Mathematics for Elementary Teachers: Real Numbers

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These course notes represent my approach to a one-semester course (15 weeks) in a two-semester sequence required of students who intend to teach at the elementary level. The course is subtitled, “Real Numbers,” and the topic list includes: numeration; the operations of addition, subtraction, multiplication, and division in the decimal system; integers; rational numbers, including decimal fractions; proportions; percent; algebra arising from linear patterns.

I have taught the course more than a dozen times and these notes evolved from a supplement to a standard text into the current form, in which they are the entire course and no text is used. The approach here is an emphasis on visually modeling numbers and the operations on them. A goal in the course is to build a robust understanding of the mathematical validity of the operations represented by opaque terms, such as “borrowing,” “carrying,” “moving over,” and “goes into,” that are used frequently in elementary mathematics.
To the Instructor

Welcome to Mathematics for Elementary Teachers! This course is a chance for your students to develop a deeper understanding of the mathematics taught in elementary grades, and to develop an appreciation for the difficulties their students experience as learners of mathematics.

The author has taught this course more than ten times. The typical course enrollment is a little more than 30 students, with a typical range of about 20 to 40. (I once taught the course to more than 50 students, but I do NOT recommend it.)

Before going into details about the course, a few remarks about the context in which the course is offered. Students at California State University Dominguez Hills interested in teaching elementary school typically take 3 courses in mathematics: a general education course, this course, and a second course for elementary teachers, covering geometry and very basic probability and statistics. Therefore, those topics are not covered here. Moreover, students who are interested in teaching mathematics at the middle school level may take several other courses including one on problem solving, another on rational numbers and functions, one on algebraic thinking, and another course that takes a more rigorous approach to geometry. Hence, this course is designed primarily to examine the key ideas that a K-6 teacher needs to know, excluding ideas from geometry, statistics, and probability. Now, on to the content of this course.

This course might well be subtitled “An Intuitive Approach.” Rather than approach the course with a foundation of axioms and precise definitions, the course takes the approach of informal introductions to key ideas, and relies on examples to clarify the meaning of terms. Students are often expected to use their existing understandings of ideas to help make sense of the text. Nonetheless, there is a standard of reasonableness for explanations that will emerge during the course if the instructor consistently presses for greater clarity of explanations. Often these explanations will rely on informal models, which are explained below.

The bulk of the notes consist of problems. Students are frequently asked to solve problems either via work with physical manipulatives, or via diagrams representing manipulatives. The physical manipulatives are the same
ones that teachers are sometimes asked to use in working with elementary
students. These are commercially available at, for instance,

http://www.etacuisenaire.com/

A search for “educational manipulatives,” or “educational materials,” will
produce lots of alternative sites for your shopping pleasure. In some cases,
I have also listed below alternative ways for students to create their own
manipulatives.

Manipulatives. The manipulatives needed are:

Base 10 blocks, available commercially as plastic, foam, or similar ma-
terials, or they may be printed on card stock, using a template such as the
one at


Students need about 4 of the hundred-square blocks (also called “flats”),
30 of the ten-by-one blocks (also called “longs”), and 30 of the unit-square
blocks (also called “units” or “ones”). However, I have also found that many
students prefer to skip working with actual manipulatives and instead wish
to use only drawings, which is also acceptable.

Base five manipulatives can be created using blocks on card stock by
cutting the same base 10 templates down to appropriate size (the unit is the
same, the long is now 10\textsubscript{five}, and the flat is now 100\textsubscript{five}). A convenient
commercial alternative is to use algebra tiles, since the lengths of the long
and the flat are approximately right for base five. Of course the instructor
should explain that algebra tiles are not an exact model for base five.

Cuisenaire rods are used in Chapter 4. These are rods that come in nat-
ural number lengths from 1 to 10 cm. Each length is a different color. In
my experience, students have difficulty using algebra, and in particular rea-
soning with variables (as opposed to solving equations). Cuisenaire rods
give the students a chance to use a concrete referent to describe phenomena
such as being a multiple of 2. An alternative to the commercial rods is to
create rods in the same way that base 10 blocks were created, using card
stock. The most important rods students need are some 1 cm rods (which
are white or cream-colored), 2 cm rods (red), 3 cm rods (light green), 4 cm
rods (purple), and 5 cm rods (yellow). Having at least 10 of each of the 1
cm, 2 cm, and 3 cm rods, and at least 4 of each of the 4 cm and 5 cm rods is
sufficient.

Chapter 5 deals with integer operations. Two models, a hot/cold model
and a number line model, are used. For the hot/cold model, there are two
sorts of physical objects that can be used. A hot chip can be modeled phys-
ically in the classroom as either a red counter (such as a bingo counter, or
via counters that can be purchased at educational websites) or as a “+,” (tile
spacers that look like this are used for laying bathroom or kitchen tile and
are available at home improvement stores), and on paper by a +, which may be drawn in red for additional clarity. A cold chip can be modeled physically in the classroom as either a blue counter (again, a bingo counter can be used, or blue counters can be purchased via educational supply stores), or as a “−” (the same tile spacers can be cut to create negative signs), and on paper are recorded as a −, in blue ink if desired.

Moreover, a significant part of the work the students in the course are asked to do is to learn how to utilize the area model for multiplication. The area model is introduced for base ten multiplication, but becomes more important in Chapter 6, where students learn how to represent the multiplication of two rational numbers. No manipulatives are used to model multiplication of rational numbers–only drawings are needed.

Journals. In addition to problems, during the semester 6 writing assignments or “journals” are assigned. These assignments consist of readings, primarily from the journal Teaching Children Mathematics, and serve two major purposes: first, they are often used to either preview or reinforce the concepts in class, and second, they introduce the future teachers to thinking about student learning and other approaches to teaching mathematics besides the traditional drill-and-practice. The journals are an important opportunity for the students to express their developing understanding of what it means to know mathematics. The articles referenced may be available from your university library, or can be obtained via a membership in NCTM and a subscription to the journal (which comes with access to back issues online).

Course structure and meetings. The typical pattern at a class meeting is that the students work in groups to solve problems. They are asked to work on all the problems in the section, but then specific groups or individuals are given primary responsibility for presenting solutions to particular problems. This way, everyone is expected to have spent time thinking about the problems, and to have meaningful input on solutions that are presented. Then, through subsequent discussion or sometimes a key problem, the class abstracts the main principles from the problems. Then there are a few remaining problems which solidify or slightly extend the concepts in class. Students must write up all problems for the homework that is due each week.

The grading in the class consists of weekly homework, occasional journals, 1 quiz, 2 exams, presentations and group work, and a final. Currently, individuals are asked to present problems, and those individuals are graded for the clarity and accuracy of their work. In addition, groups are sometimes asked to turn in written work on a problem assigned during class–this helps to ensure that groups are using their time in class productively.
To the Student

As a future teacher of mathematics, you need specialized mathematics knowledge to teach math well. Teaching elementary school mathematics is difficult, because the concepts are fundamental. Think about how hard it is to learn, for example, Calculus. That’s what a grade schooler feels when he or she first learns addition. It’s a whole new world to them! What is easy and obvious for you is abstract and difficult for students.

One of the objectives of this course is to encourage you to go beyond the basic algorithm. You know how to add and subtract using the standard algorithms. Algorithms, however, are things that can be programmed on a computer. This means that you don’t need to understand what you are doing to compute using these algorithms. In math, understanding what is going on is the whole point. Training the mind to think is the objective, not just getting to the answer quickly.

You should have two minds in this class, one for yourself as a learner of mathematics, and another as a teacher of mathematics.

**Class Rule: Don’t ever look up answers!** It’s OK to be stuck. Say it to yourself several times a day, “It’s OK to be stuck.” If you cannot solve a problem, write down what you have tried and where you are stuck. The next time we meet in class, we can help each other. You will learn more by allowing yourself to struggle through being stuck.
Chapter 1

Bases

Journal 1. Write a 1-page response to the following: 1) Write a 1- or 2-paragraph math autobiography. What has been your experience taking math classes from K-college? 2) Describe the benefits of understanding in mathematics in your own words and how you think understanding is different from memorizing. 3) Describe two challenges in being a student in a classroom that promotes understanding, and how you personally will attempt to manage these challenges in this class.

Using the decimal system is something you are so used to that it is hard to teach. It’s obvious. It’s intuitive. This chapter is designed to put you into a frame of mind similar to your future students, which will enable you to conceptualize and understand the learning issues children face when learning place value and the decimal system.

The decimal system is also called base ten. We group numbers in powers of ten. So the number 5678 represents 5 thousands (10^3), 6 hundreds (10^2), 7 tens (10^1), and 8 ones (10^0, or 1).

1.1 Base Five

Use your base five blocks or drawings.

Notation 1. Base Five. Instead of grouping by powers of ten, we will group using powers of five. The powers of five are 5^0 = 1, 5^1, 5^2, 5^3, 5^4, .... We will use subscripts to differentiate between numbers in base five and numbers in base ten. For example, the number 1234\text{five} is 1(5^3) + 2(5^2) + 3(5^1) + 4(1) = 194 (in base ten). In base five, for each position a numeral among 0, 1, 2, 3, or 4 can be used.

Notation 2. In any base, the blocks are the unit or one, □, the long, and the flat.
Example 3. In base five, the long is: 

and the flat is: 

As with children learning our decimal system, the first thing we will learn in base five is how to count. Then we will learn some simple addition, and move on from there.

Problem 4. List the counting numbers in base five from $1_{five}$ to $100_{five}$. Include diagrams of the base five blocks for each number.

Problem 5. What is the next base five counting number after each of the following? Justify your answer with a diagram.

1. $104_{five}$
2. $244_{five}$
3. $444_{five}$
4. $1000_{five}$
5. $1044_{five}$

Problem 6. What is the previous base five counting number to each of the following? Justify your answer with a diagram.

1. $120_{five}$
2. $200_{five}$
3. $214_{five}$
4. $340_{five}$
5. $1000_{five}$
6. $2000_{five}$

For Problems 7 to 17, do not convert anything to base ten. Work strictly within base five. You may solve the problems with your blocks, in which case your written work should include diagrams of the blocks, or you may find ways to adapt what you know from working base ten to work in base five.

Problem 7. Compute $3_{five} + 2_{five}$. 
Problem 8. Compute $23_{five} + 12_{five}$.

Problem 9. Compute $33_{five} + 34_{five}$.

Problem 10. Compute $10_{five} - 3_{five}$.

Problem 11. Compute $32_{five} - 14_{five}$.

Problem 12. Compute $122_{five} - 43_{five}$.

Problem 13. Compute $2_{five} \times 3_{five}$.

Problem 14. Compute $2_{five} \times 23_{five}$.

Problem 15. Compute $12_{five} \times 32_{five}$.

Problem 16. Compute $13_{five} \times 24_{five}$.

Problem 17. Compute $3_{five} \times 223_{five}$.

1.2 Other Bases: Base Twelve and Base Sixty

Notation 18. Base twelve is a new challenge, because we need symbols beyond the digits 0 through 9. We will adopt the symbol $T$ (from ten) and $E$ (from eleven), so that the digits in base twelve are, in order, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E. Similar to base five, we will use a subscript of “twelve” to denote the base.

For Problems 19 to 24, do not convert anything to base ten. Work strictly within base twelve.

Problem 19. Compute $E_{twelve} + T_{twelve}$.

Problem 20. Compute $EE_{twelve} + 2E_{twelve}$.

Problem 21. Compute $9E_{twelve} + 20_{twelve}$.

Problem 22. Compute $3_{twelve} \times 4_{twelve}$.

Problem 23. Compute $12_{twelve} \times 75_{twelve}$.

Problem 24. Compute $2T_{twelve} \times E0_{twelve}$.

You have already worked with a system which is somewhat like base sixty. Since 60 seconds are in 1 minute, and 60 minutes are in 1 hour, this is a base sixty system. (This system of denoting time does not work at the next level since there are NOT 60 hours in 1 day.) We separate the places with a colon, as in 8:14:23, to mean 8 hours, 14 minutes, 23 seconds. We will concern ourselves only with addition and subtraction in this system, and not multiplication or division.

1.3 Conversion Between Bases

Problem 33. Convert the numbers from $1_{\text{five}}$ to $100_{\text{five}}$ to base ten.  
Problem 34. Convert $233_{\text{five}}$ to base ten.  
Problem 35. Convert $2434_{\text{five}}$ to base ten.  
Problem 36. Convert $12011_{\text{five}}$ to base ten.  
Problem 37. Without looking up the answers from a previous problem, convert the following numbers from base ten to base five.  

1. 13  
2. 49  
3. 127  
4. 299
Chapter 2

Addition, Subtraction, Multiplication, Division

2.1 The Standard Algorithms For Addition And Subtraction

Read the article:

**Journal 2.** Your responses to the following questions should be based on the article by Kari and Anderson. This journal should be about 1 page typed, double spaced.

1. What is the relationship between understanding place value and understanding addition (not just knowing the process called “add”)? Explain.

2. Discuss three things you learned from the article, focusing at least one of these on student learning.

**Notation 38. Place value in base ten.**

The rightmost digit of a positive whole number has a place value of one, and every other digit has a place value ten times that of the place value of the digit to its right. The *expanded form* of a number is the sum of the digits times their respective place value.

**Example 39.** The expanded form of 6187 is $6 \times 1000 + 1 \times 100 + 8 \times 10 + 7$.

**Problem 40.** Write each of the following numbers in expanded form.

1. 22
2. 345
3. 4573
The terms “carrying” and “borrowing” are misnomers. These are words that do not explain or make evident the underlying mathematical concepts. For instance, when one “borrows” in subtraction, one is not borrowing anything. In fact, these terms can inhibit learning. In Problems 41 and 42, explain how to compute the sums without using the word “carrying” (or other forms of the verb ‘to carry’) as best you can. Try using “regrouping” instead. In Problems 43 and 44, explain how to compute the differences without using the word “borrowing” as best you can. You may again want to use “regrouping.”

Problem 41. $27 + 19$.

Problem 42. $57 + 47$.

Problem 43. $24 - 9$.

Problem 44. $621 - 84$.

Problem 45. A student, Pat, does the following. Pat is adding $44 + 27$

Pat first adds 3 to 27. Then adds 30 to 44 to obtain 74. Then Pat subtracts 3 from 74 to get 71 as the final result. Explain what Pat could be thinking.

The following problems are about understanding the standard algorithms. Being able to compute is only one facet of mathematical ability. It must be mastered, but it is not everything. Students must also understand the concepts behind the computations, otherwise they will never own it and they will be less likely to like Mathematics. It is extremely difficult to like something you do not understand. Think about this. People usually like a subject because it makes sense to them, because they are successful at it, and because they find it compelling.

Problem 46. Compute the following sums using the standard algorithm. Then model each step of the standard algorithm with base-10 block drawings.

1. $32 + 9$
2. $26 + 25$
3. $165 + 87$

Problem 47. Compute the following differences using the standard algorithm. Then use base-10 block drawings to model each step.

1. $52 - 28$
2. $107 - 59$

3. $207 - 159$

**Problem 48.** Examine your work for $207 - 159$ in Problem 47, Part 3. In the standard subtraction algorithm, you cross out the 2 and the 0, writing 1 and 9 in their places, respectively, and “make the 7 a 17.” Explain why this is mathematically valid.

Students are capable of inventing their own algorithms for arithmetic. However, sometimes their limited understanding of the meaning of the base ten system causes them to invent algorithms that are not mathematically valid. For Problems 49 to 52, (a) describe the child’s algorithm, and then (b) explain why the algorithm is not mathematically valid.

**Problem 49.** Student’s work:

```
  34  6
+ 28  
----
  9  1
```

**Problem 50.** Student’s work: $36 + 28 = 514$.

**Problem 51.** Student’s work: $128 + 133 = 2511$

**Problem 52.** Student’s work:

```
  311  6  5
+  28  7
------
  7  4  2
```

For each of Problem 53 through Problem 57, (a) describe the child’s algorithm, and then (b) explain why the algorithm is not mathematically valid.

**Problem 53.** Student’s work: $76 - 29 = 53$

**Problem 54.** Student’s work: $76 - 29 = 57$

**Problem 55.** Student’s work: $70 - 48 = 38$

**Problem 56.** Student’s work: $70 - 48 = 32$

```
\[ \begin{array}{c}
  7^6 \\
- 4 \\
\hline
  3 \quad 2 \quad 8
\end{array} \]
```

**Problem 57.** Student’s work:

```
\[ \begin{array}{c}
  \not{7} \\
- 4 \\
\hline
  2 \quad 2 \quad 8
\end{array} \]
```

**Problem 58.** How would you help a student who says the following in discussing her solution to the problem in Table 2.1, “You can’t take 6 from 0 and you can’t take 3 from 0, so you have to borrow from the 4. Cross out 4 and write 3, cross out the zeros and write 10.”

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2.2 Alternative Algorithms For Addition

Read the article:


Journal 3. After reading the article, answer the following questions:

1. What is fluency and why is it important?

2. What are some potential advantages to letting children in a classroom use algorithms of their own choosing?

3. Mathematics education researchers have found that knowing how to compute with the standard algorithm and knowing why the algorithm works are different forms of knowledge. What is your reaction to this finding?

Algorithm 59. Adding up algorithm. Start with the leftmost place and find the sum in each place. If the sum is less than 10, just record the sum at the top. If the sum is 10 or more, put the digit from the ones place of the sum above and then cross out the number to its left and increase it by 1. For example, in the sum 357 + 496, first the 3 and 4 are added and the 7 is recorded at the top; then the 5 and 9 are added, and since the sum is 14, the 7 to the left becomes 8, and the 4 is recorded; finally, 7 and 6 are added, and since the sum is 13, the 4 is increased to 5, and the 3 is recorded. The final answer appears at the top, 853. Note that the problem is shown in three stages only for demonstration—one need only show the rightmost version.

\[
\begin{array}{c c c c c c}
7 & 3 & 5 & 7 & + & 4 & 9 & 6 \\
\hline
3 & 5 & 7 \\
7 & 8 & 4 \quad 4 & 9 & 6 \\
3 & 5 & 7 \\
\end{array}
\]

Algorithm 60. Partial sums. Add in each place, recording the results for each place on its own line, and then sum the results. For example, in the sum 357 + 496, the sums are 7 + 6 = 13, 50 + 90 = 140, and 300 + 400 = 700:

\[
\begin{array}{c c c c c c}
7 & 3 & 5 & 7 & + & 4 & 9 & 6 \\
\hline
3 & 5 & 7 \\
3 & 5 & 7 \\
3 & 5 & 7 \\
\end{array}
\]
Algorithm 61. **Lattice.** Begin at either end. Draw diagonals for each place value. Record the digits of the sum for each place value on either side of the diagonal. Then add the numbers along each diagonal. For example, in the sum 357 + 496 shown in Table 2.2, the ones sum to 13, so the 1 is written above the diagonal in the ones place column and the 3 appears below the diagonal. The tens sum to 14, so the 1 is written above the diagonal in the tens place column, and the 4 is written below that same diagonal. Then the hundreds sum to 7, so a 7 is recorded below the diagonal in the hundreds place column. Finally, the numbers are added along the diagonal directions to get the sum: the 3 below the diagonal at the right is recorded at the bottom, the 1 above the ones diagonal is added to the 4 in the tens column to get 5 tens, the 1 above the tens diagonal is added to the 7 hundreds to get 8 hundreds.

Table 2.2: Lattice Algorithm For Addition

<table>
<thead>
<tr>
<th>3</th>
<th>5</th>
<th>7</th>
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<tbody>
<tr>
<td>+</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

**Problem 62.** For each of the methods in Algorithms 59, 60, and 61:

1. Use the method to compute $967 + 385$.

2. Explain why the method works. This is different from explaining how it works. A successful explanation of why an algorithm for whole numbers works will use place value words and/or base 10 block diagrams and explain any regrouping.

**Problem 63.** When you are asked to explain why an algorithm works, you need to clearly explain the process of regrouping for the ones, tens, and hundreds (and thousands, etc., if the problem includes more place values), including both when and where the regrouping occurs. The explanations of why the algorithm works on the problem $827 + 196$ below come from previous students in this course. Please read each one with regard to the criteria above and look for what is done well and what needs to be clearer, more detailed, or revised.
1. **Lattice**

7 + 6 = 13, which is one ten, three ones. We have to write 3 in the ones diagonal and 1 in the tens diagonal.

2 + 9 = 11, which is one ten, one in the ones. We have to write one in the tens diagonal and one in the hundreds diagonal.

8 + 1 = 9, which is nine in the hundreds.

Then we add all the ones diagonal to get 3. We add all the tens diagonal 1 + 1 = 2, and all the hundreds 1 + 9 = 10, so = 1023.

2. **Lattice**

7 + 6 = 13 which is one ten and three ones. I write the number 3 in the ones diagonal and a 1 in the tens diagonal.

20 + 90 = 110, which is one tenths and one hundredths. I write the number 1 in the tens diagonal and a 1 in the hundredths diagonal.

800 + 100 = 900, which is 9 hundredths. I write a 9 in the hundredths diagonal.

Now I add the numbers in each diagonal. There is only 3 ones so the sum of the ones diagonal is 3. Next I add the numbers in the tens diagonal. 1 tenth + 1 tenth = 2 tenths. The sum of the tenths diagonal is 2. Finally I add the numbers in the hundredths diagonal. 1 hundredths + 9 hundredths = 10 hundredths. The total sum is 1023.

3. **Partial sums**

You add 7 + 6 = 13, then you add a zero on the next row and then add 2 + 9 and so on. Also, you are adding the ones, then the tens, and the hundreds.

4. **Partial sums**

First you have to start in the ones place value. You add 7 + 6 = 13, bring 13 down, 1 in the tens column and 3 in the ones. Then look at the tens column and add 2 + 9 = 11, then add a zero to the ones column. Then look at the hundreds column and add 8 + 1 = 9, then place it on the hundreds column, following two zeroes next to it. Then add the ones column and bring the 3 down. Then look at the tens column and add 1 + 1 = 2 and place it in the tens column. Then add the hundreds column, 9 + 1 = 10 placing 0 in the hundreds and 1 in the thousands. Giving 1 thousand, 0 hundreds, 2 tens, and 3 ones.

**Problem 64.** Suppose you are a teacher, and your school is proposing to teach struggling students a second algorithm besides the standard algorithm as an alternative to help them succeed with addition. Give at least one advantage and at least one disadvantage of each of the three alternative algorithms, Adding Up, Partial Sums, and Lattice. Describe the advantages and disadvantages in terms of Bass’s criteria outlined in your reading.
2.3 Alternative Algorithms For Subtraction

As with addition, here you will explore some alternative ways to perform subtraction. 17

**Algorithm 65. Compensate.** Round off the number being subtracted to a convenient value, then subtract the rest. For example, with $754 - 362$:

$$
754 - 362 = 754 - 350 - 12 = 404 - 12 = 392
$$

**Algorithm 66. Subtract in each place, with negatives.** For example, with $754 - 362$, 7 – 3 hundreds is recorded as 400, 5 – 6 tens is recorded as $-10$, and 4 – 2 is recorded as 2. $400 - 10 = 390$, and $390 + 2 = 392$.

$$
\begin{array}{c}
7 & 5 & 4 \\
- & 3 & 6 & 2 \\
\hline
4 & 0 & 0 \\
-1 & 0 \\
\hline
3 & 9 & 2
\end{array}
$$

**Algorithm 67. Missing addend.** Rewrite the subtraction problem as an addition problem with the second addend missing. Starting from the ones and proceeding to the left, put what is missing into each place value and regroup as needed. For example, with $754 - 362$, the solution starts with 2 plus what is 4, and so 2 is put into the ones column. Then in the next column, 6 plus how many will give an answer with 5, which can only happen with 9. Write 9 in that column and regroup 1, so that the last step is: 4 plus how many is 7, and so 3 is put into the last remaining place. Note that 392 is the answer.

$$
\begin{array}{c}
7 & 5 & 4 \\
- & 3 & 6 & 2 \\
\hline
3 & 6 & 2
\end{array} + \begin{array}{c}
3 & 6 & 2 \\
\hline
? \\
\hline
3 & 9 & 2
\end{array} = \begin{array}{c}
7 & 5 & 4 \\
\hline
3 & 9 & 2
\end{array}
$$

Problem 68 through Problem 70 refer to Algorithms 65, 66, and 67 as the three methods.

**Problem 68.** For each of the three methods, apply the method to the problem $937 - 468$.

**Problem 69.** For each of the three methods, apply the method to the problem $805 - 739$.

**Problem 70.** For each of the three methods, explain why the method works.
2.4 Multiplication

This section will introduce you to ways of performing and modeling multiplication. Read the following article:


Journal 4. After reading the article by Wallace and Gurganus, explain in detail three things you learned (3 or more paragraphs).

Problem 71. 1. Place the digits 2, 4, 6, and 9 in the blanks to create the largest possible answer: \[ \underline{ } \times \underline{ } \]

2. Explain how an understanding of place value is important to solving part 1.

Definition 72. The Area Model for Multiplication. Let \( a \) and \( b \) be whole numbers. Then the number \( ab \) \( (a \times b) \) is equal to the area of a rectangle having length \( a \) and width \( b \). For example, \( 27 \times 34 \) is shown in Table 2.3.

Algorithm 73. Partial Products. Much like partial sums for addition, partial products are a way of solving multiplication problems. There are different ways to write the partial products, but the basic idea is illustrated in the example of \( 27 \times 34 \) below. Each product for each pairing of digits, \( 7 \times 4, 4 \times 2, 3 \times 7, \) and \( 3 \times 2 \), is written on its own line, and then these are added to get the final sum of 918.

\[
\begin{array}{c}
2 & 7 \\
\times & 3 & 4 \\
\hline
2 & 8 \\
8 & 0 \\
2 & 1 & 1 & 0 \\
6 & 0 & 0 \\
\hline
9 & 1 & 8
\end{array}
\]

Algorithm 74. Lattice Algorithm. The lattice algorithm is similar to the one for addition. An example for \( 27 \times 34 \) is shown in Table 2.4. Each product is written so that the digits are split across the diagonal. Then digits along each diagonal are added: first the 8 ones are recorded, then along the next diagonal \( 1 + 2 + 8 = 11 \), so 1 is recorded at the bottom, and the other 1 is recorded along the next diagonal, and finally \( 2 + 6 + 1 = 9 \), so the 9 is recorded to get the final sum of 918.

Problem 75. Using each of the Area, Partial Products, and Lattice methods, find the following products: (a) \( 12 \times 3 \); (b) \( 37 \times 25 \); (c) \( 43 \times 26 \). Then, compute them using the standard algorithm.
Table 2.3: Area Model for $27 \times 34$

Table 2.4: Lattice Algorithm for Multiplication of $27 \times 34$
Problem 76. How are the Area Model, Partial Products, Lattice method, and standard algorithm related? What similarities do you observe in the calculations done with each method, as you did in Problem 75?

Problem 77. Suppose you are a teacher, and your school is considering teaching either the Area Model, Partial Products, or Lattice method before teaching the standard algorithm. Write one or two paragraphs that explain which method you would choose. As part of your explanation, you may wish to describe the advantages and disadvantages in terms of Bass’s criteria outlined in your reading for Journal 3: accuracy, generality, efficiency, ease of accurate use, and transparency.

Problem 78. Explain why, when using the standard algorithm, we “move over” one place when multiplying the digit 2 to the number 37 in the product $37 \times 25$.

Problem 79. Suppose a student computes $5 \times 23$ in the following way. The student computes $5 \times 20$ to get 100. Then computes $5 \times 3$ to get 15. Finally the student computes $100 + 15 = 115$ to get the final answer. (a) Describe the child’s algorithm, and then (b) explain whether the algorithm is mathematically valid, and why you think so.

Problem 80. Suppose a student computes $5 \times 23$ and obtains $15 + 10 = 25$. (a) Describe the child’s algorithm, and then (b) explain whether the algorithm is mathematically valid, and why you think so.

Definition 81. Repeated Addition Model for Multiplication. Let a and b be integers, and $a > 1$. Then $ab$ (a times b) is equal to adding b repeatedly a times. If $a = 1$, then $ab = 1b = b$.

Problem 82. To use the repeated addition model for $15 \times 34$, we would write

$$34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34 + 34.$$

Explain whether the repeated addition model leads to an efficient algorithm for addition.

Problem 83. For Table 2.5 and Table 2.6, (a) describe the child’s algorithm, and then (b) explain whether the algorithm is mathematically valid, and why you think so.

Problem 84. If we are asked to find the sum $28 + 18$, we could solve this by transforming the problem into the equivalent problem $30 + 16$. However, when multiplying $28 \times 18$, we CANNOT transform it into $30 \times 16$. Why not? You may wish to explain using an area model.

Problem 85. Give an example of an equivalent multiplication problem into which you can transform $28 \times 18$. 

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Table 2.5: One Student’s Multiplication

\[
\begin{array}{c}
4 \\
6 \\
\times \\
2 \\
8 \\
\hline
3 \\
2 \\
4 \\
8 \\
8 \\
1 \\
2 \\
\hline
4 \\
0 \\
6 \\
0
\end{array}
\]

Table 2.6: Another Student’s Multiplication

\[
\begin{array}{c}
4 \\
6 \\
\times \\
2 \\
8 \\
\hline
3 \\
6 \\
8 \\
\hline
9 \\
2 \\
\hline
4 \\
6 \\
0
\end{array}
\]

Problem 86. Ms. Falcon reads 8 pages of Charlotte’s Web to her class every day. How many pages does she read to them in a 5-day school week?

Problem 87. Davis is going to lay tile in his kitchen. If the kitchen floor measures 12 feet long by 7 feet wide, how many square feet of tile does Davis need?

Problem 88. Luke bought 3 identical pizzas. Each pizza was cut into 8 slices. How many slices of pizza did Luke buy?

Problem 89. For each of Problems 86 to 88, determine whether the context reflects the repeated addition model or the area model of multiplication. Justify your choice in a sentence or two about each problem.

Problem 90. Write your own word problem illustrating the repeated addition model of multiplication.

Problem 91. Write your own word problem illustrating the area model of multiplication.

2.5 Division

Read the article:


Journal 5. After reading the article, answer the following:

1. Explain in detail three things you learned (3 or more paragraphs).

2. Reflect on the difference between using the standard division algorithm and understanding the mathematical concepts hidden by the algorithm.
Problem 92.  1. Place the digits 2, 4, 6, and 9 in the blanks to create the largest possible answer: \[ \square \square \square \div \square \]

2. Explain how an understanding of place value is important to solving part 1.

Problem 93.  Mrs. Jones has 108 pieces of candy and 36 students. How many pieces of candy can she give to each student?

Problem 94.  Mrs. Jones has 108 pieces of candy. She wants to give 3 pieces of candy to each student. How many students will get candy from Mrs. Jones?

Problem 95.  Dr. Jones brings 300 sheets of paper to an exam. He has 25 students. How many sheets of paper can he give to each student?

Problem 96.  Dr. Jones has 300 sheets of paper to the copier. He is making handouts that use 12 sheets of paper. How many students will be able to receive the handout?

Definition 97.  Repeated Subtraction Model for Division. In this model for division one repeatedly subtracts a given amount from another amount until it cannot be done anymore. This model is used in Problems 94 and 96.

Definition 98.  Partitioning or Partitive Model for Division. In this model of subtraction, one first sets up the appropriate number of groups (partitions), and fills each group. The number in each group is the result of the division. This is the model used Problems 93 and 95.

For Problems 99 to 103, determine which type of division would be used for the following problems. Draw a diagram of your solution and explain how it illustrates the model.

Problem 99.  A class has 20 children. The class is to be divided into 4 teams. How many children are on each team?

Problem 100.  A class has 20 children. The class is to be divided into teams of 4. How many teams are there?

Problem 101.  Melissa, Vanessa, Corissa, and Valerie bought some cloth that is 24 yards long. If they share the cloth equally, how many yards of cloth does each person get?

Problem 102.  Jeannie is making popsicles. If she has 24 ounces of juice, and each popsicle takes 4 ounces, how many popsicles can she make?

Problem 104. Write your own word problem illustrating the repeated subtraction model of division. Be sure that your word problem is phrased as a question, and that you demonstrate the solution and how it relies on repeated subtraction.

Problem 105. Write your own word problem illustrating the partitive division model. Be sure that your word problem is phrased as a question, and that you demonstrate the solution and how it relies on partitive division.

Problem 106. If you divide a number larger than 75 by a number that is less than 5, which of the following can you conclude about the quotient? Explain your answer.

- It must be greater than 15.
- It must be less than 15.
- It will be between 5 and 75.
- You cannot be certain of any of the above.

Problem 107. How many buses are needed to take 334 students on a trip if each bus can take 25 passengers?

Problem 108. Eric rented a professional studio and computer equipment to record his band, The Divide. It cost them $1500 to create the master file that will be used to burn CDs. If it costs an additional $1.50 to burn each CD, and Eric wants to sell the CDs for $9, how many CDs must he sell to break even?
**Definition 109. Intermediate division algorithm**: Study the algorithm used to do the division problem 3144 ÷ 12 as shown in Table 2.7. The procedure can be described as follows. First, make an estimate of 3144 ÷ 12. The first estimate shown is 200 (200 is sometimes referred to as a partial quotient). Then, the product 12 × 200 = 2400, so 2400 is subtracted from 3144, leaving 744. At the next stage, make an estimate of 744 ÷ 12. The next partial quotient shown is 50. Therefore, 12 × 50 = 600 is subtracted from 744, leaving 144. At the next step, estimate 144 ÷ 12. The partial quotient shown is 10. Since 12 × 10 = 120, then 120 is subtracted from 144, leaving 24. Now, 12 × 2 = 24, so 2 is recorded as a partial quotient at the top, and 24 − 24 = 0, meaning there is no remainder. The quotient is the sum of the partial quotients, 200 + 50 + 10 + 2 = 262.

<table>
<thead>
<tr>
<th>Table 2.7: Intermediate Division Algorithm</th>
</tr>
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<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>200</td>
</tr>
<tr>
<td>12) 3144</td>
</tr>
<tr>
<td>− 2400</td>
</tr>
<tr>
<td>744</td>
</tr>
<tr>
<td>− 600</td>
</tr>
<tr>
<td>144</td>
</tr>
<tr>
<td>− 120</td>
</tr>
<tr>
<td>24</td>
</tr>
<tr>
<td>− 24</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

**Problem 110.** Apply the intermediate division algorithm to 108 ÷ 3.

**Problem 111.** Apply the intermediate division algorithm to 4284 ÷ 14.

**Problem 112.** Apply the standard algorithm to 4284 ÷ 14. Then reflect on your solution by answering the following questions:

1. When we write 3 in the quotient, and then multiply 3 by 14 to get 42, what does the 42 represent? It may be helpful to refer back to your solution using the intermediate division algorithm.

2. Explain the mathematical reason why you have to “move to the right” as you do the algorithm.

3. One confusing part of this process for children who are just learning it is why the answer is 306 instead of 36. Explain clearly why a 0 must be put into the answer. “Because otherwise you get the wrong answer,” is NOT an explanation. You may find it helpful to refer to the intermediate algorithm in your explanation.
Problem 113. Apply the standard algorithm to $451 \div 7$. Then reflect on your solution by answering the following questions:

1. In the first step you put a 6 at the top above the 5. What is the place value of this 6?

2. In the second step you “bring down the 1.” What does that mean mathematically? Explain.

3. In the next step you put a numeral up top to the right of the 6. What is the place value of this numeral? Then explain the meaning of this number in terms of base 10 blocks.

4. At the end, you have a remainder. What does this remainder represent mathematically?
Chapter 3

Number Theory

3.1 Multiples And Factors

Definition 114. Let $c$ and $b$ be two integers, where $c \neq 0$. We say that $c$ is a factor of $b$ if and only if there is an integer $n$ such that $cn = b$.

Remark 115. There are other ways to state the same concept. With the same situation, in which $cn = b$, we can say that $c$ divides $b$, or that $b$ is a multiple of $c$, or that $b$ is divisible by $c$.

Definition 116. An even number is an integer that is a multiple of 2. An odd number is an integer that is not a multiple of 2.

Problem 117. Use the definition to explain why 12 is a multiple of 4.

Problem 118. Use the definition to explain why 3 is a factor of 30.

Problem 119. Use the definition to explain why 32 is not a multiple of 3.

Problem 120. If $m$ is a natural number, is it true that the factors of $m$ must all be smaller than $m$? Explain.

Problem 121. If $r$ is a natural number, is it true that the multiples of $r$ must all be larger than $r$? Explain.

Problem 122. Using Cuisenaire rods:

1. Build the numbers 3 through 12 using only red rods, if possible, or red rods and one white cube.

2. Describe the relationship between being odd or even and how a number can be built using rods.

Problem 123. Using Cuisenaire rods:

1. Build the numbers 3 through 12 using only light green rods, if possible, or light green rods and one or two white cubes. Describe how 100 and 1000 would be built using this method.
2. Describe the relationship between being a multiple of 3 or not a multiple of 3 and how a number can be built using rods.

Use your solutions to the Problems 122 and 123 to help you answer Problems 124 to 128. Use rods or variables to explain each answer.

**Problem 124.** Will the sum of an even number and an even number always be even?

**Problem 125.** Will the sum of an even number and an odd number always be odd?

**Problem 126.** Will the sum of two odd numbers always be even?

**Problem 127.** Will the sum of two multiples of 3 always be a multiple of 3?

**Problem 128.** Will the sum of two numbers that are not multiples of 3 always be a multiple of 3?

**Problem 129.** Generalize what you learned from even numbers and multiples of 3: If \( b \) and \( c \) are multiples of \( m \), will the sum \( b + c \) also be a multiple of \( m \)? Start by using examples and then try to explain why, using either rods or algebra.

**Problem 130.** If \( m \), \( b \), and \( c \) are integers, and if \( b \) is a multiple of \( m \), will \( bc \) also be a multiple of \( m \)? Explain.

### 3.2 Divisibility Rules

There are some rules for divisibility that make it simpler to tell whether a number is a multiple of something. For instance, we may decide whether a number is a multiple of 2 (even) by looking to see whether the last digit is 0, 2, 4, 6, or 8. In the next problem, we will look for an explanation of why this rule for being a multiple of 2 works.\(^{24}\)

**Problem 131.** Complete the following:

1. Use the definition of multiple to explain why 0, 2, 4, 6, and 8 are multiples of 2, and why 1, 3, 5, 7, and 9 are not multiples of 2.

2. Use the definition of multiple to explain why 10 is a multiple of 2.

3. Explain why any multiple of 10 will also be a multiple of 2.

4. Let \( n \) be any integer. Explain why \( n \) can be written as a multiple of 10 plus a one-digit number (think back to the idea of expanded form). For instance, \( 327 = 320 + 7 = 32 \cdot 10 + 7 \).
5. Explain why you can tell whether any number is a multiple of 2 just by checking whether the last digit is 0, 2, 4, 6, or 8. (Hint: Parts 1 to 4 may be helpful.)

Problem 132. In this problem, we will look at a rule for 4.

1. List the multiples of 4 at least until 60. You are looking for a pattern in the multiples or the digits of the multiples that may make it simpler to determine whether a number is a multiple of 4 than actually dividing.

2. Test your rule on another number less than 100. Did it work?

3. Test your rule on these numbers: 90, 132, 174, 184, 194, 196. Did the rule work?

4. Based on the results of your testing, revise and refine your description of how to tell when a number is a multiple of 4.

5. Use the definition of multiple to explain why 100 is a multiple of 4.

6. Explain why you can check whether any number is a multiple of 4 by just checking its last two digits.

Problem 133. In this problem, we look at why the rule for divisibility by 5 works. Explain why you can tell whether any number is a multiple of 5 just by checking whether the last digit is 0 or 5. If you get stuck on this problem, you may find it helpful to study your work in Problems 131 and 132.

Problem 134. In this problem, we will look at a rule to see whether a number is a multiple of 3. The rule is, take any number and take the sum of its digits. If the sum of the digits is a multiple of 3, so is the original number (and vice versa). For example, the sum of the digits in 87 is $8 + 7 = 15$. I can apply the rule again to 15, $1 + 5 = 6$. Since 6 is a multiple of 3, so is 15, and then so is 87.

1. Apply the rule for divisibility by 3 to 48 and to 115.

2. Use the definition of multiple to explain why 9 and 99 are multiples of 3.

3. Write 48 in expanded form. Then draw 48 using base 10 blocks. Looking at the expanded form and the blocks, why is being a multiple of 3 dependent on whether or not $4 + 8$ is a multiple of 3?

4. Write 115 in expanded form. Then draw 115 using base 10 blocks. Looking at the blocks, why is being a multiple of 3 dependent on whether or not $1 + 1 + 5$ is a multiple of 3?

5. Explain why the rule for 3 works.
Problem 135. Can you come up with a rule for 9 that is similar to the rule for 3, using the same idea? Explain why it works.

Problem 136. Come up with a divisibility rule for 25. Explain how it works. Then, explain why you can check whether any number is a multiple of 25 by just checking its last two digits.

3.3 Primes and Factoring

Definition 137. A natural number is prime if it has exactly 2 different natural number factors, one and itself. A natural number is composite if it has more than 2 different natural number factors.

Problem 138. Explain why 1 is NOT prime, according to the definition.

Problem 139. Explain why 5 is a prime number.

Problem 140. Explain why 6 is not a prime number.

Problem 141. A student asks you if all primes are odd numbers. How do you respond?

Problem 142. A student asks you if all odd natural numbers are prime. How do you respond?

Problem 143. A student asks you if all composite numbers are even. How do you respond?

Problem 144. A student asks you if all even natural numbers are composite. How do you respond?

Problem 145. Make a list of all the factors of each of the numbers 2 through 25.

Problem 146. Write each of the numbers 2 through 25 as a product of prime numbers. This is called the prime factorization of the number.

Problem 147. Use your factorization of 24 to find all the factors of 48. Then describe your method.

3.4 Greatest Common Factor

Definition 148. Let b and c be two natural numbers. Then b and c are said to have a common factor if there is a natural number x that is a factor of b and also a factor of c. The greatest common factor of b and c, written \( \text{gcf}(b,c) \), is the largest natural number that is a factor of both b and c.
Problem 149. Use the definition to explain why 4 is a common factor of 24 and 40.

Problem 150. Use the definition to explain why 6 is not a common factor of 24 and 40.

Problem 151. Use the definition to explain why 8 is the greatest common factor of 24 and 40.

Problem 152. Find a pair of numbers whose greatest common factor is 6. Find more pairs. Describe your process in finding these pairs.

Problem 153. Find a number so that the greatest common factor of your number and 48 is 12. Find another number. Describe your process in finding these numbers.

Problem 154. Use the definition to explain why each of the following could not be the greatest common factor of 48 and 54.

- 2
- 3
- 5
- $48 \times 54 = 5092$

Use your work in Problem 146 and/or Problem 145 to help you find the greatest common factor for each of the pairs of numbers in Problems 155 through 158.

Problem 155. 16, 24
Problem 156. 20, 22
Problem 157. 18, 48
Problem 158. 48, 60

Problem 159. Are the prime factorizations of two numbers helpful in finding their greatest common factor? If so, how?

Problem 160. For each of the 4 pairs of numbers for which you found the gcf, take the positive difference of the numbers. How does this number compare to the gcf? Explain why this happens.

3.5 Least Common Multiple

Definition 161. Let $b$ and $c$ be two natural numbers. Then an integer $m$ is a common multiple of $b$ and $c$ if $m$ is a multiple of $b$ and $m$ is also a multiple of $c$. A positive integer $m$ is the least common multiple of $b$ and $c$ if it is the smallest positive integer that is a common multiple of $b$ and $c$. 
In Problems 162 through 171, find the least common multiple of the numbers given using any strategy.

**Problem 162.** 8, 12

**Problem 163.** 9, 13

**Problem 164.** 11, 18

**Problem 165.** 6, 8, 12

**Problem 166.** 7, 9, 12

**Problem 167.** 16, 24

**Problem 168.** 20, 22

**Problem 169.** 18, 48

**Problem 170.** 48, 60

**Problem 171.** 48, 54

**Problem 172.** One student, Graciela, described her method for finding least common multiples this way, using 48 and 60 as her example: Write the prime factorizations for each of the numbers. Line them up above each other, like this (see Table 3.1). Then take one number from each column: there are four columns with a 2, one column with a 3, and one column with a 5, so you get $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 240$.

- *Try Graciela’s method on at least two of Problems 162 to 171.*
- *Why does Graciela’s method work?*

**Problem 173.** Another student, Marcus, described his method for finding least common multiples this way, using 48 and 60 as his example: Find the greatest common factor of the numbers. In this case, that is 12. Then write each number using that factor. So $48 = 4 \cdot 12$, and $60 = 5 \cdot 12$. Then the least common multiple is the product of the common part, 12, with the uncommon parts, 4 and 5. So the least common multiple is $12 \cdot 4 \cdot 5 = 240$.

- *Try Marcus’ method on at least two of Problems 162 to 171.*
- *Why does Marcus’ method work?*
Problem 174. A third student, Julieann, described her method for finding least common multiples this way, using 48 and 60 as her example: Find the greatest common factor of the numbers. In this case, that is 12. Then write each number using that factor. So $48 = 4 \cdot 12$, and $60 = 5 \cdot 12$. Then the least common multiple is the product of the first number with the uncommon part of the other number, 5. So the least common multiple is $48 \cdot 5 = 240$.

- Try Julieann’s method on at least two of Problems 162 to 171.
- Why does Julieann’s method work?
Chapter 4

Integers

In this chapter there are two models, the hot/cold chip model and the number line model, for understanding integers.

4.1 Addition And Subtraction

Definition 175. A pair of integers are called zero pairs, if they add up to zero.

A Physical Model for Integers. Temperature is a familiar concept. In this model you have a tank of water. A blue chip, which can be recorded on paper as $-$, will lower the temperature by one degree. (It is called blue to signify cold, which drops the temperature.) A red chip, which can be recorded on paper as $+$, will raise the temperature by one degree. (It is called red to signify heat, which raises the temperature.) The blue chip is a physical model for $-1$, and the red chip is a physical model for $+1$.

In this model, you are always allowed to put zero pairs into the tank, since this has no effect on the temperature in the tank. An example of this is shown in Table 4.1.

Problem 176. Refer to Table 4.1. Suppose you remove 5 red chips ($+$) from the tank. What happens to the temperature? Record the removal of 5 red chips using a diagram and also explain in words.

Problem 177. Refer to Table 4.1. Suppose you now need to remove 7 blue chips ($-$) from the tank. (Hint: Use zero pairs to make the removal of 7

<table>
<thead>
<tr>
<th>+</th>
<th>+</th>
<th>+</th>
<th>+</th>
<th>+</th>
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<tbody>
<tr>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
</tbody>
</table>
Table 4.2: Showing $+5 - 3$ in the hot/cold tank model

<table>
<thead>
<tr>
<th>+5</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Result: $+2$

Blue chips possible. What happens to the temperature? Record the removal of 7 red chips using a diagram and also explain in words.

**Problem 178.** Suppose you add 5 red (+) chips and 3 blue (−) chips into the tank. Mathematically we write this as $5 - 3$. Use Table 4.2 to help you explain what happens.

For Problems 179 through 182, represent and solve the problem using chips. Include drawings of the chips. Then record your work numerically.

**Problem 179.** $-7 + -5$

**Problem 180.** $-7 + 5$

**Problem 181.** $7 + -5$

**Problem 182.** $7 + 5$

**Problem 183.** When you add two integers, how can you decide whether their sum will be positive, negative, or zero?

**Problem 184.** Describe how to add any two integers.

**Problem 185.** Laura wants to solve $-3 - (-5)$ using chips. She started with 3 blue (−) chips, but she is stuck because she cannot take away 5 blue (−) chips to represent subtracting $-5$. How can Laura show $-3$ in a way so that she can remove 5 negative chips? What is $-3 - (-5)$? Explain how you got your answer.

**Problem 186.** Diego wants to find $5 - 7$ using chips. How can he show 5 in the tank so that he can remove 7 red (+) chips to represent subtracting 7? What is $5 - 7$? Explain how you got your answer.

For Problems 187 to 191, use chips to find each sum or difference. Where applicable, be sure to record the number of chips you used to represent the first integer so that you could take away chips representing the second integer.

**Problem 187.** $-7 - (-5)$

**Problem 188.** $-7 - 5$

**Problem 189.** $7 - 9$
Problem 190. $7 - (-5)$

Problem 191. $-5 - (-3)$

The Number Line Model for Integers. The number line model uses arrows to depict quantities, and arrows are combined tip-to-tail in addition. So $+5 - 3$ is shown as an arrow of length 5 with its tail at 0 and its tip at 5, and $-3$ is combined with it by drawing an arrow of length 3 with its tail at +5 and its tip at +2, as shown in Figure 4.1. The result is where the tip of the second arrow points, which is +2 here.

![Figure 4.1: Showing the sum $+5 + (-3)$ on the number line](image)

Problem 192. Return to Problems 179 to 182 and use the number line to solve them.

Problem 193. If you are given a positive integer, and you subtract a negative number from it, is the difference always, sometimes, or never a positive integer? Give examples to support your thinking.

Problem 194. If you are given a negative integer and you subtract a negative integer from it, is the difference always, sometimes, or never a positive integer? Give examples to support your thinking.

Problem 195. Study the pattern in the equations, and then answer the questions below.

\[
\begin{align*}
10 - 3 &= 7 \\
10 - 2 &= 8 \\
10 - 1 &= 9 \\
10 - 0 &= 10
\end{align*}
\]

1. Describe any patterns you observe in the way the differences change as the integers subtracted from 10 get smaller.

2. Use the patterns you observed to predict the answer to $10 - (-1)$. Check your prediction by using chips or a number line.

3. Predict the answer to $10 - (-4)$. Explain your reasoning.

Problem 196. Without actually calculating the difference, how can you decide if the difference of two integers will be positive? Negative? Zero?
4.2 Multiplication And Division

For multiplication in the hot/cold model, the first factor is used to mean “groups of,” with the sign of the number indicating “put in,” or “take out.” The second factor determines what is in the group. 31

For instance, \(3 \times (-5)\) would mean, “Put in three groups of 5 cold \((-)\) chips.” This is shown in Table 4.3.

Solve Problems 197 to 199 using a hot/cold model.

**Problem 197.** \(-3(5)\)

**Problem 198.** \(3(5)\)

**Problem 199.** \(-3(-5)\)

**Problem 200.** Explain why a positive number times a negative number is negative. You may want to use a chip model or another model to help you explain.

**Problem 201.** Below are five sample explanations for Problem 200 from previous students in this course. Please read each one and look for what is done well and what needs to be clearer, more detailed, or revised. Describe the key features that distinguish a good explanation.

- In multiplying \(3(-5)\), we are adding 5 negative chips 3 times. So our answer must be negative because a negative number plus a negative number will always be a negative number. Multiplication is repeated addition and when you are repeatedly adding negative numbers, your answer must also be negative.

- A product of a positive and a negative can be expressed as the number of groups of negative numbers (blue chips). This leads to addition of negative numbers. Multiples of blue chips can’t create any red chips. [Shows drawing of example of \(3 \times (-5)\) as \(5 + 5 + 5\) blue chips, or 15 blue chips, \(-15\).]

- A positive number times a negative number is always negative because in that case the signs don’t change. You are simply taking the original negative number and adding on more of the same number in several
groups. [Shows example of $3 \times (-5)$ with chips and as three arrows pointed in the negative direction.]

- A positive number times a negative number is always negative, because when we multiply same signs either negative times negative or positive times positive, the answer is always positive, but if we multiply different signs the answer is always negative.

- When we are multiplying a positive number with a negative number the answer is always negative because the negative is stronger than the positive.

**Problem 202.** Observe the equations below, and use them to help you answer the questions below.

\[
\begin{align*}
5 \times 4 &= 20 \\
5 \times 3 &= 15 \\
5 \times 2 &= 10 \\
5 \times 1 &= 5 \\
5 \times 0 &= 0
\end{align*}
\]

1. Describe any patterns you observe in the way the products change as the integers multiplied by 5 get smaller.

2. Use the patterns you observe to predict $5 \times -1$. Explain your reasoning.

3. Write the next four equations in the pattern.

**Problem 203.** Complete the equations below, and use them to help you answer the questions that follow.

\[
\begin{align*}
4 \times -4 &= -16 \\
3 \times -4 &= ? \\
2 \times -4 &= ? \\
1 \times -4 &= ? \\
0 \times -4 &= ?
\end{align*}
\]

1. Describe any patterns you observe in the way the products change as the integers multiplied by $-4$ get smaller.

2. Use the patterns you observe to predict $-1 \times -4$. Explain your reasoning.

3. Write the next four equations in the pattern.

**Problem 204.** A student, Sara, was asked to explain why $-3(-4) = 12$. Sara came up with what is shown in Figure 4.2. Explain Sara’s reasoning.
Figure 4.2: Sara’s explanation of $-3(-4)$

\[-3 \cdot (4 - 4) = -3 \cdot 0 = 0\]

And,

\[-3 \cdot (4 - 4) = -3 \cdot 4 + -3 \cdot (-4) = -12 + -3 \cdot (-4)\]

Therefore,

\[0 = -12 + -3 \cdot (-4)\]
\[12 = -3 \cdot (-4)\]

**Definition 205. Missing Factor Model of Division.** For the missing factor model, $a \div b$ is solved by finding the number $q$ so that $bq = a$.

Use the missing factor model of division to solve Problems 206 to 209.

**Problem 206.** $12 \div 3$

**Problem 207.** $12 \div (-3)$

**Problem 208.** $(-12) \div 3$

**Problem 209.** $(-12) \div (-3)$

**Problem 210.** Use the missing factor model to explain why a negative number divided by a negative number results in a positive number.
Chapter 5

Fractions

Fractions are an often-feared part of mathematics. Many students reach this stage of mathematics and decide to just follow rules without any attempt to understand them, while some others give up at mathematics because they do not understand. In contrast, in this chapter, we will see that it is possible for fractions to make sense. For this reason, you are asked to draw a diagram wherever possible for the problems in this chapter. From the diagrams, you will have the opportunity to make sense of fractions and of the ways in which we operate with them, just as you did in earlier chapters with whole numbers.

5.1 Understanding Units

Problem 211. Javier has $\frac{1}{2}$ of a six-pack of soda in his car. How many cans of soda is that? How much of a 12-pack of soda is that? How much of a case of 24 cans is that?  

Problem 212. A submarine sandwich is cut into 5 pieces of equal size. If I eat 3 of these pieces, what fraction of the sandwich did I eat?

Problem 213. Three identical submarine sandwiches are going to be split equally among 5 people. How much of a sandwich does each person get?

Problem 214. A grocer has some apples. He gives half of them away to his sister. Then he gives one to his nephew. Then he gives half of what he has left to his daughter. After that, he has 4 apples left. How many apples did he start with?

Problem 215. We know $\frac{1}{2} + \frac{1}{2} = 1$. How is it possible that the grocer in Problem 214 gave away half of his apples twice yet still had something left?

Problem 216. A store has a sale in which all sweaters are one half off. They also have a coupon which offers the customer an additional one half off. Does that mean that the customer can get the items free? Explain.
Problem 217. In working with fractions as in Problems 212 through 216, there is an implicit unit of measurement, or a “whole” that is being broken into pieces. Explain why the unit of measure is important when working with fractions.

Problem 218. A family orders 3 large pizzas. The teenager in the family eats $\frac{1}{3}$ of the family’s order. How much pizza does he eat?

Problem 219. Joe ate half a large pepperoni pizza and one-third of a large veggie pizza. How much pizza did Joe eat? Use a diagram to model how to solve this problem.

Problem 220. A square yard is a unit of measurement for area. A square yard is an amount of area equivalent to a 1-yard by 1-yard square. Jane buys $\frac{1}{2}$ of a square yard of cloth with a striped pattern, and $\frac{1}{3}$ of a square yard of cloth with a floral pattern. How much total cloth did Jane purchase? Use a diagram to model how to solve this problem.

Problem 221. Suppose you have half a chocolate bar left. You give an amount equal to one-third of a whole chocolate bar to your friend. How much of the chocolate bar remains? Use a diagram to model how to solve this problem.

Problem 222. Suppose you have half a chocolate bar left. You give an amount equal to one-third of the remaining chocolate bar to your friend. How much of the chocolate bar remains? Use a diagram to model how to solve this problem.

5.2 Equivalent Fractions

Definition 223. A rational number is a number of the form $\frac{c}{d}$, where $c$ and $d$ are integers and $d \neq 0$. The number $c$ is called the numerator and $d$ is the denominator.

Definition 224. Any two rational numbers $\frac{c}{d}$ and $\frac{a}{b}$ are equivalent if there are nonzero integers $m$ and $n$ so that both the numerator and denominator of $\frac{ma}{mb}$ are the same as $\frac{nc}{nd}$.

Problem 225. Explain why $\frac{2}{3}$ is equivalent to $\frac{8}{12}$, using the definition.

Problem 226. Explain why $\frac{2}{3}$ is equivalent to $\frac{8}{12}$, using the context of a dozen eggs and a diagram.

Problem 227. Explain why $\frac{6}{8}$ is equivalent to $\frac{9}{12}$, using the definition.

Problem 228. Explain why $\frac{6}{8}$ is equivalent to $\frac{9}{12}$, using the context of a 24-pack of soda.
Problem 229. Explain why \( \frac{6}{8} \) is equivalent to \( \frac{9}{12} \), using the context of a 12-pack of soda.

Problem 230. Explain why \( \frac{6}{8} \) is equivalent to \( \frac{9}{12} \), using the context of a 4-pack of yogurt.

Problem 231. Explain why \( \frac{2}{3} \) is NOT equivalent to \( \frac{3}{5} \), using the definition of equivalence.

Problem 232. Explain why \( \frac{2}{3} \) is NOT equivalent to \( \frac{3}{5} \), using the context of a classroom of 30 students.

Problem 233. Explain why \( \frac{6}{10} \) is equivalent to \( \frac{9}{15} \), using the context of a rectangular cake.

Problem 234. Explain why \( \frac{21}{70} \) is equivalent to \( \frac{6}{20} \), using the definition of equivalence.

Problem 235. Sometimes simplest form is called “reduced.” However, this has been found to cause confusion for students. Why do you think the term reduced causes confusion?

Problem 236. A typical elementary or middle school text defines two fractions as equivalent if they “represent the same amount.” What might be some advantages and some disadvantages of this definition?

Problem 237. One day, the school math club orders 5 identical pizzas. At Table A, 5 people share 3 of the pizzas, while at the Table B, 3 people share 2 pizzas.

1. Which table gets more pizza per person? Explain using a diagram.

2. Rebecca solved the problem this way: At Table A, 5 people sharing 3 pizzas is the same as 10 people sharing 6 pizzas. At Table B, 3 people sharing 2 pizzas is the same as 9 people sharing 6 pizzas. Since more people have to share at Table A, Table B gets more pizza per person. Explain what Rebecca is thinking.

3. Use Rebecca’s method to compare Table A with 2 pizzas shared by 7 people to Table B with 4 pizzas shared by 13 people.

5.3 Addition And Subtraction Of Fractions

Problem 238. A baker buys 2 dozen eggs. He has one cake recipe calling for \( \frac{2}{3} \) of a dozen eggs, and a muffin recipe calling for \( \frac{3}{4} \) of a dozen eggs.

1. How many eggs will he use if he makes both recipes?

2. How many dozen eggs will he use in making both recipes?
3. What fraction of a 12-egg carton can he fill with the eggs that are left over?

4. What fraction of the eggs that he originally bought have been used?

5. Which of the situations in this problem corresponds to $\frac{2}{3} + \frac{3}{4}$? Why?

Problem 239. Complete the following.

1. Kendra runs $\frac{3}{5}$ of a mile, stops to drink water, and then runs $\frac{4}{5}$ of a mile. How far did she run in total?

2. A basketball player makes $\frac{3}{5}$ of the five free throws she shoots in the first half. Then she makes $\frac{4}{5}$ of the five free throws she shoots in the second half. What fraction of her free throws did she make overall in the game?

3. Why are the answers to Parts 1 and 2 in this problem different?

4. What fraction of her total run did Kendra complete before stopping for water?

Problem 240. A pee wee league basketball team goes out for pizza and orders 3 large pizzas. Their tallest player eats $\frac{1}{4}$ of the first pizza, $\frac{1}{2}$ of the second pizza, and $\frac{1}{4}$ of the third pizza.

1. How much pizza did the tallest player eat?

2. What fraction of the order of pizza did the tallest player eat?

3. Which of the previous parts of this problem corresponds to $\frac{1}{4} + \frac{1}{2} + \frac{1}{4}$? Why?

Problem 241. Complete the following:

1. Estimate $\frac{7}{8} + \frac{5}{6}$ without writing down any computations.

2. Compute $\frac{7}{8} + \frac{5}{6}$.

3. Write a word problem in which the solution is to compute $\frac{7}{8} + \frac{5}{6}$. Be sure to solve your problem and show the meaning of the solution in context.

4. Use your work in this problem to explain why $\frac{7+5}{8+6}$ does not give a reasonable answer to $\frac{7}{8} + \frac{5}{6}$.

Problem 242. Complete the following:

1. Estimate $\frac{3}{5} + \frac{4}{7}$ without writing down any computations.

2. Compute $\frac{3}{5} + \frac{4}{7}$.
3. Write a word problem in which the solution is to compute \( \frac{3}{5} + \frac{4}{7} \). Be sure to solve your problem and show the meaning of the solution in context.

4. Use your work in this problem to explain why \( \frac{3 + 4}{5 + 7} \) does not give a reasonable answer to \( \frac{3}{5} + \frac{4}{7} \).

**Problem 243.** Two students each buy identical pads of paper. Late in the semester, Leticia has used \( \frac{9}{11} \) of her paper, while Margaret has used \( \frac{3}{4} \) of her paper. Who has used a larger fraction of their paper? How much more of the pad has she used?

**Problem 244.** For this problem, do not use a calculator. Compute \( \frac{17}{35} - \frac{4}{77} \) using the common denominator, \( 35 \cdot 77 = 2695 \). Then re-compute the difference using the least common denominator of 35 and 77. Which method do you find easier? Why do you think students are taught to add and subtract fractions using the least common denominator?

### 5.4 Multiplication Of Fractions

**Problem 245.** A square mile is a unit measurement of area equal to a square with sides that measure one mile. For example, the Greater Los Angeles Region has an physical area of 4,850 square miles of land. The shape of the region is of course not a square. The area of the region is equivalent to 4,850 one mile by one mile squares. Suppose a farm is \( \frac{3}{4} \) mile by \( \frac{1}{5} \) of a mile. How big is the farm, measured in square miles?

As with whole number multiplication, the area model is a way to find the product of two fractions so that there is a visual way to interpret the product. For instance, to find the product \( \frac{2}{5} \times \frac{4}{3} \), we will draw a rectangle. The rectangle is then cut vertically into thirds, and horizontally into fifths, and finally, the rectangle is shaded so that what is shaded has a width of \( \frac{2}{5} \) of the width of the rectangle, and the height is \( \frac{4}{3} \) of the height of the rectangle, as shown in Table 5.1. Notice that the resulting rectangle has the whole cut into fifteenths, and eight of them are shaded, so that the answer is \( \frac{2}{5} \times \frac{4}{3} = \frac{8}{15} \).

**Problem 246.** A new parking lot is being built next to a stadium. The lot measures \( \frac{3}{10} \) of a mile by \( \frac{2}{3} \) of a mile. How big is the lot, measured in square miles? Use a diagram to explain your answer.

**Problem 247.** Show how to solve each multiplication below using an area model.

1. \( \frac{1}{3} \times \frac{1}{4} \)
Table 5.1: A rectangle cut vertically into thirds, then horizontally into fifths, and finally shaded.

\[
\begin{array}{c}
\frac{2}{3} \\
\frac{3}{5} \\
\frac{4}{5}
\end{array}
\]

2. \(\frac{5}{3} \times \frac{1}{4}\)
3. \(\frac{3}{5} \times 4\)
4. \(\frac{5}{7} \times 4\)
5. \(\frac{7}{8} \times \frac{2}{3}\)
6. \(\frac{8}{7} \times \frac{3}{2}\)

**Problem 248.** Explain the mathematical reason we “multiply straight across” when multiplying fractions. The area model may be helpful to you in your explanation.

**Problem 249.** Write your own word problem involving multiplication. Be sure that your problem is phrased as a question and that you include the solution. The solution to the problem should illustrate that multiplication does not always “make things larger.”

**Problem 250.** A mixed number is a number such as \(5\frac{3}{4}\), which has an integer and a fraction side by side. The number \(5\frac{3}{4}\) is a way of writing \(5 + \frac{3}{4}\). Usually, when we multiply mixed number fractions, we are told to first convert to an improper fraction. The area model can be used to multiply fractions without converting to an improper fraction. Try it out on these examples:

1. \(1\frac{2}{3} \times \frac{1}{4}\)
2. \(3 \times 1\frac{1}{6}\)

**Problem 251.** In a small town, \(\frac{2}{3}\) of the men are married to \(\frac{3}{5}\) of the women. (Assume one-man-to-one-woman marriages only.) What portion of the town is married?

### 5.5 Division Of Fractions

For each of Problems 252 to 263,
• Represent and solve the problem using either a diagram or a number line.

• Clearly label the unit of measure (one cup, one tablespoon, etc.) in your diagram of the problem.

Problem 252. Enrique has $2\frac{1}{4}$ cups of orange juice. He drinks $\frac{3}{4}$ of a cup of orange juice every morning. How many days will Enrique’s orange juice last?

Problem 253. Tran has a recipe calling for $\frac{2}{3}$ tablespoon of vanilla. She has $3\frac{1}{3}$ tablespoons of vanilla. How many times can she make the recipe?

Problem 254. Delia has $7\frac{1}{2}$ pounds of concrete mix. If it takes $1\frac{1}{2}$ pounds of concrete mix to anchor a fence post, how many posts can she anchor?

Problem 255. Norma has $3\frac{1}{2}$ ounces of frosting. It takes $\frac{1}{6}$ ounce of frosting to make a baby rose. How many baby roses can Norma put on her cake?

Problem 256. Review your answers so far. Look for a relationship between the fractions used and the answers to each problem. Describe what you observe. You are looking for a new way of dividing fractions without “invert-and-multiply.”

Problem 257. Crystal plans to run 4 miles. Each day, she runs $\frac{1}{2}$ of a mile. How many days will it take to run 4 miles?

Problem 258. Faith has a hot tub 3m wide. If she wants to place new tiles along the width of the pool (just one side), and the tiles are $\frac{3}{5}$ of a meter long, how many tiles will she need? How many of the same kind of tiles would she need if she were to put them along one length of her pool, which is 15m?

Problem 259. Miguel has $4\frac{1}{2}$ gallons of gasoline for his chainsaw. If the chainsaw uses $\frac{1}{3}$ of a gallon per hour, for how many hours can Miguel use the chainsaw?

Problem 260. Maria has $5\frac{1}{6}$ ounces of perfume. She has a supply of bottles of which each holds $\frac{5}{8}$ of an ounce. How many bottles can she fill? How much perfume goes in the last bottle? What fraction of the last bottle will be filled?

Problem 261. Dina notices that it takes $\frac{2}{5}$ of a minute to fill a water bottle. If she spends 5 minutes filling water bottles, how many bottles will she fill? How full is the last water bottle?

Problem 262. Alberto solved Problem 261 and got an answer of $7\frac{1}{2}$. Bianca solved the same problem and got an answer of 7 remainder $\frac{1}{3}$. How would you respond to each student?
Problem 263. It takes $1\frac{2}{3}$ pizzas to feed a junior league basketball team. If the league gets a donation of $5\frac{2}{3}$ pizzas, how many teams can be fed?

Problem 264. Reflect on the problems in this section by answering the following: When dividing fractions, how do you find the remainder? How do you deal with remainders? How do you get your answer as a mixed number?

Problem 265. Write your own word problem involving division. Be sure that your problem is phrased as a question and that you include the solution. The solution to the problem should illustrate that division does not always result in something smaller.

Read the article:

Journal 6. After reading the article by Gregg and Gregg, answer the following:

1. A student says, “I heard that when you divide the answer always gets smaller.” What could this student’s experiences have been with division, such that the student would have this misconception?

2. You have experienced and read about the common-denominator method for division of fractions. What are some advantages and disadvantages of this method compared to the invert-and-multiply method? (You may refer to Bass’ criteria.)

3. How do the partitive (also called fair-sharing) and repeated subtraction (also called measurement) models of division help students understand division with fractions?

5.6 Decimals

Informally, when we say “decimals,” we mean a number written with a decimal point. Ultimately, decimals are a special form of fraction, where instead of writing, for instance, $\frac{43}{100}$, we write 3.43. There are two main ideas in this section. The first is to understand how different ways of writing a decimal are equivalent, using what you know about rational numbers. The second main idea is to gain some insight into where the “rules” for multiplying and dividing decimals come from.45

Problem 266. Show that .30 is equivalent to .3, using one of the definitions of equivalent fractions.
Problem 267. Recall that with base ten blocks, we had the flat, the long, and the unit. Suppose now that the flat has a value of 1. What would be the value of the long in this case? What would be the value of the unit?

Problem 268. Use base ten blocks to explain why .30 is equivalent to .3.

Problem 269. Show that .43 is equivalent to .4300, using the definition of equivalent fractions.

Problem 270. Suppose that gum is sold in packs with 10 sticks of gum. A carton of gum contains 10 packs, a case of gum contains 10 cartons, and a pallet of gum contains 10 cases. In this context, what would it mean to have .255 pallets of gum?

Problem 271. Show that .43 is equivalent to .4300, using the context of a pallet of gum.

Problem 272. One day, Aidan, a second grader, said, “There is no point ten because after .9 comes 1.” How would you respond to Aidan?

Problem 273. Diana says that you can add as many zeroes to a decimal as you want, and it does not change the value. She says this is because when you add zero, you are adding nothing. How would you help Diana distinguish between something like .43 + 0 = .43 and something like .43 = .430? What language might you suggest instead of “adding zero” in the second case?

Problem 274. Look closely at the pattern in Table 5.2. Describe in words the relationship between the number of factors of 10 (left and center columns) and the number of 0’s following the 1 in the product (right column).

<table>
<thead>
<tr>
<th>n</th>
<th>10^n</th>
<th>10^n = 10^0 = 1.0</th>
<th>= 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10^0 = 1</td>
<td>1 = 1.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10^1 = 10</td>
<td>10 = 10.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10^2 = 10 · 10</td>
<td>100 = 100.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10^3 = 10 · 10 · 10</td>
<td>1000 = 1000.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10^4 = 10 · 10 · 10 · 10</td>
<td>10000 = 10000.</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10^5 = 10 · 10 · 10 · 10 · 10</td>
<td>100000 = 100000.</td>
<td></td>
</tr>
</tbody>
</table>

Problem 275. Look closely at the pattern in Table 5.3. Then:

1. Describe in words the relationship between the number of factors of $\frac{1}{10}$ (left and center columns) and the number of 0’s preceding the 1 in the product (right column).

2. Why does this formula seem a bit different than the previous problem?

3. How many times do we “move the decimal” from its position at the right of the 1 in $10^0$? Does this make the formula seem more consistent with the previous problem?
Table 5.3: Powers of 10–Negative exponents

<table>
<thead>
<tr>
<th>$10^0$</th>
<th>$1$</th>
<th>$=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>$\frac{1}{10}$</td>
<td>$=.1$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$\frac{1}{100}$</td>
<td>$=.01$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$\frac{1}{1000}$</td>
<td>$=.001$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>$\frac{1}{10000}$</td>
<td>$=.0001$</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>$\frac{1}{100000}$</td>
<td>$=.00001$</td>
</tr>
</tbody>
</table>

**Problem 276.** Use a diagram to explain how an area model can be used to show $0.3 \times 0.7$. Be sure to indicate the unit of measure in your diagram.

**Problem 277.** Explain, using what you learned about powers of 10 and fraction multiplication in the previous problems, the reason that in multiplications involving decimals, we “count the total number of decimal places in the problem and then move the decimal that many places in the answer.”

**Problem 278.** Students sometimes get the idea that in order to multiply a number by 10, they can just annex a zero, as in $25 \times 10 = 250$. Does this work for $2.5 \times 10$? Explain.

**Problem 279.** Compute $0.6 \div 0.03$. Explain why each of the following are equivalent to $0.6 \div 0.03$.

1. $\frac{6}{10} \div \frac{3}{100}$
2. $6 \div \frac{3}{10}$
3. $60 \div 3$

**Problem 280.** Compute $0.78 \div 0.6$. Then explain why each of the following are equivalent to $0.78 \div 0.6$.

1. $\frac{78}{100} \div \frac{6}{10}$
2. $\frac{78}{10} \div 6$

**Problem 281.** When dividing decimals, we are usually told to “move the decimal in the dividend and divisor to the right by the number of places in the divisor, and line up the decimal in the dividend over the decimal in the quotient.” For instance, in Problem 279, we would rewrite $0.6 \div 0.03$ by “moving the decimal” in both numbers to get $60 \div 3$. Explain why is valid, using the fraction form of the decimals to help you if necessary.
6.1 Proportion and Ratio

**Notation 282.** The ratio\(^4^7\) of two quantities \(c\) and \(d\) may be written as \(\frac{c}{d}\) or \(c : d\).

**Definition 283.** A proportion is a statement that two given ratios are equal. For example \(\frac{2}{3} = \frac{4}{6}\) is a proportion since the two fractions are equivalent.

**Problem 284.** On a map, 1 mile is represented by 1 centimeter (cm). If a traveler measures a distance of 3 cm on the map, how far apart are the two locations? Justify with a diagram.

**Problem 285.** If you travel 12 miles in 15 minutes, what is your speed in miles per hour? Justify with a diagram.

**Problem 286.** Suppose in a class there are 28 students. The ratio of girls to boys is 4:3. How many girls and boys are there in the class? Justify with a diagram.

**Problem 287.** While reading a novel, Mrs. Jones finds she can read 75 pages in 60 minutes. How long will it take her to read a book that has 400 pages?

**Problem 288.** Suppose a person 6 feet tall is standing next to a tree. The tree has a shadow measuring 21 feet. The person’s shadow measures 7 feet. How tall is the tree?

**Problem 289.** An actor and his agent agree to split his contract for the next movie in the ratio 9:1. If the contract is worth $3 million, how much money does each person get?

**Problem 290.** The top three finishers in a tournament split the prize money in the ratio 3:2:1. If the total prize money is $90,000, how much money does each player earn?
Problem 291. A farmer estimates that out of every 100 seeds of corn planted, 85 ears of corn will be harvested. Each ear of corn is about 1 foot long. If 7500 ears of corn are to be harvested, how many seeds must he plant?

Problem 292. The gravitational force on the moon is less than the gravitational force on Earth. If a person who is 6 feet tall weighs 175 pounds on Earth and weighs 28 pounds on the moon, then how much does a 5-foot tall, 115-pound person weigh on the moon?

Problem 293. A dog chases a raccoon that has a 150-foot headstart. If the dog runs 9 feet for every 7 feet run by the raccoon, how long will it take the dog to catch the raccoon?

Problem 294. In a coffee shop, two types of beans are combined to form the house blend. One bean sells for $8 per pound, while the other type sells for $14 per pound. They mix up batches of 60 pounds of the house blend at a time, and sell them for $9.50 per pound. How many pounds of each type of coffee go into the house blend?

Problem 295. For every 100 attendees at a concert, the band expects to sell 27 T-shirts.

1. If they have sold 1350 tickets to their upcoming show, about how many T-shirts can they expect to sell?

2. If each T-shirt sells for $20, what can the band expect the total sales value of the T-shirts at the concert to be?

Problem 296. At a candy store, fancy chocolates sell for $12 per pound. If Debra picks out chocolates weighing 10 ounces, how much will she pay for them? (There are 16 ounces in a pound.)

Problem 297. Eight workers can pave a 1-mile stretch of road in 6 hours. If crews of eight workers all work at the same rate, how many workers are needed to pave an 8-mile stretch of road in 12 hours?

Problem 298. Six workers can build a house in 3 days. Assuming that all of the workers work at the same rate, how many workers would it take to build the house in 1 day?

6.2 Percent

Definition 299. The term percent literally means per hundred. Thus N percent means that there are N objects per 100 total objects. Percent is denoted by the symbol %.

Problem 300. Using the definition of percent, explain what it means if 28% of the population in a city is under the age of 30.
Problem 301. 1. Express 28% as a fraction. Use the definition of percent to justify your answer.

2. Express 28% as a decimal. Justify your answer.

3. Explain why percents, decimals, and fractions might be called “three ways of expressing the same thing.”

Problem 302. Using the definition of percent, explain what it means if the price of gold is 5.5% higher this month compared to last month.

Problem 303. Using the definition of percent, explain what it means if 0.5% of the toys produced in a factory have a defect.

Problem 304. Using the definition of percent, explain what it means if the price of gasoline is 130% higher today than it was 6 years ago.

Use Figure 6.1 for Problems 305 to 309, and justify your answers:

Problem 305. Show 50% of 40.

Problem 306. Show 25% of 40.

Problem 307. Show 20% of 40.

Problem 308. Show 10% of 40.

Problem 309. Show 5% of 40.

Use a number line or area model to answer Problems 310 to 315. The point of these problems is not just to compute an answer, but to use your understanding of the meaning of percent to illustrate the answer with a diagram.

Problem 310. What is 35% of 80?

Problem 311. What is 28% of 125?

Problem 312. What percent of 40 is 18?

Problem 313. What percent of 60 is 51?

Problem 314. 33 is 55% of what number?

Problem 315. 96 is 75% of what number?
Problem 316. For this problem do not use a calculator or rules that you memorized from previous math courses. Use the definition of percent to justify your answer. Part (a): If the value added tax (VAT) in France is 1 euro for every 5 euros spent, then what percent is the VAT? Part (b): If the VAT was 1 euro for every 4.6 euros spent, then what percent is the VAT?

For Problems 317 to 320, compute the answer using any method:

Problem 317. 27 is what percent of 89?
Problem 318. 89 is what percent of 27?
Problem 319. What is 0.05% of 25?
Problem 320. What is 128% of 30?

6.3 Problems Involving Percent

Problem 321. There is a sale at your favorite store. Prices were 25% off yesterday. Today you can take an extra 25% off the reduced price. Is this the same as taking 50% off the original price?

Problem 322. A store is closing. Originally, prices were reduced by 75% from the full retail price. Now, they have reduced prices an additional 50% off the reduced prices. Does that mean the store is paying you to take the clothes?? If not, how much do you save compared to the original price?

Problem 323. An importer pays $28 per sweater, and then marks up the price 50% when she resells to the wholesaler. The wholesaler takes his purchase price and marks it up 100% when he sells to the retailer. The retailer marks her purchase price up 150% to sell the sweaters to customers. What is the price the customer is expected to pay? What is the overall percent markup of the sweater?

Problem 324. Suppose you put $100 into a savings account paying 2.5% interest each year (compounded annually). What percent will you earn if you leave the money in the account for 3 years?

Problem 325. Explain why the answer to Problem 324 is NOT 2.5% × 3 = 7.5%.

Problem 326. One student offered the following as a solution to Problem 324: 100 × 1.025 × 1.025 × 1.025 = 107.69. 107.69 − 100 = 7.69. So the answer is 7.69%. Explain this student’s work.

Problem 327. A newspaper story said that on any given day, 7 percent of all Americans eat at McDonalds. There are about 9000 restaurants in the U.S. and at the time of the article, around 275 million Americans. According to a former employee, it takes about 3-4 minutes to fill an order. Do you think the newspaper claim could be true? If not, what might explain the claim?
Problem 328. (Use a calculator for this problem.) Suppose you invest $1000 into an individual retirement account (IRA). You will retire in about 30 years. Your IRA grows on average about 7% per year. How much money will be in your IRA in 30 years?

Problem 329. Alana invested $3000 in stocks in 2001 and sold them for $4000 in 2008. What percent profit did she make on her investment?

Problem 330. Bart bought Alana’s stocks for $4000 in 2008, but he had to sell them for just $3000 in early 2009. What percent of Bart’s investment did he lose?

Problem 331. Do the answers to Problems 329 and 330 match? Explain why or why not.

Problem 332. Fantasy School District has two schools. In a recent poll, 309 of 602 students at Da Vinci Academy selected mathematics as their favorite subject, while 283 of 361 students at Euler (pronounced to rhyme with boiler, not ruler) Elementary selected mathematics as their favorite subject.

1. In which school is mathematics more popular, based on this data? Explain how you determined your answer. Why might someone choose the other answer?

2. The Da Vinci student newspaper wants to report the poll results using “friendly” percentages. Explain one way that they might report the poll findings.
Chapter 7

Patterns and Variables

Nancy is a landscape artist. Her specialty is a square pond that is surrounded by hand-painted tiles. Customers can order the pond in any size square, starting with sides measuring 2 feet, and available in one-foot increments thereafter (sides of 3 feet, 4 feet, 5 feet, etc.). The tiles are 1-foot square, and are placed edge-to-edge along the entire outer perimeter of the pond. Problems up through 341 refer to this context.

Table 7.1: Pond with 2-foot sides and square tile border

<p>| | | | |</p>
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</table>

**Problem 333.** Make a table showing the number of border tiles needed for different pond sizes, from 2-foot sides up through 6-foot sides.

**Problem 334.** Describe how the number of border tiles needed for one pond size grows as the length of the sides of the pond increase. Explain why the number of tiles is increasing according to this pattern.

**Problem 335.** How many tiles are needed for a pond with sides of length 12 feet?

**Problem 336.** If Nancy orders 64 tiles for an upcoming job, how large is the pond the customer wants?

**Problem 337.** Describe in words how to find the number of tiles needed for the border of a pond.

**Problem 338.** A student, Jada, described the number of border tiles this way: “If the customer tells you the length of the pond’s sides, you times the length by 4 for the sides, and then add 4 more for the corners.” Use Jada’s method to find the number of tiles needed for a pond with sides that measure 12 feet.
Problem 339. Another student, Karina, described how to find the number of border tiles this way: “If you know the side length, then you times the side length by itself. Then, you take the side length and add 2, and times that new number by itself too. Then you subtract the bigger number from the smaller number.” Use Karina’s method to find the number of tiles needed for a pond with sides that are 12 feet.

Problem 340. Will Jada’s and Karina’s methods always give the same answer? Explain why or why not.

Problem 341. Describe how many tiles would be needed for a border around a pond with sides of length n feet.

Problem 342. Table 7.2 shows three number patterns, f(n), g(n), and h(n). For each function, fill in the blanks in the table, and make a graph of the values in the table. Then, write a function equation that gives the output value corresponding to any input. Be sure to state any assumptions you make about the way the functions behave.

<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
<th>g(n)</th>
<th>h(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>14</td>
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<td>3</td>
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<td>4</td>
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<td>6</td>
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<td></td>
<td>2</td>
</tr>
<tr>
<td>20</td>
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</tbody>
</table>

Table 7.3: Checkers’ Price for a medium pizza

<table>
<thead>
<tr>
<th>Number of toppings</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (plain)</td>
<td>6.00</td>
</tr>
<tr>
<td>1</td>
<td>6.50</td>
</tr>
<tr>
<td>2</td>
<td>7.00</td>
</tr>
<tr>
<td>3</td>
<td>7.50</td>
</tr>
<tr>
<td>4</td>
<td>8.00</td>
</tr>
</tbody>
</table>

Checkers’ Pizza offers medium pizzas at prices according to Table 7.3. Problems 343 to 348 refer to Table 7.3.

Problem 343. Graph the Checkers’ Pizza pattern in the table.

Problem 344. If Ana wants a medium pizza with 6 toppings, how much will it cost, assuming the pattern in the table continues?

Problem 345. If Rey spends $10.50 on a medium pizza, how many toppings did he order?
Problem 346. Write a function equation that describes the price of a medium pizza for a given number of toppings.

Problem 347. If Keon has $9.85 to spend on a pizza, how many toppings can he get? Why might the function equation or the graph be misleading in answering this question?

Problem 348. Checkers’ competitor, Papi Juan’s pizza, plans to open a restaurant in the same neighborhood. Papi Juan wants his plain pizza price to be lower than Checkers’ price. Juan also knows that the two-topping pizza is the most popular, so he wants to set his two-topping price higher than Checkers’ two-topping price. However, Juan doesn’t want his two-topping price to be higher than Checkers’ three-topping price. Describe some possible pricing solutions for Papi Juan, assuming he uses a fixed price for each additional topping.

A child is building a tower out of cubes, stacking one on top of another. She notices that with one cube, there are 5 exposed faces, and with two cubes, there are 9 exposed faces. Solve Problems 349 to 357 by referring to the tower of cubes context.

Problem 349. Make a table to show the number of exposed faces for towers built of as many as 5 cubes.

Problem 350. Graph the points listed in your table.

Problem 351. Write a function equation to describe the pattern shown in the table.

Problem 352. Where on the graph can you find the numbers that appear in the function equation?

Problem 353. Can a tower have 25 painted faces? If so, how many cubes must it use?

Problem 354. Can a tower have 31 painted faces? If so, how many cubes must it use?

Problem 355. Can a tower have exactly 31 painted faces? If so, how many cubes must it use?

Problem 356. Write an equation that describes how many cubes are needed to have a tower with exactly $F$ faces.

Problem 357. On another day, the child builds a tower with a different set of blocks. If she uses $n$ blocks, the tower has $6n + 1$ exposed faces. What might the blocks look like?

Problem 358. The tables in Figure 7.1 each describe rectangular arrays of tiles. For each table, assume that the width and height functions each grow by a fixed number from one array number to the next. Complete parts (1)-(5) for each pattern.
1. Draw diagrams of the arrays.

2. Write descriptions of the number of tiles as a function of the array number.

3. Graph the values of the function described. Also graph the width and height as functions of the array number.

4. Write equations to describe each of the three functions from above, then graph the equations.

5. What do you observe about the relationship between the width and height functions and the number-of-tiles function?
Notes to the Instructor

The students begin the semester with base five. The primary purpose of this is to get students to think about what counting in a base-system means, and to realize the difficulty children face in attempting to learn the algorithms taught in primary grades for addition, subtraction, multiplication, and division. Base sixty work reinforces this theme. Base twelve underlines the idea that the number of symbols needed is directly related to the base. As noted in “To The Instructor,” you will need to have students print base five blocks onto card stock or have some alternative such as algebra tiles ready.

Students may choose to interpret multiplication as repeated addition, for instance by seeing $2_{five} \times 3_{five}$ as $3_{five} + 3_{five}$. As they proceed to problems such as 16, they may continue to use repeated addition, or they may find ways to use an understanding of bases, the distributive property, and place value to do something like,

$$13_{five} \times 24_{five} = 10_{five} \times 24_{five} + 3_{five} \times 24_{five} = 240_{five} + (3_{five} \times 20_{five} + 3_{five} \times 4_{five}) = 240_{five} + (110_{five} + 22_{five}) = 240_{five} + 132_{five} = 422_{five}$$

Generally, the students are left to their own inventions, but always with the requirement to justify their work to themselves, their group, and to the class if they present their solution. Moreover, students are allowed to choose the methods that make sense to them, rather than push everyone to the “most efficient” approach. This is all laying groundwork for understanding how to approach the teaching of base ten to elementary students.

Problems such as 25 lead to interesting conversations about whether the units are hours and minutes or minutes and seconds, and whether it matters.

The goal of introducing expanded form is primarily to begin to get the students to focus on place value for later explorations of algorithms for the operations on natural numbers and again in number theory when exploring divisibility rules. In this unit, you may choose to use base ten blocks, similar to the use of base five blocks in chapter 1, or you may allow the students to draw the blocks without actually using them. It is absolutely necessary to at least have the drawings for students to gain an appreciation for the process of regrouping and to prepare for the area model of multiplication.

The errors from students in this section are designed to help the students realize the many ways in which place value is at the root of elementary students’ problems in executing the standard algorithms. Occasionally, students come up with other explanations for
why certain errors occur. Generally, since these are fictitious students, any explanation is acceptable, but the point is to draw from the entire discussion to highlight the importance of understanding the role of place value.

9 Here the student regrouped the 4 to the tens place, rather than the 1.

7 Here the student did not regroup the 1 ten with the 5 tens, instead leaving everything on one line.

5 Here the student again did not regroup the 1 ten with the 5 tens, instead leaving everything on one line.

9 Here the student regrouped both the 1 ten and the 1 hundred to the hundreds place, whereas the 1 ten (from $6 + 8$) should have been in the tens place.

10 Here the student proceeded to “subtract the smaller from the larger,” so that in the ones, since 9 is larger than 6, the student took $9 - 6 = 3$ in the ones place. Therefore no regrouping was necessary in the tens place.

11 Here the student subtracted in the ones without taking a ten from the 7 tens in 76.

12 Here the student proceeded to “subtract the smaller from the larger,” as before.

13 Here the student subtracted the ones correctly with regrouping, but did not take a ten from the 7 tens in the regrouping process.

14 Here the student regrouped one hundred from the 7 hundreds as 10 tens AND as 10 ones, rather than as 9 tens and 10 ones.

15 This assignment is very important and within it, partial sums is the most important algorithm. The goal is to build in students a flexibility in understanding what constitutes a valid way of performing addition, as well as to help them reason with unfamiliar but mathematically valid approaches to performing computations. Partial sums is a vehicle for helping students realize that numbers are added by summing within each place value, and that all other algorithms essentially use a shorthand for this process. Please note that explanations of the algorithms provided to students are purposely not emphasizing place value, as the students’ role is to explain how place value is relevant to each algorithm.

16 This problem is provided so that a clear discussion of the shortcomings of student attempts can be had, where the students in your class do not have to be the owners of the errors. This sometimes allows students to be more openly critical, which is what is needed if the students are to become effective in communicating their mathematical reasoning. By the way, these are actual descriptions given by students in my classes.

17 In this section, students are asked to examine alternative ways to subtract whole numbers in the decimal system. First, the goals of this section parallel those of addition, namely to build understanding of different valid ways of performing computation, and to build skill in communicating mathematical reasoning. Note that the compensate algorithm is not, strictly speaking, an algorithm, because it is meant to be used flexibly, where different students may apply it differently to the same problem. Next, note that the compensate and missing addend methods are transliterations of mental subtraction algorithms. Also, the subtraction with negatives and other approaches to subtraction were introduced in the Bass
Building on the goals of the previous sections, in multiplication, the students must understand and reason with unfamiliar methods of computation. In addition, it is of particular importance that the students understand the area model, as it will become an essential tool for representing multiplication of fractions. The students are given the opportunity to build a base ten area model (either physical blocks or a diagram) and to connect the algorithms offered to the base ten area model, so that they understand how the four multiplications in a two-digit-by-two-digit problem are represented and combined in different methods for multiplication. The area model shown in Table 2.3 purposely does not have the subsections labeled. One of the main realizations for the students to have is that the model can be separated into 4 sections corresponding to $20 \times 30$, $7 \times 30$, $20 \times 4$, and $7 \times 4$.

In a somewhat different approach than addition and subtraction, instead of the students explaining the algorithms in terms of place value, students are asked to connect the methods to each other. Nonetheless, place value should enter the discussion, and the students should be able to understand why the different methods work.

For problem 85, students often come up with $28 \times 18 = 28 \times 10 + 28 \times 8$, which uses the distributive property. However, since the addition problem can be thought of as using the associative property: $28 + 18 = 28 + (2 + 16) = (28 + 2) + 16$, $28 \times 18 = 28 \times (2 \times 9) = (28 \times 2) \times 9$ is a closer analogy to the addition problem. However, both questions are accurate response to the prompt.

The intermediate algorithm is central to the goal of having students understand how long division works. As an alternative, the partial quotients can be written to the right of the division, because having the partial quotients at the top is sometimes confusing for students.

In spite of this algorithm and the associated journal, students still have trouble explaining what a zero in a place in the quotient means.

This section concludes the first cohesive unit of the course, which could be subtitled “primary grades mathematics,” and is a good place to put an exam.

For problem 106, the answer will depend in part upon what sorts of numbers students consider. In particular, the conclusion “It must be greater than 15,” depends on the students assuming that the number is positive, in which case the conclusion is correct.

The number theory assignments are the most challenging portion of the course. As elsewhere, the emphasis is on visual representation. So, in this case rods (such as the colored Cuisenaire rods) are used to demonstrate multiples. Then, students can draw generic diagrams to illustrate what it means to be a multiple of a number, e.g. 12 is a multiple of 4 because a length of 12 can be made from rods of length 4, or alternatively because 12 can be made from 4 rods of same length (where the length is a whole number). Students may move back and forth between visual representation and the more abstract algebraic notation throughout this chapter. However, students are typically more adept at explaining that adding red rods with red rods produces a length that is measurable with red rods than they are at explaining how $2x + 2y = 2(x + y)$ shows that the sum of two even numbers is even. Also, although the initial definitions for even and for divisibility allow for all integers, the
chapter is primarily concerned with whole numbers 0, 1, 2, 3, . . . .

Again, a major goal of this section is to build students’ mathematical reasoning, along with helping them understand the basic terminology, and understanding how visual representations can assist in developing understanding of concepts.

This section extends the ideas about multiples and combines it with place value to stretch students’ understanding of both concepts. This section is also the one which most represents Doing Mathematics, because students come up with their own rules and justify them. The mathematics of this chapter is not essential to later material. This section is provided mostly as an opportunity for students to understand that familiar rules can be built on understanding.

Each problem is built to help students discover why the rule for divisibility works, and the sequence of problems is designed so that later problems put more of the burden of building an explanation on the students.

Students may come up with one of two different rules. One is that a number is a multiple of 4 if and only if the last two digits form a multiple of 4. A different description is that a number is a multiple of 4 if and only if, when the tens digit is even, then the ones digit is 0, 4, or 8, or when the tens digit is odd, then the ones digit is 2 or 6.

This section deals only with basic ideas of primes and factorizations. It is background for the work on greatest common factor and least common multiple.

Notice that at this point the discussion is limited to natural numbers.

The difficulty of greatest common factor for students is the trouble of keeping track of a divisor or factor as opposed to a common multiple. Beyond this, students are mostly reviewing a concept that they know. However, there are opportunities for students to learn from each other the different ways of computing the gcf. This goes to reinforce the need for the students as future teachers to know different ways of computing the gcf, since different approaches will help different students.

On Problem 160, students should come to the realization that the gcf divides the difference of the numbers. This is a nice way of reviewing and extending what they learned from the section on divisibility.

In this section, the goals parallel those for gcf, particularly the idea of learning multiple ways of computing the lcm. Students are first given numbers wherein it is reasonable to find the lcm by listing multiples of each number; however, the problems build so that the need for a more sophisticated strategy presents itself. A few strategies are presented in Problems 172 to 174. Depending on the strategies the students invent for themselves, some of these might be named for students in the class rather than what appears in the text. This gives students ownership of their mathematics.

The main goal for students is to give them concrete ways of understanding what many have simply memorized as “the rules” for integers. Although each model has its weaknesses, they both allow students a grounding upon which to reason about how to operate with integers.

Two main models are presented to the students as a way to understand integers. Each
of these has strengths and weaknesses.

The hot/cold chip model is the following. The temperature of a tank of water is affected by putting in or taking out hot or cold chips. A hot chip can be modeled physically in the classroom as either a red counter (such as a bingo counter) or as a “+,” (tile spacers used for laying bathroom or kitchen tile are available at home improvement stores), and on paper by a +. A cold chip can be modeled physically in the classroom as either a blue counter (again, a bingo counter can be used, or blue counters can be purchased via educational supply stores), or as a “−” (the same tile spacers can be cut to create negative signs), and on paper are recorded as a −.

As is explained in the student notes, a single chip changes the temperature of the tank by either +1 degree or −1 degree. The actual temperature of the tank is not necessarily important, but instead it is the net change in temperature that is used. So if 5 hot chips are put in the tank, the change in temperature is +5. If 5 hot chips are put in along with 3 cold chips, as in Problem 178, then this is recorded on paper as +5 − 3 and the net effect is to change the temperature by +2. To see this more clearly, students are given the tool of zero pairs. Zero pairs are simply a pairing of equal numbers of hot and cold chips, which then have 0 net effect on the temperature. In Table 4.2, the problem +5 − 3 is shown as the students might be asked to do it, with + and − paired, so that the end result of +2 is clear in the “water tank” shown.

In Problems 183 and 184, students are asked to abstract from their process while still remaining grounded in the model of the hot-cold tank. That is, in the first case, they should be able to argue from their experience with the tank that in Problem 183, when adding two integers, whichever color or sign being added to the tank is more numerous will determine the sign of the sum. They might then work more abstractly in Problem 184 to describe how to use the absolute values of the summands to determine the sign of the sum, or to refer to summing integers of opposite sign as equivalent to subtraction, with the sign of the sum being the sign of the summand with larger absolute value. Note that since the term absolute value is not introduced in the notes, the instructor may want to introduce this language into students’ explanations during classroom discussion.

For subtraction, zero pairs become an important tool in the model. Zero pairs can be put into a tank without affecting the temperature in the tank. This allows subtraction to be modeled using the idea of taking away. For instance, if students are looking at Problem 185, students may initially start with −3 in the tank, but will need to put extra zero pairs in the tank in order to be able to take away −5. This is shown in steps in Table 7.4.

Table 7.4: Showing $-3 - (-5)$ in the hot/cold model

<p>| | | |</p>
<table>
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<th></th>
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<tbody>
<tr>
<td>$-3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+$</td>
</tr>
</tbody>
</table>

Result: +2
Table 7.5: Showing $-3(-5)$ in the hot/cold model

First show 15 zero pairs

| $- - - - -$ | $+ + + + +$ |
| $- - - - -$ | $+ + + + +$ |
| $- - - - -$ | $- - - - -$ |
| $+ + + + +$ | $- - - - -$ |

Then remove 3 groups of $-5$

| $- - - - -$ | $-3(-5)$ |
| $- - - - -$ | $+ + + + +$ |
| $- - - - -$ | $- - - - -$ |
| $+ + + + +$ | $- - - - -$ |

Result: $+15$

Subtraction on the number line is not demonstrated in the text to offer the instructor a choice of ways to model subtraction. Subtraction with the number line can be modeled as “the opposite of addition.” So $-3 - (-5)$ would be modeled by first showing a $-3$ arrow with its tail at 0, and then taking the opposite of a $-5$ arrow, which is a $+5$ arrow, and putting it with its tail at $-3$. This is shown in Figure 7.2.

An alternative to this is to model subtraction as the difference (or missing addend) and ask students to find the arrow that indicates the difference between the arrows $-3$ and $-5$. However, the students may have difficulty orienting the arrow in the correct direction (i.e. it is the vector with tail at $-5$ and tip at $-3$). I have not tried this, but offer it as an alternative.

The most difficult concept is multiplication of a negative by a negative. In the hot/cold model, Problem 4.2, $-3(-5)$, would be modeled as, “Take out 3 groups of 5 cold (−) chips.” This requires the use of zero pairs, as shown in Table 7.5.

Although not contained in the notes, it is also possible to use a number line to model integer multiplication. In this case, the first factor is the number of arrows of length and direction determined by the second factor, with a negative sign on the first factor indicating to “take the opposite.” So $3 \times (-5)$ is an indication to put 3 arrows that each point left with length 5 units, to reach $-15$, while $-3 \times (-5)$ would mean putting those same arrows, but then taking the opposite of the result, to get 15. Essentially, $-3(-5)$ is taken as $-(3 \times -5)$.
In addition, students are given the chance to see integer multiplication as a logical extension of multiplication with positive integers, using the idea of patterns, and to look at how the phenomenon of the product of two negative numbers yielding a positive number can be understood via the idea of additive inverse (see Problem 204).

Finally, students are asked in Problems 206 to 209 to use the missing factor model to extend their understanding of integer operations to division.

The goal of this chapter is to build in students an understanding of the procedures used for fraction operations as well as understanding of the connection between questions asked in context and the operation(s) required to solve them. Many students have limited tools for attacking word problems, beyond such beginning strategies such as “How many more?” means “subtraction.” Beginning with this assignment, diagrams are a primary tool for justifying that answers make sense.

In this particular section, a goal is to help students understand that a single object can be measured using different units, and that a question can request a specific unit of measurement.

The point of Problem 211 is to get students to begin to think of the same quantity as being measurable using different units.

Problem 213 asks for an answer in terms of sandwiches. However, students will often answer in terms of the entire lunch, or will say that each person gets \( \frac{1}{5} \) of each sandwich. While both of these answers accurately describe what a person gets, they do not answer the question as asked.

Problem 214 is emphasizing the point that a fraction is always a fraction in comparison to some unit of measurement, and that if the unit changes, then the meaning of, in this case, \( \frac{1}{2} \), changes.

In Problems 221 and 222, students will sometimes ask if there is a typo—are both questions the same? The answer is no. Students have to read carefully to distinguish \( \frac{1}{3} \) of a whole bar from \( \frac{1}{3} \) of the remaining bar.

One of the difficulties in writing these notes was in choosing an appropriate way to develop the idea of equivalence of fractions. Elementary textbooks use the convenient definition that two fractions are equivalent if they “represent the same amount.” This seemed unacceptable to me. College textbooks that deal with fractions sometimes use the cross-multiplication definition, which does not connect with most students’ intuitive way of understanding equivalence. Even after eliminating these choices, several possible definitions seemed within reach. I chose the definition presented because common denominators are important to being able to perform operations with two fractions.

The problems presented serve the dual goals of helping students to develop both a formal mathematical understanding of how to apply a definition and to develop skill in providing a kid-friendly way of explaining equivalence.

In Problem 235, more than one reason is possible, but the primary one is that “reduced” suggests that the fraction is getting smaller, whereas an equivalent fraction is still the same amount.
In Problem 236, one key issue in the elementary definition is that students may be confused by the implicit unit conversions, that is, equivalent fractions represent the same amount using the same unit of measure. For instance, \( \frac{1}{2} \) dozen eggs represents the same amount as 6 eggs, but we don’t want to declare \( \frac{1}{2} \) equivalent to 6—the problem is that the unit in the former case is a dozen eggs, and in the latter case it is one egg. A more insidious example is that if students are asked to compare, say \( \frac{2}{8} \) and \( \frac{4}{8} \), they may draw a smaller pizza to show \( \frac{2}{8} \) than the one they draw for \( \frac{4}{8} \), so that two fractions that should be equivalent do NOT appear equivalent based on the drawings. In contrast, Definition 224 resorts to computations, thereby removing any doubt about how to determine that fractions are equivalent.

The definition of fraction equivalence can be used as the basis for comparing fractions by common denominator, but in Problem 237, comparison by common numerator is also valid, and also arises out of Definition 224.

In this section, a major goal is for students to confront some of the major misconceptions surrounding addition and subtraction of fractions. In particular, they see scenarios, such as Problem 238, Part 4, where it IS permissible to get a common denominator, and then add the numerators and add the denominators. It is advisable for the instructor to point out this occurrence and contrast it with the usual addition of fractions that occurs in Part 2 of the same problem. The addition of fractions and misconceptions about the process are revisited in several of the problems.

Also, Problem 243 is designed to serve as motivation for subtracting fractions. Namely, just as with whole numbers, subtraction can be used to answer the question, How much more?

In this section, the goal is for students to extend the idea of the area model of multiplication to the multiplication of fractions. A key component of this way of modeling multiplication is to keep track of the whole, and the number of pieces that into which the whole is broken. Another issue that students may or may not raise is whether the whole must be a square. That is, it is often inconvenient to draw a square as the whole, since typically more cuts are going to be made in the one direction than in the other direction. So in that sense, the drawing is not a scale drawing of an area measured in square units.

Students will run into issues when dealing with multiplication involving improper fractions, or a fraction times a whole number. In particular, there are (at least) three ways to represent the multiplication involved in Problem 247, Part 2, namely those shown in Table 7.6, Table 7.7, and Table 7.8. In Table 7.6, the entire multiplication is represented as a single block. In Tables 7.7 and 7.8, the whole is separated clearly, so as to assist in identifying the denominator (here, twelfths) in the multiplication. In the two-block versions, it is perhaps easier to see the result as \( \frac{3}{12} + \frac{2}{12} = \frac{5}{12} \).

Notice that in the Problem 247, many of the diagrams are “the same,” until the diagram is labeled with appropriate units and shaded.

Answers to Problem 248 should describe something like how the product of the numerators is “how many pieces you have,” while the product of the denominators is “the unit of measurement,” or “how big the pieces are as a fraction of the whole.”

The major goal of this assignment is for students to realize that there is a common de-
nominator method for dividing fractions. Students are allowed to draw any sort of diagram that makes sense to them. Some prefer number lines, while others draw rectangles and cut them up and group them in various ways. If necessary, the instructor can help the students discover the common denominator idea by transcribing the numeric work of Problems 252 to 255. For example, in 252, \[ \frac{9}{4} \div \frac{3}{4} \], then \[ \frac{20}{6} \div \frac{4}{6} \], and so forth.

The remainder issue that comes up beginning with Problem 259 helps to clarify further the importance of unit when dealing with fractions. Problem 260 exposes students who rely solely on the memorized method of “invert-and-multiply,” because they find they cannot answer how much perfume goes in the last bottle directly from their invert-and-multiply answer.

After this section, students usually take their second exam.

45 Students will have the opportunity to use base-10 blocks and the area model again to reinforce and extend earlier understandings.

46 The problems associated with Table 5.2 and Table 5.3 are designed to lead in to Problem 277.

47 Again, visuals are a helpful tool for suggesting justifications for solutions. Many students can do the computation but cannot explain why it gives a reasonable answer. A diagram, such as a number line, or at least a ratio table, helps them to see proportions as scaling a ratio up or down.

48 As with ratios, students may be able to do simple problems but not understand them. In particular, they have a common error of always turning answers that are percents into numbers between 1 and 100. The main goals here are to understand percent as a particular fraction, and to realize that percents are always relative to some unit (as in the shopping problems).
Table 7.8: Representing $\frac{5}{2} \times \frac{1}{4}$ in TWO blocks another way

In Problems 305 to 309, the idea is for students to visually cut 40 into sections using their understanding of percents. For instance, 50%, being equivalent to $\frac{1}{2}$, would mean cutting 40 into two equal sections, and taking 1. Ten percent of 40 might be found by thinking of cutting 40 into ten sections. Five percent of 40 might be gotten by cutting 10% into two sections, or thinking of 40 in twenty sections, or even by thinking of 50% cut into ten sections.

The problems in this section build on earlier work to get students to be consciously aware of a percent as having a designated unit as 100%.

The main goal of this unit is to work on some typical algebra or pre-algebra level problems in context, with the idea that simple patterns can be represented in multiple ways that offer different insights about the underlying behavior.