Linear Topology

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Dedication

To Lee Mahavier, for her support during my first semester of graduate school.
To the Student

Topology is an area of mathematics, just as Algebra, Analysis and Geometry are areas of mathematics. Like other areas Topology is generally defined heuristically or not at all. The kind of problems we shall consider first are those that have to do with the concept of a limit point of a set of real numbers or the limit of a convergent sequence of real numbers as defined in an Analysis course such as Calculus. There the definition is made in terms of the distance between points and involves the concept of numbers being “near” one another. Precisely, the number $x$ is a limit point of the set $M$ of numbers if for every positive number $\varepsilon$, there is a point of $M$ that is different from $x$ and whose distance from $x$ is less than $\varepsilon$. We start by defining a limit in a more abstract setting. We do this by introducing a notion of “nearness” that does not depend on having a distance between points. So we might consider this part of topology the study of those concepts that might be defined in terms of limit points.

While topological spaces can be defined in very general settings, this sequence is restricted to the study of linear topology, that is the study of the topological properties of the real line. Many of the results that we will prove for the line hold in general topological spaces and often the proofs given in class will not use any properties of the line and thus are actually proofs for general topological spaces. In brief, our goal is an understanding of the following topics: open and closed sets, limit points, compactness, connectedness, measure of a set, sequences, convergence, least upper bound and greatest lower bound. While investigating these topics, we will be developing the tools that are needed for such courses as general topology, measure theory, functional analysis, differential equations, and so forth.
Problem Sequence

Definition 1. By a **point** is meant an element of the real numbers, \( \mathbb{R} \).

Definition 2. By a **point set** is meant a collection of one or more points.

Definition 3. The statement that the set \( S \) is a **topological space** means that there is a collection of subsets of \( S \), called regions, such that

1. if \( p \) is in \( S \) then there is a region that contains \( p \), and
2. if \( U \) and \( V \) are two regions having \( p \) in common then there is a region that contains \( p \) and is a subset of \( U \cap V \).

Definition 4. The statement that the point set \( M \) is **linearly ordered** means that there is a meaning for the words “less than (\( < \)),” “less than or equal to (\( \leq \)),” “greater than (\( > \)),” and “greater than or equal to (\( \geq \)).” If each of \( a \), \( b \) and \( c \) is in \( M \), then

1. if \( a \leq b \) and \( b \leq c \) then \( a \leq c \)
2. one and only one of the following is true:
   i. \( a < b \),
   ii. \( b < a \), or
   iii. \( a = b \).

Axiom 5. \( \mathbb{R} \) is linearly ordered.

Axiom 6. If \( p \) is a point there is a point less than \( p \) and a there is a point greater than \( p \).

Axiom 7. If \( p \) and \( q \) are two points then there is a point between them, for example, \((p+q)/2\).

Axiom 8. If \( a < b \) and \( c \) is any point, then \( a + c < b + c \).

Axiom 9. If \( a < b \) and \( c > 0 \), then \( a \cdot c < b \cdot c \). If \( c < 0 \), then \( a \cdot c > b \cdot c \).

Axiom 10. If \( x \) is a point, then \( x \) is an integer or there is an integer \( n \) such that \( n < x < n + 1 \).
Axiom 11. If \( n \) is an integer, then there is no integer other than \( n \) between the integers \( n - 1 \) and \( n + 1 \).

Definition 12. The statement that the point set \( O \) is an open interval means that there are two points \( a \) and \( b \) such that \( O \) is the set of all points between \( a \) and \( b \).

Definition 13. The statement that \( I \) is a closed interval means that there are two points \( a \) and \( b \) such that \( p \in I \) if and only if \( p = a \), \( p = b \), or \( p \) is between \( a \) and \( b \).

Notation: We use the notation \((a, b)\) to denote the open interval consisting of all points \( p \) such that \( a < p < b \). Similarly we use the notation \([a, b]\) to denote the closed interval determined by the two points \( a \) and \( b \) where \( a < b \). We do not use \((a, b)\) or \([a, b]\) in case \( a = b \), although many mathematicians do.

Problem 14. Determine whether \( \mathbb{R} \) is a topological space if regions are defined to be sets containing exactly one point. I.e. \( R \) is a region if and only if \( R = \{ p \} \) for some \( p \in \mathbb{R} \).

Problem 15. Determine whether \( \mathbb{R} \) is a topological space if the only region is the entire space, \( \mathbb{R} \).

Problem 16. Determine whether \( \mathbb{R} \) is a topological space if only closed intervals are regions.

Problem 17. Consider \( \mathbb{R} \) where \( R \) is a region if and only if there are numbers \( a \) and \( b \) with \( a < b \) such that \( R = \{ x | a \leq x < b \} \) (i.e. regions are “half” open intervals). Show that this is a topological space. This space is referred to as the Sorgenfrey line.

Theorem 18. \( \mathbb{R} \) is a topological space if only open intervals are regions.

From this point on we interpret \( \mathbb{R} \) to mean the topological space where regions are defined to be open intervals. This topological space would be referred to as the usual topology on \( \mathbb{R} \) or the Euclidean topology on \( \mathbb{R} \).

Definition 19. Two sets are said to be mutually disjoint if they have no points in common.

Definition 20. The statement that the set \( S \) is a Hausdorff space means that \( S \) is a topological space and if \( p \) and \( q \) are two (distinct) elements of \( S \) then there are mutually disjoint regions \( U \) and \( V \) containing \( p \) and \( q \) respectively.

Theorem 21. \( \mathbb{R} \) is a Hausdorff space.

Definition 22. The statement that the sequence \( p_1, p_2, p_3, \ldots, \) denoted \( (p_i) \), converges to the point \( p \) means that if \( R \) is a region containing \( p \), then there is a positive integer \( n \) such that if \( m \) is a positive integer and \( m > n \), then \( p_m \) is in \( R \).
Definition 23. The statement that the sequence \((p_i)\) converges, means that there is a point \(p\) such that \((p_i)\) converges to \(p\).

Problem 24. For each positive integer \(n\), let \(p_n = 1 - 1/n\). Show that the sequence \((p_i)\) converges to 1.

Problem 25. If \(m\) is a positive, odd integer then \(p_m = 1/m\) while if \(m\) is a positive, even integer then \(p_m = (m+1)/m\). Show that the sequence, \((p_i)\) does not converge to zero.

Problem 26. For each positive integer \(n\), let \(p_{2n} = 1/(2n-1)\) and \(p_{2n-1} = 1/2n\). Show that the sequence \((p_i)\) converges to 0.

Definition 27. If \(M\) is a point set and \(p\) is a point, the statement that \(p\) is a limit point of \(M\) means that every region containing \(p\) contains a point of \(M\) different from \(p\).

Definition 28. If \(M\) is a point set, then the closure of \(M\), denoted by \(\text{Cl}(M)\), is the set to which the point \(p\) belongs if and only if \(p\) is a point of \(M\) or \(p\) is a limit point of \(M\).

Definition 29. The statement that the topological space \(S\) is regular at the point \(p\) means that if \(U\) is a region containing \(p\), there is a region \(V\) containing \(p\) such that \(\text{Cl}(V) \subseteq U\).

Definition 30. The statement that the topological space \(S\) is regular means that \(S\) is regular at each of its points.

Theorem 31. \(\mathbb{R}\) is a regular space.

Definition 32. If \((p_i)\) is a sequence, then the set \(\{p_i: i \text{ is a positive integer}\}\) denotes the range of the sequence. That is, \(\{p_i: i \text{ is a positive integer}\}\) denotes the point set to which the point \(p\) belongs if and only if there is a positive integer \(n\) such that \(p = p_n\).

Problem 33. Show that if the sequence \((p_i)\) converges to the point \(p\), and, for each positive integer \(n\), \(p_n \neq p_{n+1}\), then \(p\) is a limit point of the range of the sequence.

Theorem 34. If \(p\) is a limit point of the point set \(M\) then there is a sequence of points \(p_1, p_2, p_3, \ldots\) of \(M\), all different and none equal to \(p\), that converges to \(p\).

Definition 35. The statement that the set \(M\) is finite means that there is a positive integer \(n\), such that \(M\) has \(n\) points and does not have \(n+1\) points.

Definition 36. The statement that the set \(M\) is infinite means that \(M\) is not finite.

Definition 37. A rightmost point of a point set \(M\) is a point, \(r\), such that \(r \in M\) and \(r \geq m\) for all \(m \in M\). Leftmost point is defined analogously.
Definition 38. A first point to the right of a point set $M$ is a point, $r$, such that $r > m$ for all $m \in M$ and there is no point $s < r$ satisfying $s > m$ for all $m \in M$. First point to the left of $M$ is defined analogously.

Theorem 39. If $M$ is a finite point set then $M$ has a leftmost and a rightmost point.

Theorem 40. If $p$ is a limit point of the point set $M$, then every region containing $p$ contains infinitely many points of $M$.

Problem 41. Show that if $c$ is a number and $(p_i)$ is a sequence that converges to the point $p$, then the sequence $(cp_i)$ converges to $cp$.

Problem 42. Show that if the sequence $(p_i)$ converges to $p$ and the sequence $(q_i)$ converges to $q$, then the sequence $(p_i + q_i)$ converges to $p + q$.

Theorem 43. If $p$ is a limit point of the point set $H$ and $H$ is a subset of the point set $K$, then $p$ is a limit point of $K$.

Theorem 44. If $H$ and $K$ are point sets and $p$ is a limit point of $H \cup K$, then $p$ is a limit point of $H$ or $p$ is a limit point of $K$.

Definition 45. The statement that two point sets in a topological space are mutually separated means that neither contains a point nor a limit point of the other.

Definition 46. The statement that the point set $M$ in a topological space $S$ is connected means that $M$ is not the union of two mutually separated sets.

Theorem 47. If a connected set $M$ is a subset of the union of two mutually separated point sets $H$ and $K$, then $M$ is a subset of one of $H$ and $K$.

Theorem 48. If the sequence $(p_i)$ converges to the point $p$ and $q$ is a point different from $p$, then $(p_i)$ does not converge to $q$.

Theorem 49. If the sequence $(p_i)$ converges to the point $p$ and $q$ is a point different from $p$, then $q$ is not a limit point of the range of the sequence $(p_i)$.

Definition 50. The statement that the point set $M$ is open means that if $p$ is a point of $M$, then there is a region $R$ such that $p \in R \subseteq M$.

Every region is an open set. Can you construct some other open sets?

Definition 51. The statement that the point set $M$ is closed means that if $p$ is a limit point of $M$, then $p \in M$.

Theorem 52. If $H$ and $K$ are closed point sets then $H \cup K$ and $H \cap K$ are closed.

Theorem 53. If $H$ and $K$ are regions then $H \cup K$ and $H \cap K$ are open.
Definition 54. If $M$ is a point set then $M^c$ is the point set $M^c = \{x \in \mathbb{R} : x \notin M\}$.

Theorem 55. If $M$ is a point set and $M$ is closed, then $M^c$ is open.

Theorem 56. If $M$ is a point set and $M$ is open, then $M^c$ is closed.

Theorem 57. If $\mathcal{G}$ is a finite collection of regions, each containing the point $p$, then the set of all points that are in all the sets in $\mathcal{G}$ is an open point set.

Theorem 58. If $\mathcal{G}$ is an arbitrary (i.e. possibly infinite) collection of closed point sets, each containing the point $p$, then the set of all points that are in all the sets in $\mathcal{G}$ is a closed point set.

Notation: If $\mathcal{G}$ is a collection of point sets, then the union of the members of $\mathcal{G}$ is denoted by $\cup\{G|G \in \mathcal{G}\}$ or $\cup_{G \in \mathcal{G}} G$ or, more simply, by $\mathcal{G}^*$. Similarly, the set of points common to the members of $\mathcal{G}$, called the intersection of the members of $\mathcal{G}$ is denoted by $\cap\{G|G \in \mathcal{G}\}$ or $\cap_{G \in \mathcal{G}} G$.

Theorem 59. If $\mathcal{G}$ is a finite collection of closed point sets, then $\mathcal{G}^*$ is closed.

Theorem 60. If $\mathcal{G}$ is an arbitrary collection of open sets, then $\mathcal{G}^*$ is open.

The previous four theorems are often stated as follows:

1. The collection of open sets is closed under arbitrary union,
2. the collection of open sets is closed under finite intersection,
3. the collection of closed sets is closed under arbitrary intersection, and
4. the collection of closed sets is closed under finite union.

Most mathematicians would use these facts as the definition of a topological space. They might define a topological space as an ordered pair, $(S, \tau)$ where $S$ is a set and $\tau$ is a collection of subsets of $S$, called open sets, such that

1. $S$ and $\emptyset$ are elements of $\tau$,
2. $\tau$ is closed under the operation of finite intersection, and
3. $\tau$ is closed under the operation of arbitrary union.

If we were using this definition for a topological space, then the regions in Definition 3 would be referred to as a basis for the topological space. Either way, what we now have is a collection of points, regions, and open sets with the properties that:

\[1\] We have avoided introducing the empty set in these notes to avoid addressing set theory and to move to topology as quickly as possible.

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1. every point is contained in a region,
2. every point contained in two regions is contained in a region that is a subset of the original two regions, and
3. every open set is the union of regions.

**Theorem 61.** If \( H \) and \( K \) are two mutually disjoint closed point sets, they are mutually separated.

**Theorem 62.** If \( H \) and \( K \) are connected point sets having a point in common, then \( H \cup K \) is connected.

**Theorem 63.** If \( H \) is a connected point set and \( K \) is a point set and every point of \( K \) is a limit point of \( H \), then \( H \cup K \) is connected.

**Definition 64.** The statement that the point set \( M \) is **bounded above** means that there is a point to the right of every number in \( M \). The statement that \( M \) is **bounded below** is defined similarly.

**Definition 65.** \( M \) is **bounded** means that \( M \) is bounded above and below.

**Theorem 66.** If the sequence \((p_i)\) converges to the point \( p \), then the range of this sequence is bounded.

**Definition 67.** The statement that the sequence \((p_i)\) is an **increasing** sequence means that for each positive integer \( n \), \( p_n < p_{n+1} \). The statement that the sequence \((p_i)\) is **non-decreasing** means that for each positive integer \( n \), \( p_n \leq p_{n+1} \). We define **decreasing** and **non-increasing** sequences similarly.

**Theorem 68.** If \((p_i)\) is a non-decreasing sequence and there is a point to the right of each point of the sequence, then the sequence converges to some point.

**Theorem 69.** If \( G \) is a collection of connected point sets and one of them intersects all the others, then \( G^* \) is connected.

**Definition 70.** If \( M \) is a point set in a topological space \( S \), then by a **component** of \( M \) is meant a connected subset of \( M \) that is not a subset of any other connected subset of \( M \).

**Theorem 71.** If \( M \) is a point set and \( p \) is a point of \( M \), then there is exactly one component of \( M \) that contains \( p \).

Any theorems that use the forthcoming definition of a Cauchy sequence assume, in addition to our other axioms, that we have a distance between the points in our space. On the number line, the distance from the point \( a \) to the point \( b \) is \(|a - b|\). Recall that \(|p|\) is either \( p \) or \(-p\) depending on to whether \( p \) is non-negative or negative.
Definition 72. The statement that the sequence \((p_i)\) is a Cauchy sequence means that if \(\varepsilon\) is a positive number, then there is a positive integer \(N\) such that if each of \(m\) and \(n\) is an integer with \(m > N\) and \(n > N\), then \(|p_m - p_n| < \varepsilon\).

Theorem 73. The sequence \((p_i)\) is a Cauchy sequence if and only if it is true that for each positive number \(d\), there is a positive integer \(n\) such that if \(m\) is a positive integer and \(m \geq n\), then \(|p_m - p_n| < d\).

Theorem 74. If \((p_i)\) is a sequence converging to the point \(p\), then the sequence \(p_1 - p_2, p_2 - p_3, \ldots\) converges to 0.

Problem 75. Show that the converse to Theorem 74 is false by providing a counterexample.

Axiom 76. If \(M\) is a point set and there is a point to the right of every point of \(M\), then \(M\) has either a rightmost point or a first point to the right.

The point described in Axiom 76 is called the least upper bound of \(M\). The point that would be described if left and right were interchanged is called the greatest lower bound of \(M\). These are usually denoted by \(\text{lub}(M)\) and \(\text{glb}(M)\). We have already been using the concept of \(\text{lub}(M)\), but calling it either the rightmost point of \(M\) or the first point to the right of \(M\), depending on whether the point was an element of \(M\) or not. Axiom 76 is called the Completeness Axiom and applies only to linearly ordered sets.

A linearly ordered space that satisfies Axiom 76 would be called a complete space. At this point we could say that \(\mathbb{R}\) is a complete, linearly ordered space that has no minimal or maximal element. We will mark with (CA) those theorems that require the Completeness Axiom. We assume of course that a statement similar to Axiom 76, but with right and left reversed also holds. We might restate Axiom 76 as follow: “If \(M\) is bounded above then \(M\) has a least upper bound.”

Theorem 77. If \(M\) is a point set having least upper bound \(b\) that is not in \(M\), then \(b\) is a limit point of \(M\).

Theorem 78. If the sequence \((p_i)\) converges to a point \(p\), then \((p_i)\) is a Cauchy sequence.

Theorem 79. If \((p_i)\) is a Cauchy sequence, then the range of \((p_i)\) is bounded.

Theorem 80. If \(H\) and \(K\) are bounded sets and \(H \subseteq K\) then \(\text{lub}(H) \leq \text{lub}(K)\) and \(\text{glb}(H) \geq \text{glb}(K)\).

Theorem 81. If \(H\) and \(K\) are bounded sets and \(L\) is the set to which the number \(x\) belongs if and only if there are numbers \(h \in H\) and \(k \in K\) such that \(x = h + k\), then \(\text{glb}(H) + \text{glb}(K) = \text{glb}(L)\).

Theorem 82. If \((p_i)\) is a Cauchy sequence, then the range of \((p_i)\) does not have two limit points.
Theorem 83. If \((p_i)\) is a Cauchy sequence, then the sequence \((p_i)\) converges.

Definition 84. If \((a,b)\) is a segment, then by the **length** of \((a,b)\) is meant \(b-a\).

**Notation** If \(\mathcal{G}\) is a collection of segments, then \(L(\mathcal{G})\) denotes the set of all numbers that are the sums of the lengths of finite subsets of \(\mathcal{G}\).

**Problem 85.** Show that if \(\mathcal{G}\) is a finite collection of segments, then \(\text{lub}(L(\mathcal{G}))\) is the sum of the lengths of the segments in \(\mathcal{G}\).

**Definition 86.** The statement that \(\mathcal{G}\) is a **summable** collection of segments means that \(\mathcal{G}\) is a collection of segments such that \(L(\mathcal{G})\) is bounded.

**Definition 87.** If \(\mathcal{G}\) is a summable collection of segments then by the **sum of the lengths** of the members of \(\mathcal{G}\) is meant \(\text{lub}(L(\mathcal{G}))\).

**Problem 88.** Show that if \(\mathcal{G}\) is the summable collection of segments defined by \(\mathcal{G} = \{(a_i, b_i)\}_{i=1}^{\infty}\) then the sequence \(\{\sum_{i=1}^{\infty} (b_i - a_i)\}_{i=1}^{\infty}\) converges to the \(\text{lub}(L(\mathcal{G}))\).

**Definition 89.** The statement that the collection \(\mathcal{G}\) of point sets **covers** the set \(K\) means that if \(p\) is a point of \(K\), then there is an element \(g \in \mathcal{G}\) such that \(p \in g\). We call \(G\) a **cover** of \(K\).

**Problem 90.** Find a collection \(\mathcal{G}\) of open intervals covering the open interval \((a,b)\) such that no finite subset of \(\mathcal{G}\) covers \((a,b)\).

**Problem 91.** Find a collection \(\mathcal{G}\) of closed intervals covering the open interval \((a,b)\) such that no finite subset of \(\mathcal{G}\) covers \((a,b)\).

**Problem 92.** Find a collection \(\mathcal{G}\) of closed intervals covering the closed interval \([a,b]\) such that no finite subset of \(\mathcal{G}\) covers \([a,b]\).

**Theorem 93.** (AC) If \(\mathcal{G}\) is a collection of open intervals covering the closed interval \([a,b]\), then some finite subset of \(\mathcal{G}\) covers \([a,b]\).

**Definition 94.** If \(\mathcal{L} = \{L(\mathcal{G}) : \mathcal{G}\) is a collection of open intervals covering \(M\}\) then the **outer measure** of a point set, \(M\), is defined by \(m(M) = \text{glb}(\mathcal{L})\).

**Problem 95.** Show that the outer measure of the open interval \((a,b)\) and the closed interval \([a,b]\) is \(b-a\).

**Problem 96.** Show that if \(M\) is a finite set, then \(M\) has outer measure zero.

**Definition 97.** The statement that the point set \(M\) is **countable** means that \(M\) is finite or there is a sequence \(\{p_i\}_{i=1}^{\infty}\) of distinct points such that for each point \(x\) in \(M\), there is a positive integer \(i\) such that \(p_i = x\).

**Problem 98.** Show that if \(M\) is a countable point set then \(M\) has outer measure zero.
Problem 99. Find a sequence $S_1, S_2, S_3, \ldots$ of open intervals such that for each positive integer $n$, $S_{n+1} \subseteq S_n$, and $\bigcap_{n=1}^{\infty} S_n$ is an open interval.

Problem 100. Find a sequence $S_1, S_2, S_3, \ldots$ of open intervals such that for each positive integer $n$, $S_{n+1} \subseteq S_n$, and $\bigcap_{n=1}^{\infty} S_n$ is not an open interval.

Problem 101. Find a sequence $I_1, I_2, I_3, \ldots$ of closed intervals such that for each positive integer $n$, $I_{n+1} \subseteq I_n$ and $\bigcap_{n=1}^{\infty} I_n$ is a closed interval.

Problem 102. Find a sequence $I_1, I_2, I_3, \ldots$ of closed intervals such that for each positive integer $n$, $I_{n+1} \subseteq I_n$ and $\bigcap_{n=1}^{\infty} I_n$ is not a closed interval.

Problem 103. Find an example of a sequence of closed point sets, each containing the next, such that there is no point common to all the sets of the sequence.

Definition 104. The statement that the sequence of sets $(M_i)$ is nested means that $M_{i+1} \subseteq M_i$ for all $i = 1, 2, 3, \ldots$

Lemma 105. (CA) If $(M_i)$ is a nested sequence of closed intervals then $\bigcap_{i=1}^{\infty} M_i$ is either a single point or a closed interval.

Lemma 106. If $(M_i)$ is a nested sequence of closed sets and there is a point common to all of them and $p$ is a limit point of the set of all such points then $p$ is a limit point of $M_k$ for all $k = 1, 2, \ldots$

Theorem 107. If $(M_i)$ is a sequence of closed point sets and there is a point common to all the sets of the sequence $(M_i)$, then the set of all such points is a closed point set.

Definition 108. The statement that the point set $M$ is conditionally compact means that if $K$ is an infinite subset of $M$, then $K$ has a limit point.

Problem 109. (CA) Show that the open interval $(a,b)$ is conditionally compact.

Problem 110. Find an example of a nested sequence of conditionally compact sets such that there is no point common to all the sets of the sequence.

Lemma 111. Show that if $M$ is conditionally compact then $M$ is bounded.

Theorem 112. If $(M_i)$ is a nested sequence of closed and conditionally compact point sets then there is a point common to all the sets of the sequence $(M_i)$ and the set of all such points is a closed and conditionally compact point set.

Definition 113. The statement that the point set $M$ is compact means that if $\mathcal{G}$ is a collection of regions covering $M$, then some finite subset of $\mathcal{G}$ covers $M$.

Problem 114. Construct a point set $M$ such that $M$ is closed and bounded, every point of $M$ is a limit point of $M$, and $M$ is not an interval.

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Theorem 115. (CA) If \( M \) is an infinite and bounded point set then \( M \) has a limit point.

Together, Theorem 115 and Lemma 111 imply that a set is conditionally compact if and only if it is bounded.

Lemma 116. Show that if \( M \) is a countable set then there exists a nested sequence of closed intervals \( I_1, I_2, I_3, \ldots \) such that \( \bigcap_{i=1}^{\infty} I_k \) contains no point of \( M \).

Theorem 117. If \( (p_i) \) is a sequence of distinct points in the closed interval \([a, b]\), then there is a point in \([a, b]\) that is not in the sequence.

Theorem 118. If \( (p_i) \) is a sequence of distinct points in the closed interval \([a, b]\) then the range of the sequence has a limit point.

Definition 119. The statement that the set \( K \) is dense in the set \( M \) means that every point of \( M \) is a point or a limit point of \( K \).

This definition would probably most often be given by saying that \( K \) is dense in \( M \) means that \( \text{Cl}(K) = M \). These are equivalent.

Theorem 120. There is a sequence \((p_i)\) of distinct points in the interval \([a, b]\) such that the range of the sequence is dense in \([a, b]\).

Problem 121. Construct a set \( M \) such that \( M \) is closed and bounded, every point of \( M \) is a limit point of \( M \), and \( M \) contains no interval.

Theorem 122. No countable and closed point set \( M \) has the property that every point of \( M \) is a limit point of \( M \).

This theorem guarantees that the set you created in Problem 114 is not countable. Can you find a rational number in this set? Can you find a point of this set that is not an endpoint of one of the intervals used in the construction of the set? Can you find an irrational number in the set?

Theorem 123. If \( M \) is a countable subset of the interval \([a, b]\) then every point of \( M \) is a limit point of the set of points in \([a, b]\) that are not in \( M \).

It follows from the previous theorem that the set of all irrational numbers in the interval \([a, b]\) is dense in the interval \([a, b]\).

Congratulations! You have completed a one-semester introduction to topology.