be familiar with Leibniz (\( \frac{d}{dx} \)) notation. It is equivalent to writing with primes and we believe it is fair to assume your instructor and classmates can follow you, if you are more comfortable with that notation.

Exercise 3. Show that each of the following are differentiable at the given point.
1. \( f(x) = x^5, \ x_0 = 2 \).
2. \( g(x) = \sqrt{x}, \ x_0 = 1 \).
3. \( h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \ x_0 = 0 \).

**Prove or Disprove 4.** The value of \( f'(x_0) \) is unique.

**Prove or Disprove 5.** The function \( f \) is differentiable at \( x_0 \) if and only if \( f \) is continuous at \( x_0 \).

**Prove or Disprove 6.** If \( f \) is differentiable at \( x_0 \), then the function \( cf \) is differentiable at \( x_0 \) for any constant \( c \).

**Prove or Disprove 7.** If \( f \) and \( g \) are both differentiable at \( x_0 \), then their sum, \( (f+g) \), is differentiable at \( x_0 \).

**Remark 8.** A third way to define the derivative is via sequences. This can be helpful in proving conjectures. We say \( f \) is differentiable at \( x_0 \) if for every sequence \( (x_n) \subset I \setminus \{x_0\} \) with \( x_n \to x_0 \) we have

\[
\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}
\]

exists, is finite, and has the same value.

**Theorem 9.** (Product and Quotient Rules) If \( f \) and \( g \) are both differentiable at \( x_0 \), then so are \( fg \) and \( f/g \) with

\[
(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)
\]

and

\[
(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}, \text{ assuming } g(x_0) \neq 0
\]

**Theorem 10.** (Chain Rule) Let \( f \) and \( g \) be functions with the domain of \( f \) containing the range of \( g \). If \( g \) is differentiable at \( x_0 \) and \( f \) is differentiable at \( g(x_0) \), then \( f \circ g \) is differentiable at \( x_0 \) and

\[
(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0).
\]

**Definition 11.** Suppose \( f \) is a function on an open interval \( (a,b) \). Then \( f \) has a local maximum (local minimum) at \( x_0 \in (a,b) \) if there exists \( \delta > 0 \) such that for all \( y \in (a,b) \cap (x_0 - \delta, x_0 + \delta) \), \( f(y) \leq f(x_0) \) (\( f(y) \geq f(x_0) \)).

**Theorem 12.** Suppose \( f \) is a function on \( (a,b) \). If \( f \) has a local maximum or a local minimum at \( x_0 \in (a,b) \) and if \( f \) is differentiable at \( x_0 \), then \( f'(x_0) = 0 \).

**Theorem 13.** (Rolle’s Theorem) If \( f \) is continuous on \([a,b]\), differentiable on \((a,b)\), and \( f(a) = f(b) \), then there exists an \( x_0 \in (a,b) \) such that \( f'(x_0) = 0 \).
Exercise 14. Show that if the hypothesis “differentiable on \((a, b)\)” is dropped from Rolle’s Theorem, the conclusion is false.

Theorem 15. (Mean Value Theorem) If \(f\) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists an \(x_0 \in (a, b)\) such that
\[
f'(x_0) = \frac{f(b) - f(a)}{b - a}.
\]

Exercise 16. Give an example to show that if \(f\) has even one point of discontinuity in \((a, b)\), then the conclusion of the Mean Value Theorem may not be true.

Exercise 17. Give an example to show that if \(f\) has even one point of non-differentiability in \((a, b)\), then the conclusion of the Mean Value Theorem may not be true.

Prove or Disprove 18. If \(f\) is differentiable on \((a, b)\) with \(f'(x) = 0\) for all \(x \in (a, b)\), then \(f\) is constant on \((a, b)\).

Definition 19. A function \(f\) is increasing (decreasing) on the interval \((a, b)\) if for any \(x, y \in (a, b)\) with \(x < y\) we have \(f(x) \leq f(y)\) (\(f(x) \geq f(y)\)). If the \(\leq\) in the conclusion is replaced with \(<\) we say \(f\) is strictly increasing (strictly decreasing).

Prove or Disprove 20. If \(f'(x) \geq 0\) for all \(x \in (a, b)\), then \(f\) is increasing on \((a, b)\).

Definition 21. Let \(f\) be a function. We say \(f\) is bounded on the interval \((a, b)\) (\(a\) and \(b\) can be finite or infinite) if there exists real numbers \(m, M\) such that \(m \leq f(x) \leq M\) for all \(x \in (a, b)\).

Definition 22. Let \(D \subseteq \mathbb{R}\) and \(f : D \rightarrow \mathbb{R}\). We say \(f\) is uniformly continuous if for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that for every \(x, y \in D\) with \(|x - y| < \delta\) we have
\[
|f(x) - f(y)| < \varepsilon.
\]

Theorem 23. Let \(f\) be differentiable on the interval \((a, b)\) with \(f'\) bounded. Then \(f\) is uniformly continuous on \((a, b)\).

Exercise 24. Show that \(f(x) = \sin(x)\) is a uniformly continuous function on \(\mathbb{R}\).

Exercise 25. Construct a differentiable, uniformly continuous function \(f\) such that \(f'\) is unbounded. This shows that it is not necessary that \(f'\) be bounded in order that \(f\) be uniformly continuous.

Remark 26. Recall the following two definitions from the first semester.

Let \(X \subseteq \mathbb{R}\) and let \(\{f_n\}\) and \(f\) be real functions with domain \(X\). We say \(f_n\) converges pointwise to \(f\) if for each \(x_0 \in X\) we have
\[
\lim_{n \to \infty} f_n(x_0) = f(x_0);
\]
that is, for every $x_0 \in X$ and every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n > N$

$$|f_n(x_0) - f(x_0)| < \varepsilon.$$ 

We write this as $f_n \xrightarrow{p} f$. Additionally, we say $f_n$ converges uniformly to $f$ if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n > N$ we have

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in X$. We denote this by

$$f_n \xrightarrow{u} f.$$

**Prove or Disprove 27.** Let $f_n : (a, b) \to \mathbb{R}$ be a sequence of functions. If each $f_n$ is differentiable on $(a, b)$ and $f_n \xrightarrow{p} f$, then $f$ is differentiable on $(a, b)$.

**Definition 28.** A function $f$ is said to be continuously differentiable on an interval $I$ if it is differentiable on $I$ and the function $f'$ is continuous on $I$.

**Exercise 29.** Find examples of functions $f$ and $g$ where $f$ is continuously differentiable and $g$ is differentiable, but not continuously so.

**Theorem 30.** (This is an exception to the rule. You need not prove this. It is included for completeness.) Let $\{f_n\}$ be a sequence of continuously differentiable functions defined on an interval $I$ and let $f$ and $g$ be functions defined on $I$ where $f_n \xrightarrow{p} f$ on $I$. If $f_n' \xrightarrow{u} g$ on $I$, then $f$ is differentiable on $I$ and $f' = g$.

**Definition 31.** A function $f$ is said to be in Baire Class One if it is the pointwise limit of continuous functions, $f_n$, but is not itself continuous; that is, a discontinuous $f$ is Baire Class One if there exists $f_n$, each of which is continuous, so that

$$f_n \xrightarrow{p} f.$$

**Exercise 32.** Find an example of a function which has a point of discontinuity, but is in Baire Class One.

**Prove or Disprove 33.** A function can have infinitely many discontinuities but be in Baire Class One.

**Theorem 34.** If $f$ is a derivative (that is, there exists a differentiable function $g$ such that $g'(x) = f(x)$ for all $x$), then $f$ is in Baire Class One.

**Theorem 35.** Let $f$ be differentiable on the interval $I$. Suppose $a, b \in I$ with $a < b$ and $f(a) \neq f(b)$. Let $z$ be any number between $f'(a)$ and $f'(b)$. Then there exists a number $c$ between $a$ and $b$ so that $f'(c) = z$. 

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**Remark 36.** This shows that every derivative has the Intermediate Value (Darboux) Property\(^1\).

**Prove or Disprove 37.** There exists a function \( f \) on \( \mathbb{R} \) such that
\[
f'(x) = \chi_{\{0\}} = \begin{cases} 
0 & x \neq 0 \\
1 & x = 0
\end{cases},
\]
the characteristic function of zero.

**Definition 38.** Let \( f : \mathbb{R} \to \mathbb{R} \). We say that \( c \) is a sequential derivate of \( f \) at \( x_0 \) if there exists sequence \( \{h_n\} \) converging to zero so that
\[
\lim_{n \to \infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n} = c.
\]

**Prove or Disprove 39.** If \( f \) is differentiable at \( x_0 \), then \( f \) has a unique sequential derivate at \( x_0 \).

**Exercise 40.** Find an example of a function which is not differentiable at a point \( x_0 \), but has sequential derivate(s) at \( x_0 \).

**Exercise 41.** Find an example of a function \( f \) that has more than one finite sequential derivate at \( x_0 \) and is continuous at \( x_0 \)?

**Prove or Disprove 42.** A function can have infinitely many sequential derivate values.

**Theorem 43.** Suppose that the functions \( f \) and \( g \) are continuous on some interval \([a, b]\) containing the point \( c \) and differentiable on \((a, b)\) (except possibly at \( c \) itself). Furthermore, suppose that
\[
1. \quad \lim_{x \to c} f(x) = 0,
2. \quad \lim_{x \to c} g(x) = 0,
3. \quad \text{for every } x \in (a, b) \setminus \{c\}, \ g'(x) \neq 0, \text{ and}
4. \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists.}
\]
Then
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.
\]

**Remark 44.** This is the first version of L'Hopital's Rule that you learn in Calculus I. There are other indeterminate forms (what are they?), but this text will not go into them.

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\(^1\)Let \( f : [a, b] \to \mathbb{R} \). We say \( f \) has the IVP if for any \( v \), is a number between \( f(a) \) and \( f(b) \), there is a point \( c \in (a, b) \) such that \( f(c) = v \).
Chapter 2

Integration

Definition 45. A partition $P$ of $[a,b]$ is a finite set of $n+1$ elements of $[a,b]$ containing at least one element of $(a,b)$ with

$$P = \{t_0 = a, t_1, t_2, \ldots, t_{n-1}, t_n = b\}$$

where $t_{k-1} < t_k$ for $k = 1, 2, \ldots, n$. If $k$ is a natural number no greater than $n$, we will denote the $k^{th}$ subinterval $[t_{k-1}, t_k]$ by $I_k$ and let $\Delta I_k = t_k - t_{k-1}$.

Define the mesh of $P$, written as $\text{mesh}(P)$, as

$$\text{mesh}(P) = \max_k \{\Delta I_k\}.$$ 

Definition 46. Let $f$ be a function defined on $[a,b]$, and let $P$ be a partition of $[a,b]$. Let

$$m_k(f) = \inf_{x \in I_k} \{f(x)\}$$

$$M_k(f) = \sup_{x \in I_k} \{f(x)\}.$$ 

Then we define the Lower and Upper Type I sums of $f$ on $[a,b]$ relative to $P$ by

$$L(f, P) = \sum_{k=1}^{n} m_k(f) \Delta(I_k)$$

$$U(f, P) = \sum_{k=1}^{n} M_k(f) \Delta(I_k).$$

Prove or Disprove 47. Let $f$ be defined on $[a,b]$ and let $P$ be a partition of $[a,b]$. Then $L(f, P) \leq U(f, P)$.

Definition 48. A partition $Q$ is a refinement of the partition $P$ if $P \subseteq Q$.

Exercise 49. Given two partitions $P_0$ and $P_1$, find a partition that is a refinement of both.
Prove or Disprove 50. Let $f$ be defined on $[a, b]$ and let $P, Q$ be partitions of $[a, b]$ with $Q$ a refinement of $P$. Then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Exercise 51. Let $f$ be the characteristic function of the rationals on $[0, 1]$; that is

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Determine $U(f, P)$ and $L(f, P)$ where $P$ is a partition of $[0, 1]$.

Exercise 52. Let $g : [0, 2] \to \mathbb{R}$ be the function

$$g(x) = \begin{cases} 2 & \text{if } x > 1 \\ 0 & \text{if } x \leq 1 \end{cases}.$$ 

Let $P$ be a partition of $[0, 2]$ with $x_k < 1 < x_{k+1}$ for some index $k$. Determine $U(f, P)$ and $L(f, P)$.

Definition 53. Let $f$ be defined on $[a, b]$. Then we define the Upper and Lower Type I integrals of $f$ by

$$U(f) = \inf \{ U(f, P) : P \text{ is a partition of } [a, b] \}$$

$$L(f) = \sup \{ L(f, P) : P \text{ is a partition of } [a, b] \}.$$ 

If $U(f) = L(f)$, then we say $f$ is Type I integrable on the interval and denote this number as $\int_a^b f$.

Exercise 54. Show that characteristic function of the rationals on $[0, 1]$ is not Type I integrable on $[0, 1]$.

Exercise 55. Show that $f(x) = x$ is Type I integrable on $[0, b]$ where $b > 0$.

Prove or Disprove 56. If $f$ is a bounded function on $I$, then

$$L(f) \leq U(f).$$

Theorem 57. The function $f$ is Type I integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition $P$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Definition 58. A Type II sum associated with $f$ on $[a, b]$ with respect to $P$ is a sum of the form

$$T(f, n, P) = \sum_{k=1}^{n} f(x_k)\Delta(I_k)$$

where $x_k \in I_k$ for $k = 1, 2, \ldots, n$. We say that $f$ is Type II integrable on $[a, b]$ if and only if there exists a real number $T(f)$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|T(f, n, P) - T(f)| < \varepsilon$$

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for every Type II sum associated with \( f \) on \([a, b]\) with respect to \( P \), where \( \text{mesh}(P) < \delta \). If \( f \) is Type II integrable on \([a, b]\), we denote \( T(f) \) as the Type II integral of \( f \).

**Theorem 59.** The function \( f \) is Type II integrable on \([a, b]\) if and only if it is Type I integrable on \([a, b]\).

**Remark 60.** Since Type I and Type II integrals are equivalent and we will denote both by \( \int_a^b f \).

**Prove or Disprove 61.** If \( f \) is integrable on \([a, b]\), then \( f \) is integrable on \([c, d]\) where \( a < c < d < b \).

**Prove or Disprove 62.** If \( f \) is monotone on \([a, b]\), then \( f \) is integrable on \([a, b]\).

**Prove or Disprove 63.** If \( f \) is continuous on \([a, b]\), then \( f \) is integrable on \([a, b]\).

**Exercise 64.** Show the differences in Prove or Disprove 62 and 63. Find a monotone function on \([a, b]\) that is not continuous and a continuous function on \([a, b]\) that is not monotone.

**Prove or Disprove 65.** If \( f \) and \( g \) are integrable on \([a, b]\), then their sum \( f + g \) is also integrable on \([a, b]\).

**Prove or Disprove 66.** If \( f \) is integrable on \([a, b]\) and \( c \) is any real number, then \( cf \) is integrable on \([a, b]\).

**Prove or Disprove 67.** If \( f > 0 \) on \([a, b]\) and integrable, then \( \int_a^b f > 0 \).

**Prove or Disprove 68.** If \( f > g \) on \([a, b]\) (i.e. \( f(x) > g(x) \) for all \( x \in [a, b] \)) and \( f, g \) are both integrable, then \( \int_a^b f > \int_a^b g \).

**Prove or Disprove 69.** If \( f \) is integrable on \([a, b]\), then \(|f|\) is integrable on \([a, b]\) and \( |\int_a^b f| \leq \int_a^b |f| \).

**Prove or Disprove 70.** If \(|f|\) is integrable on \([a, b]\), then \( f \) is integrable on \([a, b]\).

**Theorem 71.** If \( f \) is bounded and integrable on \([a, b]\), then \( f^2 \) is integrable on \([a, b]\).

**Prove or Disprove 72.** If \( f \) and \( g \) are integrable on \([a, b]\), then their product \( fg \) is integrable on \([a, b]\).

**Prove or Disprove 73.** If \( f \) and \( g \) are integrable on \([a, b]\), then \( \max\{f, g\} \) and \( \min\{f, g\} \) are integrable on \([a, b]\).

**Prove or Disprove 74.** If \( f \) is defined on \([a, b]\) and \( c \in (a, b) \) and \( f \) is integrable on both \([a, c]\) and \([c, b]\), then \( f \) is integrable on \([a, b]\) and

\[
\int_a^b f = \int_a^c f + \int_c^b f.
\]
Prove or Disprove 75. If \( f \) has finitely many discontinuities on \([a, b]\), then \( f \) is integrable on \([a, b]\).

Theorem 76. Let \( f_n \rightarrow f \) on \([a, b]\) and suppose each \( f_n \) is integrable. Then \( f \) is integrable and
\[
\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.
\]

Exercise 77. Give an example to show that uniform convergence cannot be replaced by pointwise convergence in the theorem above.

Theorem 78. (The Mean Value Theorem for Integrals) If \( f \) is continuous on \([a, b]\) then there exists a \( c \in (a, b) \) such that
\[
f(c) = \frac{1}{b-a} \int_a^b f.
\]
(Also, draw a picture to give this a geometric meaning.)

Theorem 79. (The Fundamental Theorem of Calculus I) Let \( f \) be continuous on \( I \) and differentiable on \((a, b)\). Then if \( f' \) is integrable on \([a, b]\) we have
\[
\int_a^b f' = f(b) - f(a).
\]

Theorem 80. (The Fundamental Theorem of Calculus II) Let \( f \) be integrable on \([a, b]\). Then if \( x \in [a, b] \), define
\[
F(x) = \int_a^x f.
\]
Then \( F \) is continuous on \([a, b]\). If \( f \) is continuous at \( x_0 \in (a, b) \), then \( F \) is differentiable at \( x_0 \) and \( F'(x_0) = f(x_0) \).

Definition 81. The sequence of functions \( f_n \) on \([a, b]\) converges discretely to \( f \) if for every \( x \in [a, b] \) there exists an \( N \in \mathbb{N} \) such that for \( n > N \) we have \( f_n(x) = f(x) \). We write this as \( f_n \rightarrow f \).

Prove or Disprove 82. If \( f_n \) is integrable on \([a, b]\) and \( f_n \rightarrow f \), then \( f \) is integrable on \([a, b]\).

Definition 83. Let \( P \) be a partition of \([a, b]\) and \( f \) a bounded function defined on \([a, b]\). Let \( g \) be an increasing function from \([a, b]\) into \( \mathbb{R} \). Then define \( \Delta g(I_k) \) by
\[
\Delta g(I_k) = g(t_k) - g(t_{k-1})
\]
for \( i = 1, 2, 3, \ldots, n \). We define the Lower and Upper Type I sums of \( f \) on \([a, b]\) relative to \( P \) with respect to \( g \) by
\[
L(f, g, P) = \sum_{k=1}^n m_k(f) \Delta g(I_k)
\]
and

\[ U(f, g, P) = \sum_{k=1}^{n} M_k(f) \Delta g(I_k). \]

**Exercise 84.** Suppose \( P \) is a partition of the unit interval, \( f \) is the characteristic function of the rationals, and \( g(x) = x^2 \). Find the values of \( L(f, g, P) \) and \( U(f, g, P) \).

**Definition 85.** Let \( P \) be a partition of \([a, b]\) as earlier. Let \( g \) be a strictly increasing function from \([a, b]\) into \( \mathbb{R} \). The Upper and Lower Riemann-Stieltjes integrals of \( f \) with respect to \( g \) over \([a, b]\) are defined as

\[ \int_{a}^{b} f \, dg = \inf\{ U(f, g, P) : P \text{ is a partition of } [a, b] \} \]

and

\[ \int_{a}^{b} f \, dg = \sup\{ L(f, g, P) : P \text{ is a partition of } [a, b] \} \]

**Remark 86.** When \( \int_{a}^{b} f \, dg = \int_{a}^{b} f \, dg \), we say \( f \) is Riemann-Stieltjes integrable with respect to \( g \) and denote this number as \( \int_{a}^{b} f \, dg \). We refer to the set of Riemann-Stieltjes integrable functions with respect to \( g \) by \( R_g \) or, when the interval is not understood, \( R_g[a, b] \).

**Prove or Disprove 87.** Let \( f \) be in \( R_g \) and let \( g_1, g_2 \) be increasing functions on \([a, b]\). If \( g_1 \leq g_2 \), then \( \int_{a}^{b} f \, dg_1 \leq \int_{a}^{b} f \, dg_2 \).

**Prove or Disprove 88.** Let \( f \) be in \( R_g \) and \( g \) an increasing function on \([a, b]\). Then for any constant \( c \), \( \int_{a}^{b} f \, d(cg) = c \int_{a}^{b} f \, dg \).

**Exercise 89.** Let \( f(x) = c \) on \([a, b]\). Determine the value of \( \int_{a}^{b} f \, dg \).

**Exercise 90.** Let \( f \in R_g[0, 3] \) where \( g \) is defined by

\[ g(x) = \begin{cases} 
2 & \text{if } x > 1 \\
0 & \text{if } x \leq 1
\end{cases} \]

Determine the value of \( \int_{a}^{b} f \, dg \).
Chapter 3

Series and Series of Functions

A Bit of Review.

Definition 91. A sequence is a function $f$ with domain $\mathbb{N}$ and range a subset of $\mathbb{R}$.

Definition 92. The sequence $(x_n)$ converges to the real number $L$, written

$$\lim_{n \to \infty} x_n = L,$$

or $x_n \to L$, if for every $\varepsilon > 0$ there exists a natural number $N$ such that for all $n > N$ we have

$$|x_n - L| < \varepsilon.$$

Definition 93. A sequence which does not converge is said to diverge.

This can happen in several ways. Take a moment to consider how this can happen.

Exercise 94. Determine the negation of the definition of convergence for a formal definition we use when the limit does not exist.

Remark 95. There is also the possibility that the $\lim_{n \to \infty} x_n$ exists, but it is not finite.

Definition 96. We say $\lim_{n \to \infty} x_n = \infty$ if for every $M \in \mathbb{R}$ there exists a natural number $N$ such that if $n > N$, then $x_n > M$.

Exercise 97. Formulate the analogous definition for $\lim_{n \to \infty} x_n = -\infty$.

Definition 98. We say a sequence $x$ is bounded if there exists numbers $m$ and $M$ such that for all $n \in \mathbb{N}$

$$m \leq x_n \leq M.$$

Definition 99. Let $x$ be a bounded sequence. Define sequences $(u_n)$ and $(v_n)$ by, for all $N \in \mathbb{N}$

$$u_N = \inf\{x_n : n > N\}$$
and 
\[ v_N = \sup\{x_n : n > N\}. \]

**Example 100.** Let \( x = \{1 + 1/2, 1/2, 1 + 1/3, 1/3, 1 + 1/4, 1/4, \ldots\}. \) Then \( u_N = 0 \) for all \( N \) while \( v_0 = 1 + 1/2, v_1 = 1 + 1/3, v_2 = 1 + 1/3, \) and \( v_3 = 1 + 1/4. \)

**Definition 101.** Let \( x \) be a bounded sequence and let \((u_N)\) and \((v_N)\) be defined as above. Define the limit inferior and limit superior of \( x \), denoted by \( \liminf x_n \) and \( \limsup x_n \), respectively, as
\[ \liminf x_n = \lim_{N \to \infty} u_N \]
and
\[ \limsup x_n = \lim_{N \to \infty} v_N. \]

**Example 102.** So for \( x = \{1 + 1/2, 1/2, 1 + 1/3, 1/3, 1 + 1/4, 1/4, \ldots\} \), \( \liminf x_n = 0 \) while \( \limsup x_n = 1 \).

**Definition 103.** A sequence \((x_n)\) of real numbers is called a Cauchy sequence if for every \( \varepsilon > 0 \) there exists a natural number \( N \) so that if \( n > m \geq N \), then
\[ |x_n - x_m| < \varepsilon. \]

**Definition 104.** A subsequence is a (infinite) sequence whose terms consist of some of the terms of \((x_n)\) taken in order. The usual notation is to write our subsequence as
\[ (x_{n_k}) = x_{n_1}, x_{n_2}, x_{n_3}, \ldots \]
where \( n_1 \) denotes the first term taken from \((x_n)\), \( n_2 \) denotes the second term taken from \((x_n)\), et cetera. Notice it must be true that for the value of \( n_k \) we must have \( n_k \geq k \).

**On to New Stuff.**

**Definition 105.** The symbol \( \sum_{n=m}^{\infty} a_n \) means \( a_m + a_{m+1} + a_{m+2} + \cdots \). To give this some meaning, we change this into a sequence and consider
\[ s_k = a_m + a_{m+1} + \cdots + a_k = \sum_{n=m}^{k} a_n \]
for each \( k \geq m \). The elements of \( \{s_k\} \) form the sequence of partial sums. The infinite series \( \sum a_n \) converges if and only if the sequence of partial sums \( \{s_k\} \) converges to the real number \( S \). Then we say
\[ \sum_{n=m}^{\infty} a_n = \sum a_n = S. \]

If a series does not converge, then we say it diverges.
**Theorem 106. Geometric Series**

If $|r| < 1$ and $a$ is any real number, then the series

$$
\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots
$$

converges.

**Definition 107. p-Series**

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p$ is fixed.

**Theorem 108.** This series converges if and only if $p > 1$.

This type of series is notorious for being easy to determine convergence/divergence, but difficult for finding the exact sum. For example (without proof), the so-called Basel Problem was to find the exact value of

$$
\sum_{n=1}^{\infty} \frac{1}{n^2}.
$$

It was solved by Leonhard Euler in 1735 who showed

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$

**Prove or Disprove 109.** The series $\sum a_n$ converges if and only if $a_n \to 0$.

To determine convergence (but not the sum’s value) we have many different tests.

**Theorem 110. Comparison Test**

Let $\sum a_n$ be a series where $a_n \geq 0$ for all $n$.

1. If $\sum a_n$ converges and $|b_n| \leq a_n$, for all $n$, then $\sum b_n$ converges.

2. If $\sum a_n = \infty$ and $b_n \geq a_n$, for all $n$, then $\sum b_n = \infty$.

**Prove or Disprove 111.** If $\sum |a_n|$ converges, then $\sum a_n$ converges.

**Definition 112.** A series is said to converge absolutely if $\sum |a_n|$ is a convergent series.

**Prove or Disprove 113.** Let $\{a_n\}$ be a sequence of nonzero real numbers such that $a_{n+1}/a_n$ is a constant. Then $\sum a_n$ is a geometric series.

**Theorem 114. Ratio Test**

A series $\sum a_n$ of nonzero terms

1. converges absolutely if $\limsup |a_{n+1}/a_n| < 1$,

2. diverges if $\liminf |a_{n+1}/a_n| > 1$.  

3. Otherwise, if \( \lim \inf |a_{n+1}/a_n| \leq 1 \leq \lim \sup |a_{n+1}/a_n| \) the test fails and the series may converge or diverge.

**Exercise 115.** Find examples of two series, \( \sum x_n \) and \( \sum y_n \) such that \( \sum x_n \) is convergent, \( \sum y_n \) is divergent, and The Ratio Test fails for both of them.

**Theorem 116. Root Test**

Let \( \sum a_n \) be a series and let \( \alpha = \limsup |a_n|^{1/n} \). The series

1. converges absolutely if \( \alpha < 1 \),
2. diverges if \( \alpha > 1 \).
3. Otherwise \( \alpha = 1 \) and the test fails.

**Exercise 117.** Let \( a_n \) be a sequence defined by

\[
a_n = \begin{cases} 
2r^n & \text{n odd} \\
r^n & \text{n even}
\end{cases}
\]

where \( r \) is a fixed, nonzero number. Show that this is an example of when the Ratio Test fails, but the Root Test does not.

**Theorem 118.** Let \( \sum_{n=1}^{\infty} a_n \) be a series with non-negative terms and let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function such that \( f(n) = a_n \) for all \( n \). Then the series converges if and only if

\[
\lim_{t \to \infty} \int_1^t f(x) \, dx
\]

exists and is finite.

**Remark 119.** Theorem 118 is the easy way to prove convergence for \( p=1 \)-series.

**Definition 120.** Let \( a_n > 0 \) for all \( n \). An alternating series is a series of the form

\[
\sum_{n=1}^{\infty} (-1)^n a_n \text{ or } \sum_{n=1}^{\infty} (-1)^{n+1} a_n.
\]

**Theorem 121.** An alternating series converges if and only if \( a_n \to 0 \).

**Definition 122.** Let \( f : \mathbb{R} \to \mathbb{R} \). We say that \( f \) is convergence preserving if for every convergent series \( \sum a_n \), the series

\[\sum f(a_n)\]

also converges.

**Prove or Disprove 123.** If \( f \) is convergence preserving, then \( \lim_{x \to 0} f(x) = 0 \).
Prove or Disprove 124. If \( f \) is convergence preserving, then \( f \) is continuous at \( x_0 = 0 \).

Prove or Disprove 125. If \( f \) is convergence preserving, then \( f \) is differentiable at \( x_0 = 0 \).

Definition 126. Given a sequence \( \{a_n\} \) of real numbers, the series \( \sum_{n=0}^{\infty} a_n x^n \) is called a power series. For any power series, one of the following is true:

1. The power series converges for all \( x \in \mathbb{R} \);
2. The power series converges for only \( x = 0 \);
3. The power series converges for all \( x \) in some bounded interval centered at 0.

If instead of \( x^n \) we have \((x - c)^n\) this is referred to as a power series centered at \( c \).

Theorem 127. Let \( \sum a_n (x - c)^n \) be a power series. Define \( s \) as

\[
s = \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

and let

\[
r = \begin{cases} 
0 & \text{if } s = \infty \\
1/s & \text{if } 0 < s < \infty \\
\infty & \text{if } s = 0 
\end{cases}
\]

Then \( \sum a_n (x - c)^n \) converges absolutely for all values of \( x \) that satisfy \( |x - c| < r \) and diverges for all values of \( x \) that satisfy \( |x - c| > r \). The number \( r \) is called the radius of convergence. The values of \( x \) for which the series converge are called the interval of convergence. From the theorem the interval contains \((c - r, c + r)\). The endpoints of the interval must be checked individually.

Exercise 128. Find the interval and radius of convergence for

\[
\sum_{n=1}^{\infty} n^n (x + 4)^n.
\]

Exercise 129. Find the interval and radius of convergence for

\[
\sum_{n=0}^{\infty} \frac{10^n}{n!} (x - 3)^n.
\]

Exercise 130. Find the interval and radius of convergence for

\[
\sum_{n=1}^{\infty} \frac{(x - 1)^n}{n3^n}.
\]
Definition 131. Suppose \( \sum_{k=0}^{\infty} a_k (x-a)^k \) has radius of convergence \( r > 0 \). Let \( f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \) for each \( x \in (x-r, x+r) \). Then we say \( f \) is represented by the power series \( \sum_{k=0}^{\infty} a_k (x-a)^k \) or that \( \sum_{k=0}^{\infty} a_k (x-a)^k \) is the power series representation for \( f \).

Remark 132. We now look at going in the other direction; that is, if we start with a function \( f \), can we always find a series representation for \( f \), and if not, what conditions on the function are necessary for the series to exist?

Definition 133. Let \( f \) have derivatives of all orders on an interval \( I \) and let \( c \) be an interior point of \( I \). The Taylor Series for \( f \) centered at \( c \) is

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k.
\]

When \( c = 0 \) this is called the Maclaurin Series for \( f \).

Exercise 134. Find the Maclaurin Series for each of the following and find the interval of convergence:

1. \( \frac{1}{1-x} \)
2. \( e^x \)
3. \( \sin(x), \cos(x) \)

Remark 135. For a Taylor (Maclaurin) Series, the convergence of the partial sums is uniform. This means the properties of continuity, differentiability, and integrability are passed through limits; i.e. since each term is continuous, differentiable, and integrable (on closed intervals \( [a, b] \)), then so is the series.

Exercise 136. Find the Maclaurin series for each of the following by using Exercise 134 and Remark 135. Find the interval of convergence:

1. \( \frac{1}{1+x} \)
2. \( \sinh(x), \cosh(x) \)
3. \( \ln(1+x) \)
4. \( \arctan(x) \)