Notes for a Course on Proofs

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To the Student

Your role in this course is to solve the following problems. Feel free to work out of order - sometimes the problems you will understand and be able to complete first will happen later in the problem sequence. However, when working on later problems, remember things that have already been proven - sometimes they provide hints about how to proceed with later problems.

The purpose of these notes is to guide you on a path towards critical thinking - by the end of the course, you should be able to write clear and concise solutions to the problems posed (it is your instructor’s job to provide you with feedback about your writing), be able to critically examine solutions posed by other students, and, therefore, critically evaluate your own solutions. This last is the most difficult of all - it is easy to convince yourself that your solution works, but will it convince others? Having a critical eye for your own work is a necessary and challenging skill to develop.

One of the best skills that you can learn as a mathematician is to ask good questions. For this course, you should be asking good questions of your instructor, your colleagues (in the appropriate forum as indicated by your instructor), and (mostly importantly) yourself.

Good luck and enjoy!
Chapter 1

Introduction - Thinking Outside the Box

Solve the following problems by any means necessary. Solutions to these nine problems are due at the beginning of class on (Insert date of second class here). Solutions should be written clearly, professionally, and in complete sentences in such a way that they are complete and explain the answer to the question.

1. There are three boxes of candy. One box contains mint candies, one chocolate candies, and the other is mixed. All three boxes are incorrectly labeled. What is the smallest number of candies that you need to remove and sample to be able to correctly label all three boxes? ¹

2. A sealed room contains one light bulb. Outside of the room, there are three switches, only one of which operates the bulb. You are outside the room, able to operate the switches in any way you see fit, but when the door is opened for the first time, you must determine which switch operates the light. What do you do? ²

3. Describe how one can use a four-minute hourglass and a seven-minute hourglass to measure a period of nine minutes. ³

4. Two doors are guarded by two men, one of whom always lies and one of whom always tells the truth; however, you do not know which man is which. One of the doors leads to freedom and one to captivity. Determine a single question that, if asked of one of the guards, would reveal the door to freedom with certainty.

5. A cruel calculus instructor decided to terrorize her students. The instructor announced that during the next class, the students will line up, facing away from the front of the line. The instructor will then place either a white or a gray dunce cap on each student’s head. Each student

¹ ibid.
² ibid.
³ ibid.
will be unable to see his or her own cap but will be able to see the cap colors of all those classmates who are in front of him or her.

Starting at the head of the line, each student will be asked, in turn, “What is the color of your dunce cap?” Students will only be allowed to respond by saying “white” or “gray.” The students who answer correctly will be given A’s and the student who answer incorrectly will fail the course. For each student, the instructor will respond with either “Correct! You receive an A” or “you fail! Get out of my classroom, you dunce,” depending on the correctness of the answer. All students will be able to hear all students’ answers and the instructor’s responses. Knowing this horrific fate that awaits them, the students have all night to come up with a plan. If there are $n$ students in the class, how many of them can be guaranteed to receive A’s? Your challenge is to devise a scheme that the students can employ to allow as many of them as possible to receive A’s. (Caution: No “cheating” is allowed; that is, students cannot use the tones of their voices or say additional phrases or use hand gestures to provide any additional information.)

6. Consider the following mathematical illusion: A regular deck of 52 playing cards is shuffled several times by an audience member until everyone agrees that the cards are completely shuffled. Then, without looking at the cards themselves, the magician divides the deck into two equal piles of 26 cards. The magician taps both piles of face-down cards three times. Then, one by one, the magician reveals the cards of both piles. Magically, the magician is able to have the cards arrange themselves so that the number of cards showing black suits in the first pile is identical to the number of cards showing red suits in the second pile. Your challenge is to figure out the secret to this illusion and then perform it for your friends.

7. Some number of coins are spread out on a table. They lie either heads up or tails up. Unfortunately you are blindfolded and thus both the coins and the table upon which they sit are hidden from view. Certainly you can feel your way across the table and count the total number of coins on the table’s surface, but you cannot determine if any individual coin rests heads up or down (perhaps you are wearing gloves). You are informed of one fact (beyond the total number of coins on the table): Someone tells you the number of coins that are lying heads up. You can now rearrange the coins, turn any of them over, and move them in any way you wish, as long as the final configuration has all the coins resting (heads or tails up) on the table. Your challenge is to turn over whatever coins you wish and divide the coins into two collections so

4Burger
5ibid.
that one collection of coins contains the same number of heads up coins as the other collection contains.  

8. You find yourself on a reality TV show that has you completing with other real people in totally artificial circumstances. In one scenario, you are given nine balls of clay. You are informed by the program’s B-celebrity host that hidden inside one of those clay balls is a key that will unlock a refrigerator that houses a vast quantity of food. Since the producers “thought” the ratings would be higher if the contestants were deprived of nutrition, even the thought of brussels sprouts makes your mouth water. You are told that the eight balls that do not contain the key to your dietary dreams all weigh the same. The special ball with the key insider weighs slightly more, but not enough for you to feel the difference by holding the balls in your hand. One of the program’s sponsors, Replace-Oh!, the company that manufactures one-time-use balance scales (with the slogan “Weigh aweigh then throw away!”), has agreed to provide some of its scales in exchange for a few shameless plugs throughout the program. Their scales will tell which side is heavier and then instantly self-destruct. You are only allowed to break open one clay ball to see if you can find the refrigerator key. Your challenge is to determine the fewest disposable balance scales required to guarantee that you can identify the ball with the key. Justify your answer.  

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6ibid.
7ibid.
Definition 1. A natural number is a counting number, as in 1, 2, 3, . . .

Definition 2. A whole number is a counting number or zero, as in 0, 1, 2, 3, 4, . . .

Definition 3. An integer is a positive or negative counting number or zero, as in

\[ \ldots, -2, -1, 0, 1, 2, 3, \ldots \]

9. (⋆) Prove there does not exist an equilateral triangle in the plane whose vertices are at integer lattice points \((x, y)\).

Definition 4. An integer \(n\) is even if there exists an integer \(k\) so that it can be written as \(n = 2k\).

Definition 5. An integer \(n\) is odd if there exists an integer \(k\) so that it can be written as \(n = 2k + 1\).

10. (⋆) Prove that there exist integers \(m\) and \(n\) so that \(2m + 7n = 1\).

11. (⋆) Prove that there do not exist integers \(m\) and \(n\) so that \(2m + 4n = 7\).

12. (⋆) Let \(a\) and \(b\) be two integers. Prove that if \(a + b\) is even then \(a - b\) is even.

13. (⋆) Prove that an integer \(n\) that can be written as \(n = 2k - 1\) for some integer \(k\) is odd.

14. Prove that the product of two even numbers is even.

15. Prove that the product of an even number and an odd number is even.

16. Prove that the product of two odd numbers is odd.
Chapter 3

Logic

Definition 6. A mathematical statement is a declarative sentence which is either true or false.

Notice that a mathematical statement does not need to be a statement about mathematics.

17. Which of the following are mathematical statements? Explain
   (a) \(2^3 + 3^2 = 17\)
   (b) \(8x^3 + 6x^2 - 4x + 2\)
   (c) Broccoli is green.
   (d) Will you marry me?
   (e) It is a beautiful day.

18. Which of the following are true mathematical statements? Which are false? Explain.
   (a) Broccoli is a vegetable and \(3 < \sqrt{17}\).
   (b) Mercury is the closest planet to the sun and \(3 > \sqrt{17}\).
   (c) Shakespeare was a playwright or \(|3| = |-3|\).
   (d) Chile is a country in South America or 8 is a prime number.
   (e) 7 is an odd number or \(\pi > 3\).

19. The problem above asks about two types of statements: “and” statements and “or” statements. What is necessary for an “and” statement to be true? For an “or” statement to be true?

20. (*) Suppose each of the following three statements is true:
    John is smart.
    John or Mary is ten years old.
    If Mary is ten years old, then John is not smart.
    Which of the following statements are true and why?
(a) Mary is 10 years old.
(b) John is 10 years old.
(c) Either John or Mary is not 10 years old.

**Definition 7.** The negation of a statement is a statement having the opposite truth value of the original statement. In other words, if the original statement is true, the negation is false; and if the original statement is false, the negation is true.

**Example 8.** Original Statement: $3 > \pi$.
Silly negation: $3$ is not greater than $\pi$.
Useful Negation: $3 \leq \pi$.

**Example 9.** Original Statement: Curtis is nice.
Silly negation: It's not true that Curtis is nice.
Useful Negation: Curtis is not nice.
Notice that the negation is not “Curtis is mean.” That statement is stronger than the statement that he is not nice.

21. (*) Find grammatically correct negations of each of the following statements:

(a) The number $x$ is negative.
(b) For all $x$, $f(x) > 0$.
(c) There exists an $x$ such that $g'(x)$ is undefined.

22. Write a useful negation of each of the following statements:

(a) All cows eat grass.
(b) There is a horse that does not eat grass.
(c) There is a car that is blue and weighs less than 4000 pounds.
(d) Every math book is either white or hard to read.
(e) Some cows are spotted.
(f) No car has 15 cylinders.

**Definition 10.** A truth table is a way of keeping track of the truth or falsity of a complicated statement for all possible truth values of the component statements.

**Example 11.** Let $P$ and $Q$ be statements. Then the statement “$P$ and $Q$” has the following truth values, depending upon the truth values of $P$ and of $Q$:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P$ and $Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>
Definition 12. Two statements are equivalent if their truth table match for all possible values of the component statements.

23. (*) Express the statement “Not (A and B)” as an equivalent statement involving only “not” and “or.” Then express the statement “Not (A or B)” as an equivalent statement involving only “not” and “and.” Justify your answers with truth tables.

Definition 13. A conditional statement is one of the form “If P then Q.”

24. Which of the following conditional statements are true and why?
   (a) If $1 + 1 = 3$ then $\sqrt{36} = -6$.
   (b) If $\cos \pi = 0$ then $\sin \pi = 0$.
   (c) If $\sin \pi = 0$ then $\cos \pi = 0$.
   (d) If $x = -3$ satisfies $x^2 = 9$ then $\log_2 \frac{1}{8} = -3$.

25. (*) Complete the rest of the truth table for a conditional statement “If P, then Q.”

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>If P, then Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

26. Find the negation of the statement “If $\sin \pi = 0$, then $\cos \pi = 0$.”

27. (*) What is the negation of the statement “If P, then Q”? Justify your answer with a truth table.

Definition 14. The converse of the conditional statement “If P, then Q” is “If Q, then P.”

Definition 15. The inverse of the conditional statement “If P, then Q” is “If not P, then not Q.”

Definition 16. The contrapositive of the conditional statement “If P, then Q” is “If not Q, then not P.”

28. (*) Consider an implication, its converse, its inverse, and its contrapositive. Which of these statements are equivalent to one another? Justify your answers with truth tables.

29. The converse of the inverse is another conditional statement. Which one? Explain.
30. Give an example of a true conditional statement with a false converse, or explain why no such example can exist. Explain.

31. Give an example of a true conditional statement with a false contrapositive, or explain why no such example can exist. Explain.

32. (*) Give an example of a false conditional statement with a false inverse, or explain why no such example can exist. Explain.
Chapter 4

More Number Theory and Basic Proof Techniques

**Definition 17.** A number $n$ is a rational number if it can be written as $\frac{p}{q}$ for some integers $p$ and $q$ where $q \neq 0$.

**Fact.** If we assume that $q > 0$, every rational number can be written uniquely in a reduced form (i.e., where $p$ and $q$ have no common divisors).

33. (*) Prove that $\sqrt{2}$ is not a rational number.

**Definition 18.** A natural number $n$ is prime if it has exactly two divisors, namely 1 and itself.

34. Prove or disprove: If $n$ is prime, then $2^n - 1$ is prime.

35. (*) Prove or disprove: If $n$ is a natural number, $n^2 + n + 41$ is prime.

36. (*) Prove or disprove: The square of an odd integer is odd.

**Definition 19.** If $a$ and $b$ are integers, $a$ divides $b$ if there exists an integer $k$ so that $b = ak$.

37. Prove or disprove: The product of two even integers is divisible by 4.

38. Prove or disprove: Let $a$ and $b$ be integers. If $a$ divides $b$ then $a$ divides $bc$ for all integers $c$.

39. Prove or disprove: Let $a$ and $b$ be integers. If $a$ divides $b$ and $a$ divides $c$, then $a$ divides $b - c$.

40. (*) Prove or disprove: Let $a$ and $b$ be integers. If $a$ and $b$ are both positive and $a$ divides $b$ and $b$ divides $a$ then $a = b$. Does this still hold if $a$ and $b$ are not necessarily positive? Why or why not?
41. Prove or disprove: Let $a, b,$ and $c$ be integers. If $ab$ divides $c$ then $a$ divides $c$.

42. Prove or disprove: Let $a, b,$ and $c$ be integers. If $ac$ divides $bc$ then $a$ divides $b$.

43. (*) Prove or disprove: An integer $n$ is even if and only if $n^2$ is even.

44. Prove or disprove: If $n$ is a natural number, then $n^2 + n$ is even.

**Fact.** Every integer other than 1 and $-1$ has a prime divisor.

45. (*) Prove or disprove: There are an infinite number of primes.

46. (*) Let $a, b$ and $c$ be integers such that $a^2 + b^2 = c^2$. Prove at least one of $a$ and $b$ is even.

**Definition 20.** For a natural number $n$, $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$.

47. Prove that if $n$ is a natural number greater than 1, then $n! + 1$ is odd.

48. How many pairs of primes $p$ and $q$ are there such that $q - p = 3$? Prove your answer.

49. Prove or disprove: Let $n$ be an integer. If 3 divides $n^2$, then 3 divides $n$.

50. Prove or disprove: $\sqrt{3}$ is irrational.

**Definition 21.** Let $a$ and $b$ be integers. The greatest common divisor of $a$ and $b$ is denoted by $\gcd(a, b)$ and is the natural number $d$ that satisfies the following two conditions:

(i) $d$ divides both $a$ and $b$

(ii) If $n$ is an integer that divides both $a$ and $b$ then $n$ divides $d$.

**Example 22.** Notice that for any integers $a$ and $b$, 1 is a common divisor. Therefore, the greatest common divisor always exists.

51. Let $b$ be a nonzero integer. Prove that $\gcd(0, b) = |b|$.

**Axiom:** The Division Algorithm - If $a$ and $b$ are positive integers with $b \leq a$, then there exists a natural number $q$ (called the quotient) and a nonnegative integer $r$ (called the remainder) such that $a = bq + r$ where $0 \leq r < b$.

**Example 23.** If $a = 7$ and $b = 3$ then $q = 2$ and $r = 1$ so that $7 = 3 \cdot 2 + 1$.

If $a = 20$ and $b = 5$ then $q = 4$ and $r = 0$ so that $20 = 5 \cdot 4 + 0$.

Be careful - the remainder needs to be smaller than $b$ so while it is true that $7 = 3 \cdot 1 + 4$, this is not the way we want to choose $q$ and $r$. 

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**Axiom:** The Euclidean Algorithm - Let \( a \) and \( b \) be two positive integers with \( b \leq a \). Let \( d \) be the \( \gcd(a, b) \). Then the following two statements hold:

(i) There are two lists of positive integers \( q_i \) and \( r_i \) such that

\[
\begin{align*}
    b &> r_1 > r_2 > r_3 > \cdots > r_{k-1} > r_k > r_{k+1} = 0 \\
    a &= bq_1 + r_1 \\
    b &= r_1q_2 + r_2 \\
    r_1 &= r_2q_3 + r_3 \\
    \vdots \\
    r_{k-3} &= r_{k-2}q_{k-1} + r_{k-1} \\
    r_{k-2} &= r_{k-1}q_k + r_k \\
    r_{k-1} &= r_kq_{k+1} \quad \text{(that is, } r_{k+1}=0) \\
\end{align*}
\]

and \( d = r_k \).

(ii) The GCD of \( a \) and \( b \) may be written as an integer combination of \( a \) and \( b \); that is, there exist integers \( x \) and \( y \) so that \( d = ax + by \).

**Example 24.** We can use the Euclidean Algorithm to find \( \gcd(44, 104) \).

\[
\begin{align*}
    104 &= 44(2) + 16 \\
    44 &= 16(2) + 12 \\
    16 &= 12(1) + 4 \\
    12 &= 4(3) \\
\end{align*}
\]

The last non-zero remainder is the greatest common divisor, and so \( \gcd(44, 104) = 4 \).

52. Use the Euclidean Algorithm to find \( \gcd(219, 69) \).

53. Find integers \( m \) and \( n \) so that \( \gcd(219, 69) = 219m + 69n \).

54. Use the Euclidean Algorithm to find \( \gcd(10245, 5357) \).

55. Find integers \( m \) and \( n \) so that \( \gcd(10245, 5357) = 10245m + 5357n \).

**Definition 25.** Two integers \( a \) and \( b \) are relatively prime if

\[ \gcd(a, b) = 1. \]

**Comment.** Notice that prime is a quality of one number. Relatively prime is a quality of a pair of numbers.
56. What does the Euclidean Algorithm tell us about integers \( a \) and \( b \) with \( \gcd(a, b) = 1 \)? Explain. (Notice that saying that they are relatively prime does NOT count.)

57. (*) Prove or disprove: If \( a, b \) and \( c \) are integers such that \( a \) and \( b \) are relatively prime and \( a \mid bc \) then \( a \mid c \).

58. Prove or disprove: If \( p \) is a prime and \( a \) is an integer such that \( p \) does not divide \( a \) then \( a \) and \( p \) are relatively prime.

59. Prove or disprove: If \( p \) is a prime and \( a \) is an integer, then \( \gcd(p, a) = p \) if and only if \( p \) divides \( a \).

60. (*) Prove or disprove: Let \( a \) and \( b \) be integers and let \( p \) be a prime number. If \( p \mid ab \) then \( p \mid a \) or \( p \mid b \).

61. (*) Prove or disprove: Let \( a, b \) and \( c \) be integers such that \( \gcd(a, c) = \gcd(b, c) = 1 \). Prove that \( \gcd(ab, c) = 1 \).

62. (*) Prove or disprove: Let \( a, b \) and \( c \) be integers such that \( a \) and \( b \) are relatively prime and \( c \) divides \( a + b \). Prove that \( \gcd(a, c) = \gcd(b, c) = 1 \).

**Definition 26.** Let \( a \) and \( b \) be positive integers. A natural number \( n \) is the least common multiple of \( a \) and \( b \) if it satisfies the following two properties:

(i) \( a \) divides \( n \) and \( b \) divides \( n \)
(ii) If \( m \) is any other number so that \( a \) divides \( m \) and \( b \) divides \( m \) then \( n \) divides \( m \).

**Comment:** Notice that if \( ab \) is always a common multiple of \( a \) and \( b \) and, therefore, the \( \text{lcm}(a, b) \) always exists.

63. Prove or disprove: Let \( a \) and \( b \) be natural numbers and \( \text{lcm}(a, b) = m \). Then \( m = b \) if and only if \( a \) divides \( b \).

64. (*) Prove or disprove: Let \( a \) and \( b \) be natural numbers and \( \text{lcm}(a, b) = m \). Then \( \text{lcm}(an, bn) = mn \) for all natural numbers \( n \).

65. (*) Prove or disprove: Let \( a \) and \( b \) be natural numbers with \( \gcd(a, b) = d \) and \( \text{lcm}(a, b) = m \). If \( d = 1 \) then \( m = ab \), i.e. if \( a \) and \( b \) are relatively prime then their least common multiple is the product of \( a \) and \( b \).

66. (*) Let \( d \) and \( n \) be nonzero integers. Prove or disprove: If \( d \mid n^2 \) then \( d \mid n \).

67. Prove or disprove: Let \( n \) be an integer and \( p \) be prime. Prove that if \( p \mid n^2 \) then \( p \mid n \).
68. Prove that if \( p \) is a prime, then \( \sqrt{p} \) is irrational.

69. Prove or disprove: The product of any three consecutive integers is a multiple of three.

70. Extend the previous problem and prove your extension.
Chapter 5

Induction

**Axiom:** If we want to show that a statement is true for all natural numbers, we can show that each natural number makes the statement true. However, since the set of natural numbers is infinite, we need to be clever about this. It is enough to prove a statement using the Principle of Mathematical Induction. This involves three steps:

(a) Show that the statement is true for the smallest natural number, 1 (or some other base case, if relevant).

(b) Assume that the statement holds for a certain natural number, \( k \).

(c) Show that whenever the statement is true for a certain natural number, \( k \), it is then true for the next natural number, \( k + 1 \). This shows that is is true for every natural number starting at a specific point (usually, but not always, 1).

**Example 27.** We wish to show that \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \) for every natural number \( n \).

**Proof:** We will first show that this holds for a base case (where \( k = 1 \)).

*If \( n = 1 \) this statement claims that \( 1 = 1^2 \). The right hand side is equal to one, so this is a true statement.*

We next assume that this is true for some natural number \( k \) and we show that this is true for the next natural number \( k + 1 \).

*Therefore, we assume that \( 1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2 \) and we want to show that*

\[
1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.
\]

*In other words, we want to see that*

\[
1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 2 - 1) = (k + 1)^2,
\]

*or*

\[
1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.
\]
Let us begin by examining the left hand side of the desired equation:

\[1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 1) =\]

Noticing that the first part of this sum is part of our assumption statement, we group that part together:

\[(1 + 3 + 5 + 7 + \cdots + (2k - 1)) + (2k + 1) =\]

and by substituting from our assumption statement, \(1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2\), we get:

\[k^2 + 2k + 1 = (k + 1)^2\]

as desired. Therefore, we have shown that the desired statement is true for \(k = 1\), and that whenever it is true for one natural number, it is true for the next natural number. We have, therefore, shown that it is true for all natural numbers.

71. Prove or disprove: The sum of the first \(n\) natural numbers is \(\frac{n(n+1)}{2}\), i.e.

\[1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} .\]

72. (*) Prove or disprove: For every natural number \(n\),

\[2 + 4 + 6 + \cdots + 2n = n(n + 1) .\]

73. Prove or disprove: For every natural number \(n\),

\[3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2} .\]

74. Prove or disprove: For every natural number \(n\),

\[4 + 8 + 12 + \cdots + 4n = 2n(n + 1) .\]

75. (*) Prove or disprove: For every natural number \(n\),

\[5 + 10 + 15 + \cdots + 5n = \frac{5n(n+1)}{2} .\]

76. Prove or disprove: For all natural numbers \(a\) it is true for every natural number \(n\) that,

\[a + 2a + 3a + \cdots + na = \frac{na(n+1)}{2} .\]
77. Prove or disprove: For every natural number \( n \),
\[
2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.
\]

78. Prove or disprove: For every natural number \( n \),
\[
1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

79. (*) Prove or disprove:: For every natural number \( n \),
\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.
\]

80. Prove or disprove:: For every natural number \( n \),
\[
\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1.
\]

81. Prove or disprove: For every natural number \( n \),
\[
1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.
\]

82. (*) Prove or disprove:: For every natural number \( n \),
\[
1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1).
\]

83. (*) Prove or disprove: \( 3 \mid (n^3 + 2n) \) for every natural number \( n \).

84. Prove or disprove: \( 4 \mid (13^n - 1) \) for every natural number \( n \).

85. (*) Prove or disprove: For every natural number \( n \), 6 divides \( n^3 - n \).

86. Prove or disprove: The product of any three consecutive natural numbers is divisible by 6.

87. (*) Prove or disprove: Every natural number greater than 1 is either prime or the product of prime numbers.

88. Prove or disprove: For every natural number \( n \),
\[
3 + 6 + 12 + \cdots + 3(2^n-1) = 3(2^n - 1).
\]

89. (*) Prove or disprove: For every natural number \( n \), \( 9^n - 8n - 1 \) is divisible by 64.
Chapter 6

Set Theory

Definition 28. A set is a specified collection of objects.

Example 29. The set of integers between \(-1\) and \(4\), including the endpoints, is \(\{-1, 0, 1, 2, 3, 4\}\).

Definition 30. If \(A\) is a set and \(x\) is an object that belongs to \(A\) then \(x\) is an element of \(A\), denoted \(x \in A\). If \(x\) is not an object that belongs to \(A\), we say that \(x\) is not an element of \(A\) (or is not a member of \(A\)), denoted \(x \notin A\).

90. List all of the elements in the following sets:

(a) The set of natural numbers strictly less than 6.
(b) The set of integers whose square is less than 17.
(c) The set of prime numbers less than 100.
(d) The set of rational numbers strictly between 0 and 1.

Definition 31. If \(A\) and \(B\) are sets, \(B\) is a subset of \(A\), denoted \(B \subseteq A\) if every member of \(B\) is a member of \(A\).

91. True or False, and explain your reasoning. \(\mathbb{N}\) is the set of natural numbers, \(\mathbb{Q}\) is the set of rational numbers, \(\mathbb{Z}\) is the set of integers, and \(\mathbb{R}\) is the set of real numbers.

(a) \(\mathbb{N} \subseteq \mathbb{Q}\)
(b) \(\mathbb{Z} \subseteq \mathbb{N}\)
(c) \(\mathbb{Q} \subseteq \mathbb{Z}\)
(d) \(\mathbb{N} \subseteq \mathbb{R}\)
(e) \(\mathbb{R} \subseteq \mathbb{Q}\)
(f) \((6, 9] \subseteq [6, 10)\)
(g) \([7, 10) \subseteq \mathbb{R}\)
**Definition 32.** Sets $A$ and $B$ are equal, denoted $A = B$, if $A \subseteq B$ and $B \subseteq A$.

**Definition 33.** The set with no members is called the empty set and denoted $\emptyset$.

92. True or False and explain your reasoning.

(a) $\emptyset \subseteq \mathbb{N}$
(b) $\emptyset \in \mathbb{N}$
(c) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
(d) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
(e) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
(f) $\{\emptyset, \{\emptyset\}\} \subseteq \{\{\emptyset, \{\emptyset\}\}\}$

93. True or False and explain your reasoning.

(a) For every set $A$, $\emptyset \subseteq A$
(b) For every set $A$, $\emptyset \in A$

94. Give an example, if there is one, of sets $A$, $B$, and $C$ such that the following are true. If there is no example, state such. Explain your reasoning on all problems.

(a) $A \subseteq B, B \not\subseteq C$ and $A \subseteq C$.
(b) $A \subseteq B, B \subseteq C,$ and $C \subseteq A$.
(c) $A \not\subseteq B, B \not\subseteq C$ and $A \subseteq C$.
(d) $A \subseteq B, B \not\subseteq C$ and $A \not\subseteq C$.

95. (∗) Let $A$ be a set. Prove or disprove: Then $\emptyset \subseteq A$ and $A \subseteq A$.

96. (∗) Let $A$, $B$ and $C$ be sets. Prove or disprove: If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

**Definition 34.** A subset $A \subseteq B$ is a proper subset if $A \subseteq B$ and $A \neq B$. This is denoted by $A \subset B$. 

97. List all of the subsets of the following sets. Which ones are proper?

(a) ∅
(b) {1}
(c) {1, 2}
(d) {{∅}}
(e) {∅, {∅}}

**Definition 35.** The power set of a set A is the set of all (proper and not proper) subsets of A. This is denoted \( \mathcal{P}(A) \).

98. True or False and explain your reasoning.

(a) ∅ ∈ \( \mathcal{P}(\{∅, \{∅\}\}) \)
(b) {∅} ∈ \( \mathcal{P}(\{∅, \{∅\}\}) \)
(c) {{∅}} ∈ \( \mathcal{P}(\{∅, \{∅\}\}) \)
(d) ∅ ⊆ \( \mathcal{P}(\{∅, \{∅\}\}) \)
(e) {∅} ⊆ \( \mathcal{P}(\{∅, \{∅\}\}) \)
(f) {{∅}} ⊆ \( \mathcal{P}(\{∅, \{∅\}\}) \)

99. True or False and explain your reasoning.

(a) 3 ∈ \( \mathbb{Q} \)
(b) {3} ⊆ \( \mathcal{P}(\mathbb{Q}) \)
(c) {3} ∈ \( \mathcal{P}(\mathbb{Q}) \)
(d) {{3}} ⊆ \( \mathcal{P}(\mathbb{Q}) \)
(e) {3} ⊆ \( \mathbb{Q} \)
(f) {{3}} ∈ \( \mathcal{P}(\mathbb{Q}) \)

100. (∗) Prove or disprove: Let \( n \) be a natural number and \( A \) be a set containing \( n \) elements. The number of elements in \( \mathcal{P}(A) \) is \( 2^n \).

101. Give an example, if there is one, of each of the following. If there is no example, state such. Explain your reasoning on all problems.

(a) A set \( A \) such that \( \mathcal{P}(A) \) has 64 elements.
(b) Sets \( A \) and \( B \) such that \( A ⊆ B \) and \( \mathcal{P}(B) ⊆ \mathcal{P}(A) \).
(c) A set \( A \) such that \( \mathcal{P}(A) = ∅ \)
(d) A set \( A \) such that \( \mathcal{P}(A) = \{∅\} \)
(e) Sets \( A, B, \) and \( C \) such that \( A ⊆ B, B ⊆ C \) and \( \mathcal{P}(A) ⊆ \mathcal{P}(C) \).

102. Find three sets \( A, B, \) and \( C \) such that \( A ∈ B, B ∈ C, \) and \( A ∈ C \).
For all of the following, $A$, $B$, and $C$ are sets.

103. True or False and explain your reasoning.
   (a) The empty set is a proper subset of every set.
   (b) If $A$ is a proper subset of $\emptyset$, then $A = \{17\}$.
   (c) If $A \subseteq B$ then $A = B$.
   (d) If $A = B$ then $A \subseteq B$.
   (e) Since $\emptyset$ is a member of $\{\emptyset\}$, $\emptyset = \{\emptyset\}$.
   (f) There is a set that is a member of every set.
   (g) There is a set which is a member of every power set.

**Definition 36.** If $A$ and $B$ are sets, then the union of $A$ and $B$ is the set of all objects that belong to $A$ or belong to $B$, denoted $A \cup B$. In other words,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

**Definition 37.** If $A$ and $B$ are sets, then the intersection of $A$ and $B$ is the set of all objects that belong to both $A$ and $B$, denoted $A \cap B$. In other words,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

104. Prove or disprove: $\emptyset \cap A = \emptyset$ and $\emptyset \cup A = A$

105. $\ast$ Prove or disprove: $A \cap B \subseteq A$

106. $\ast$ Prove or disprove: $A \subseteq A \cup B$

107. Prove or disprove: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

108. Prove or disprove:

   $$A \cup (B \cup C) = (A \cup B) \cup C$$

   and

   $$A \cap (B \cap C) = (A \cap B) \cap C.$$  

109. Prove or disprove: $A \cup A = A = A \cap A$

110. $\ast$ Prove or disprove: If $A \subseteq B$, then $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$.

**Definition 38.** Let $A$ and $B$ be sets. Then the complement of $A$ relative to $B$ is the set $\{x \in B \mid x \notin A\}$, denoted $B - A$.

**Definition 39.** If for a certain problem, all of the sets being considered are subsets of a given set $U$ then $U$ is called a universal set.

**Definition 40.** If $U$ is the universal set and $A \subseteq U$ then the complement of $A$ relative to $U$ is denoted $A'$ and is $U - A = A' = \{x \in U \mid x \notin A\}$
111. Prove or disprove: \((A')' = A\)

112. Prove or disprove: \((A \cup B)' = A' \cap B'\)

113. Prove or disprove: \((A \cap B)' = A' \cup B'\)

114. \((*)\) Prove or disprove: \(A - B = A \cap B'\)

115. \((*)\) Prove or disprove: \(A \subseteq B\) if and only if \(B' \subseteq A'\)

116. Prove or disprove:

\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
\]

and

\[
A \cup (B \cap C) = (A \cup B) \cap (A \cup C).
\]

117. Prove or disprove: \((A \cap B) \cup C = A \cap (B \cup C)\)

118. Prove or disprove: \((A \cup B) - (A \cap B) = (A - B) \cup (B - A)\)

**Definition 41.** Two sets \(A\) and \(B\) are disjoint if \(A \cap B = \emptyset\).

119. \((*)\) Prove or disprove: \(A \cap B\) and \(A - B\) are disjoint.

120. Prove or disprove: \(A = (A \cap B) \cup (A - B)\).

121. \((*)\) Prove or disprove: \(A - (A \cap B') = A \cap B\)

122. Prove or disprove: If \(A\) and \(B\) are sets such that \(A \cup B = A \cap B\) then \(A \cap B' = \emptyset\).

123. \((*)\) Prove or disprove: If \(A\) and \(B\) are sets such that \((A \cup B)' = A' \cup B'\) then \(A = B\).

124. \((*)\) Prove or disprove: Let \(A\), \(B\) and \(C\) be sets such that \(A \cup C \neq A \cap C\). Then \(A\) is not a subset of \(C\) or \(B\) is not a subset of \(A\).

**Definition 42.** A set of sets is called a family or collection of sets.

**Example 43.**

\(\mathcal{A} = \{\{1,2,3\}, \{3,4,5\}, \{3,6\}, \{2,3,6,7,9,10\}\}\)

is a family consisting of four sets.

125. Consider the family \(\mathcal{A} = \{\{1,2,3\}, \{3,4,5\}, \{3,6\}, \{2,3,6,7,9,10\}\}\). Answer the following true/false questions, with reasons.

(a) \(\{3,4,5\} \in \mathcal{A}\)

(b) \(3 \in \mathcal{A}\)

(c) \(\{3,4,5\} \subseteq \mathcal{A}\)
(d) \(\{3, 4, 5\} \subseteq \mathcal{A}\)

**Definition 44.** Let \(\mathcal{A}\) be an family of sets. The union over \(\mathcal{A}\) is

\[
\bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for some set } A \in \mathcal{A}\}
\]

\[
= \{x \mid x \in A \text{ for at least one set } A \in \mathcal{A}\}
\]

**Definition 45.** Let \(\mathcal{A}\) be an family of sets. The intersection over \(\mathcal{A}\) is

\[
\bigcap_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for every set } A \in \mathcal{A}\}
\]

126. Consider the family \(\mathcal{A} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{3, 6\}, \{2, 3, 6, 7, 9, 10\}\}\). Answer the following, with explanations:

(a) Find \(\bigcup_{A \in \mathcal{A}} A\)

(b) Find \(\bigcap_{A \in \mathcal{A}} A\)

127. Let \(A_n = \{1, 2, 3, \ldots, n\}\) and let \(\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}\). Answer the following, with explanations:

(a) Find \(\bigcup_{A_n \in \mathcal{A}} A_n\)

(b) Find \(\bigcap_{A_n \in \mathcal{A}} A_n\)

128. Consider the family \(\mathcal{A} = \{A_r = [r, \infty) \mid r \in \mathbb{R}\}\). Answer the following, with explanations:

(a) Find \(\bigcup_{A_r \in \mathcal{A}} A_r\)

(b) Find \(\bigcap_{A_r \in \mathcal{A}} A_r\)

129. For each natural number \(n\), let \(A_n = \left(0, \frac{1}{n}\right)\) and let \(\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}\). Answer the following, with explanations:

(a) Find \(\bigcup_{A_n \in \mathcal{A}} A_n\)

(b) Find \(\bigcap_{A_n \in \mathcal{A}} A_n\)
130. For each \( n \in \mathbb{Z} \), let \( C_n = [n, n+1) \) and let \( \mathscr{C} = \{ C_n \mid n \in \mathbb{Z} \} \). Answer the following, with explanations:

(a) Find \( \bigcup_{C_n \in \mathscr{C}} C_n \)

(b) Find \( \bigcap_{C_n \in \mathscr{C}} C_n \)

131. For each \( n \in \mathbb{Z} \), let \( A_n = (n, n+1) \) and let \( \mathscr{A} = \{ A_n \mid n \in \mathbb{Z} \} \). Answer the following, with explanations:

(a) Find \( \bigcup_{A_n \in \mathscr{A}} A_n \)

(b) Find \( \bigcap_{A_n \in \mathscr{A}} A_n \)

132. \((\ast)\) Prove or disprove: For every set \( B \) in a family \( \mathscr{A} \) of sets, \( \bigcap_{A \in \mathscr{A}} A \subseteq B \)

133. Prove or disprove: For every set \( B \) in a family \( \mathscr{A} \) of sets, \( B \subseteq \bigcup_{A \in \mathscr{A}} A \)

134. \((\ast)\) Prove or disprove: If the family \( \mathscr{A} \) contains at least one set, then \( \bigcap_{A \in \mathscr{A}} A \subseteq \bigcup_{A \in \mathscr{A}} A \).
Chapter 7

Relations

Definition 46. Let $S$ be a set. Let $a, b \in S$. An ordered pair is a double $(a, b)$ where $a$ is the first term of the ordered pair and $b$ is the second term of $(a, b)$. A relation $R$ on $S$ is a set of ordered pairs where the elements of the ordered pairs come from $S$.

Definition 47. The set of first terms is sometimes called the domain of the relation. The set of second terms is sometimes called the range of the relation.

Example 48. Let $S = \{2, 4, 6, 8, 10\}$. Then a relation on $S$ could be defined by

$$R = \{(2, 2), (4, 4), (6, 6), (6, 8), (6, 10), (8, 6), (8, 8), (8, 10), (10, 6), (10, 8), (10, 10)\}.$$ 

This relation can be described as two numbers from $S$ are related if they have the same number of divisors. It can also be written as $2R2$ meaning that 2 is related to 2 under the relation $R$.

135. Let $S = \{1, 2\}$. List all of the possible relations on $S$.

136. Let $A = \{1, 2, 3, 4, 5\}$. Write all elements of the following relations:

(a) $R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$

(b) $E = \{(a, b) \in A \times A \mid a + b \text{ is even}\}$

(c) $U = \{(a, b) \in A \times A \mid a \neq b\}$

Definition 49. Let $A$ and $B$ be sets. Then $A \cup B$ is another set, and we can define a relation on $A \cup B$ called the Cartesian product of $A$ and $B$, denoted $A \times B$, by $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. In other words, this relation is the set of all ordered pairs where the first term is an element of $A$ and the second term is an element of $B$. 
137. (*) Let $A = \{1, 2, 3\}$ and $B = \{2, 5\}$. List all elements of $A \times B$. What is the domain of this relation? What is the range of this relation?

138. Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. Show that $A \times (B \times C) \neq (A \times B) \times C$.

139. (*) Let $A$ have $n$ elements and $B$ have $m$ elements. Prove or disprove: $A \times B$ has $mn$ elements.

140. Let $A$ and $B$ be non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

**Definition 50.** Let $S$ be a nonempty set and $R$ a relation on $S$. Then

(a) The relation $R$ is **reflexive** on $S$ if for every $x \in S$, $(x, x) \in R$.

(b) The relation $R$ is **symmetric** on $S$ if for every $x, y \in S$, $(y, x) \in R$ whenever $(x, y) \in R$.

(c) The relation $R$ is **transitive** on $S$ if for every $x, y, z \in S$, whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

(d) The relation $R$ is **antisymmetric** on $S$ if for every $x, y \in S$, whenever $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.


142. Let $S = \{1, 2, 3\}$. Find a relation on $S$ that is reflexive and symmetric, but not transitive.

143. Let $S = \{1, 2, 3\}$. Find a relation on $S$ that is reflexive and transitive, but not symmetric.

144. Let $S = \{1, 2, 3\}$. Find a relation on $S$ that is symmetric and transitive, but not reflexive.

**Definition 51.** An equivalence relation on a set $S$ is a relation that is reflexive, symmetric, and transitive.

145. Let $S = \{1, 2, 3\}$. Find a relation on $S$ that is an equivalence relation.

146. Let $S = \{1, 2, 3\}$. Find a relation on $S$ that is not reflexive, not symmetric, and not transitive.

147. Let $S = \mathbb{R}$. For any real numbers $a$ and $b$, define $a \simeq b$ iff $a^2 = b^2$. Prove that $\simeq$ is an equivalence relation on $\mathbb{R}$. List all of the elements that are equivalent to $-7$.

148. Let $S = \mathbb{R} \times \mathbb{R}$. Define $(a, b) \simeq (c, d)$ iff $a^2 + b^2 = c^2 + d^2$. Prove that $\simeq$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$. List all of the elements that are equivalent to $(0, 0)$.
149. Let $\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{8\}\}$.

(a) List all elements of $\bigcup \mathcal{A}$
(b) Let $R = \{(x, y) \in \bigcup \mathcal{A} \times \bigcup \mathcal{A} \mid x \text{ and } y \text{ belong to the same member of } \mathcal{A}\}$. Prove that $R$ is an equivalence relation on $\bigcup \mathcal{A}$.
(c) List all elements that are related to 5.

**Definition 52.** Let $S$ be a non-empty set and $R$ an equivalence relation on $S$. The set of all elements equivalent (i.e. related to) an element $a \in S$ are the **equivalence class** of $a$, denoted $[a]$.

150. For points $(a, b)$ and $(c, d)$ in $\mathbb{R}^2$, define $(a, b) \sim (c, d)$ iff $a^2 + b^2 = c^2 + d^2$.

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{R}^2$.
(b) List all of the elements of $\mathbb{R}^2$ that are equivalent to $(0, 0)$.
(c) List all of the elements of $\mathbb{R}^2$ that are equivalent to $(5, 11)$

151. Let the set $S = \{(x, y) \in \mathbb{R}^2 : y - x \text{ is an integer}\}$.

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{R}^2$.
(b) List all of the elements of $\mathbb{R}^2$ that are equivalent to $\pi$.
(c) List all of the elements of $\mathbb{R}^2$ that are equivalent to $-17$.

152. For points $(a, b)$ and $(c, d)$ in $\mathbb{R}^2$, define $(a, b) \sim (c, d)$ iff $a^2 + b^2 = c^2 + d^2$.

(a) Find $[(0, 0)]$
(b) Find $[(5, 11)]$

153. Let the set $S = \{(x, y) \in \mathbb{R}^2 : y - x \text{ is an integer}\}$.

(a) Find $[\pi]$.
(b) Find $[-17]$.

**Definition 53.** A relation $\leq$ on a set $A$ is a partial ordering if $\leq$ is reflexive, antisymmetric, and transitive.

154. Prove or disprove: For any set $A$, $\mathcal{P}(A)$ under $\subseteq$ is a partially ordered set.

155. Prove or disprove: For a set $A$, define an order $a \leq b$ iff $a = b$. Then $A$ under $\leq$ is a partially ordered set.

**Definition 54.** A partially ordered set $A$ with partial order $\leq$ is said to be totally ordered if given any two elements $a$ and $b$ in $A$, either $a \leq b$ or $b \leq a$. 

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156. Prove or disprove: For any set $A$, $\mathcal{P}(A)$ under $\subseteq$ is a totally ordered set.

157. (*) Prove or disprove: $\mathcal{P}(\mathbb{N})$ has an infinite totally ordered subset (under the order $\subseteq$).

158. Prove or disprove: For two integers $a$ and $b$, define a relation by $a \sim b$ if $a \mid b$. Then this relation is not an equivalence relation.

159. Prove or disprove: For two integers $a$ and $b$, define a relation by $a \sim b$ if $a \mid b$. Then this relation is not a total ordering.

160. Prove or disprove: For two integers $a$ and $b$, define a relation by $a \sim b$ if $a \mid b$. Then this relation is a partial ordering.