Mathematical Modeling

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Chapter 1

Day 1: A Simple Population Model

Consider the following simple population model: Rabbits reproduce very quickly. Let’s assume that an adult pair of rabbits gives birth to one new pair of baby rabbits every month, a male and a female. We’ll also assume that it takes one month for a baby rabbit to become an adult, and one month’s gestation time between the time a pair of rabbits becomes adult and the time their first pair of offspring are born. For the purposes of this model, we assume that rabbits never die.

We start with a single pair of newborn rabbits at $t = 0$ months.

1. Make a chart in GeoGebra showing how many rabbit pairs are alive after $n$ months. One column should label the number of months that have elapsed, starting with $t = 0$ months, and another column should show the number of pairs of rabbits alive at that time. It may be useful to include columns showing the number of babies and adults at any given time.

2. Make a graph of the number of rabbit pairs alive after $n$ months. The easiest way to do this is to add a new column to your spreadsheet containing the coordinate pairs you would like to graph.

3. When modeling a situation, we often have data points that we would like to fit some kind of curve to in order to make predictions about what will happen in the future. Try to come up with the best function that you can find to match your data points.

4. How could we modify our model to make it more realistic?

Turn in a written report about what you did. In writing your report, you may find it helpful to note that spreadsheets from GeoGebra can be easily copied into Excel or Word. If you want to create a table in Word, first copy your spreadsheet, then select the data and goto “Insert table” in the Table menu.
Chapter 2

Some financial models

Model the following situations. In each case, you should start out by building an appropriate spreadsheet model, but then also attempt to find a mathematical description of the situation if you can.

1. Money in a savings account accrues a fixed percentage interest each year. (This is usually called the APR, the Annual Percentage Rate.) Create a spreadsheet to show how much money will be in the account after $n$ years.

2. Actually, though, interest in a savings account is normally compounded monthly. Create a model in which interest is compounded monthly. How does the effective annual interest rate compare with the official APR?

3. Create a model in which interest is compounded $n$ times a year. Can you describe what happens as $n$ gets big?

4. The Colorado Mega Millions Lottery offers winners a choice between 26 equal annual payments or else a single lump sum payment of some percentage of the won amount. According to the official lottery rules, “[t]he cash option prize shall be determined by dividing the Grand Prize amount that would be paid over 26 annual installments by a rate established by the Mega Millions Finance Committee prior to each drawing divided by the number of total jackpot winners.” The Mega Millions Finance committee has asked you as a consultant to write a report to help them identify what rate would be a fair equivalent. As you are considering this problem, it will probably be helpful to consider that either the recipient or the lottery board can put whatever money they have into a savings account. So, one approach to this problem would be to figure out how much money a recipient would have after receiving all 26 installments if they put each one immediately into a bank account and didn’t spend any of the money, and to compare this with a lump sum that is likewise immediately put into a bank account.

5. The most common type of loan is one in which a fixed amount of money, called the principal, is lent out at a fixed annual rate and paid back over a fixed period of time in equal installments. Create a model to figure out the size of each installment given information about the other variables. Using either the information from an actual loan that you have or else from a typical loan, use your model to compute what percentage of the money paid back to the bank is interest, and what percentage is repayment of the principal. You could do this for a mortgage, a car loan, a student loan, or some other kind of loan.
Chapter 3

The temperature in Ft. Collins

The United States Historical Climatology Network provides climate data from across the United States. Go to their website at

http://cdiac.ornl.gov/epubs/ndp/ushcn/ushcn.html

and download historical data about average monthly temperature in Ft. Collins.

Create a formula to model the average monthly temperature in Ft. Collins. How can you measure if your formula is a good match?
Chapter 4

More Population Growth

4.1 Part 1

1. One simple model of population growth is one in which a fixed percentage of all animals alive at one stage reproduce and add their offspring to the next stage. Model this situation.

2. A somewhat more sophisticated model, originally studied in the mid-nineteenth century, assumes that there is a maximum size of the population we are studying, called its carrying capacity. We then define the current saturation $s$ to be the fraction of the carrying capacity that the current population represents. In this model, we assume that the percentage of the current population that reproduces at any stage is directly proportional to $1 - s$.

Build a model of population growth under these conditions. How does it compare to the simpler model given in question 1?

3. Data about the world population from 1950 until the present provided by the United Nations can be found at

http://esa.un.org/unpd/wpp/unpp/panel_population.htm

How do our two models fit with this data? Can you determine which model would be better to use to predict future population numbers?

4.2 Part 2

On the UN webpage, the tab labeled “detailed indicators” gives more detailed demographic detail about the world’s population. Use some of this data to create a more detailed model of how the world population changes. Your model should divide the population into different age groups such as children, adults, and retirees, and use the number of people in each age group at one stage to predict the number of people in each age group at the next stage. For example, you might assume that a fixed percentage of the children at one stage become old enough to be considered adults at the next stage. Compare your model to the actual data from the UN webpage.
Chapter 5

Continuous Population Growth

Most of the situations we have modeled so far have divided time up into discrete time periods. This makes sense for many real-world situations, such as in animal populations that have one mating season per year. However, in many other real-world situations, some kind of change is happening continuously. For example, human populations are always growing. We can often model situations where change is happening continuously by using differential equations.

For example, rather than assuming that new births in a population are a fixed percentage of the population from one generation to the next, we can assume that the rate at which the population continuously increases is directly proportional to the current population. In the language of differential equations, we can express this by writing

\[ \frac{dP}{dt} = kP. \]

In order to build a model of this situation, we can start with a given population at time \( t = 0 \) years and then estimate the population at some time in the future using the known slopes of the curve.

5.1 Part 1

1. Build a spreadsheet model of this situation where the new population is computed at one year intervals. That is, use the population and slope at time \( t = 0 \) to predict the population at time \( t = 1 \). Then, use the population and slope at time \( t = 1 \) to predict the population at time \( t = 2 \), and so on. The proportionality constant \( k \) and the initial population should be parameters in your model.

2. In order to get a more accurate model, we can compute more intermediate values for \( P \). Compute monthly values for \( P \). How does this change our estimate of \( P \) after one year?

3. Now, add another parameter \( n \) to your model, and compute \( n \) values per year. To find the real values of our function \( P \), we can take the limit as \( n \) goes to infinity. Try to find a equation for \( P \).

4. How does this problem relate to other problems we have done so far?
5. If we measure our population in units corresponding to our starting population, then we get the initial condition \( P(0) = 1 \). Explore the relationship between \( P(1) \) and \( k \) in this case.

6. One way to help understand this problem is through the use of a slope field. (You may have studied slope fields in a calculus class; they are especially emphasized in the AP Calculus curriculum.) A slope field for a differential equation is found by plotting a line segment at each point on a grid with the slope given by the differential equation. This grid can then be used to visualize the family of curves satisfying the differential equation.

Use GeoGebra to create a slope field for this differential equation with slopes shown for all points with integer coordinates between 0 and 10. What does the slope field tell us about the possible solutions to this differential equation?

7. In Calculus, you learned how to solve differential equations like this. Solve this differential equation using techniques from Calculus, and compare your results to the work you have done so far.

### 5.2 Part 2

In the previous questions, we have extended our analysis of exponential population growth to the case where the growth happens continuously rather than in discrete generations. Now, develop a similar analysis of the logistic population growth described in Section 4.1, Problem 2. First, you will need to develop a differential equation describing logistic growth. Then, plot approximations of a solution for this equation using time intervals of a year and of a month. Next, plot a slope field for your differential equation. Finally, solve the differential equation using methods from calculus.
Chapter 6

Iterating the logistic equation

We have seen that in many modeling situations, it is helpful to iterate a function. In order to get a better understanding of iteration in general, we are going to spend some time exploring what happens when we iterate the function \( f(x) = 2x(1-x) \) for different starting values in the interval \([0, 1]\). More generally, we can examine what happens when we iterate the function \( f_k(x) = kx(1-x) \) for different values of \( k \). This equation is called the logistic equation because it represents the change in a population under logistic growth with carrying capacity 1. If \( x \) is the saturation of some population, then this equation says that the survival rate between generations is directly proportional to one minus the saturation. This differs from the logistic growth we have previously studied, in which the logistic equation gave the change in the population, rather than the new population.

Explore what happens when we iterate the logistic equation. You should develop and explore whatever questions seem interesting to you. Write a report describing your findings.

6.1 Possible questions about iteration of the logistic equation

If you are having trouble deciding what to look at, here are some possible questions to look at and things to try. Note that there are too many questions here for you to look at all of them, and your group may be interested in some other questions entirely, and that is fine. This is just a list of possibilities to get you started.

1. If \( x < 0 \), what happens to \( x \) under iteration? Likewise, what happens if \( x > 1 \)? Can you prove this will always happen?
2. If \( 0 \leq x \leq 1 \), for which values of \( k \) is \( x \) guaranteed to stay between 0 and 1 under iteration?
3. Where are the fixed points of these functions? How do they depend on \( k \)? A fixed point is a value where \( f(x) = x \). You could look at this algebraically or graphically.
4. Try graphing \( f_k(x) \) along with some of its iterates and the line \( y = x \). What does this tell you? (The iterates of \( f_k(x) \) are \( f_k(f_k(x)), f_k(f_k(f_k(x))), \text{etc.} \))
5. Try drawing cobweb diagrams for these functions. A cobweb diagram traces a path on the graph of a function showing what happens to a given point \( x_0 \) when the function is repeatedly applied to that point. To draw a cobweb diagram, a straight line is
Iterating the logistic equation traced from \((x_0, x_0)\) to \((x_0, f(x_0))\), and then from there to \((f(x_0), f(x_0))\), and then on to \((f(x_0), f(f(x_0)))\), etc. It is possible to use either Geogebra or Excel to produce cobweb diagrams.

6. Try making a table of what the function tends to for different values of \(k\) and different starting values of \(x\). For which values of \(k\) is there an attracting fixed point? For which values of \(k\) is there an attracting pair of numbers that alternates? (We call this an attracting 2-cycle.) Can you find cycles of more than 2 numbers?

7. What makes a fixed point attract or repel? What makes it more or less attractive? Look, for example, at values of \(k\) near 2 and near 2.9. What makes a cycle attracting? Try looking, for example, at \(k = 3.2\).

8. If we start with \(k = 2\) and slowly increase \(k\), how does the behavior of the function change? Are there specific points where it changes dramatically?

9. Try drawing a “Family portrait” showing what values the function tends towards: For different values of \(k\) between 0 and 4, iterate a lot of times (say, 40), then plot the point \((k, y)\) where \(y\) is each of the next ten iterates.