Mathematical Modeling
(Instructor Version)

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Acknowledgements

The materials used in these course notes have been drawn from a variety of sources. Many of these problems relate to models discussed in the first several chapters of *The Active Modeler: Mathematical Modeling with Microsoft Excel* by Neuwirth and Arganbright [2]. Many of the questions posed about iteration in Section 6.1 were based on questions written by Avery Solomon for a course at Cornell University. The article [3] that accompanies the group assignment on the sample take-home final originally appeared in the *New York Times*. These materials were originally developed in their current form as part of an NSF MSP grant.
Introduction to the Instructor

These course notes are designed to be used for a one-semester project-based course in mathematical modeling. They have been used for an online Master’s level course for in-service secondary school teachers, as well as for a face-to-face upper-level capstone course for undergraduates.

The course consists of eight interrelated projects in mathematical modeling. In my classes, we have spent approximately two weeks on each project. In general, each project goes as follows:

- I introduce the project, and we discuss it briefly as a class;
- the students then have a week to work on the project and to write it up as carefully as they can;
- I read the students’ write-ups, make detailed comments on them, and lead a class discussion on common issues that came up in the initial write-ups;
- the students then have another week to revise their initial drafts;
- finally, after the students submit their final drafts, I reread and grade them, and then lead a final class discussion about the problem, in which I try to bring out all of the best aspects of all the solutions that were submitted.

Some of the assignments are assigned as group assignments, and some are assigned as individual assignments. Even if an assignment is going to be written up individually, we still usually spend some time working on it in class in groups.

All of the assignments are designed to be done using spreadsheets. Over the first several assignments, I make sure that students are familiar with GeoGebra, Excel, and the Google-Docs spreadsheet program. All of these spreadsheet programs have different strengths and weaknesses, and so I want my students to be able to use all of them. Once they are familiar with all three, I allow them to choose which one they want to use for any given assignment. GeoGebra is particularly nice in that it combines a numerical spreadsheet representation for data with a dynamic geometry representation and an algebraic representation. However, Excel works better for very large data sets and built-in curve fitting. Finally, the Google Docs spreadsheet allows several people to work on the same spreadsheet at one time from different locations.

When this course was taught online, we met for one and a half hours per week using Elluminate, a software package that allows for synchronous classroom meetings via video conferencing. Students then spent a fair amount of time each week working on the problems outside of class time. When the course was taught face-to-face, we met for two and a half hours a week, which allowed more time for groups to work on the projects in class with my guidance.

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In both versions of the class, written assignments were due electronically on BlackBoard one or two days before the class in which we would be discussing it. This allowed me to read and give feedback on the assignments before class, and allowed me to focus the class discussion on the particular issues that came up in people’s work. This was extremely valuable, and I would encourage any instructor using these notes to consider doing the same thing.

In addition to the projects, I also assigned a series of discussion prompts for people to write about on a class discussion board in BlackBoard. The discussion prompts that I used for my online class are reproduced in Appendix C.

This instructor’s version of these course materials includes “Teaching Notes” addressed to the instructor following each project. Sample student solutions to each project are also included in a separate ancillary file which can be obtained by sending an email to me at the following address: nat@alumni.princeton.edu. (Because it contains complete solutions for all of the problems in these notes, the file is not available directly from the JIBLM website.) These are actual solutions submitted by students in my classes, and are used with their permission. These student solutions are generally much more detailed than my notes to the instructor, so I encourage instructors to read through these to see the kinds of work I expect in response to these prompts. These student solutions reflect many aspects of what was discussed in class about these problems.

In putting together these projects, I have tried to adhere to the NCTM’s curriculum principle, which states that “A curriculum is more than a collection of activities: it must be coherent, [and] focused on important mathematics . . . .” While problems from these notes may be used individually, they are interrelated in many ways, and are designed to be used together as a coherent curriculum.

The notes are appropriate for use as a capstone course, and include problems that relate to or use ideas from a wide variety of previous mathematics courses, including discrete mathematics, linear algebra, calculus, and statistics. However, most of the ideas needed from these other courses can be introduced in this course in a just-in-time fashion for students who have have forgotten them or have not seen them before.

I have drawn from a variety of sources in putting together these problems. The one that I have used the most is The Active Modeler: Mathematical Modeling with Microsoft Excel by Neuwirth and Arganbright [2]. Many of the problems in these notes are based on models discussed in this book. It would probably be helpful for anyone thinking about teaching from these notes to also look at the first several chapters of this book. I have sometimes shown the book to my students at the end of the course as a potential further resource. (I would not want them looking at the book during the course, as it contains solutions to many of the problems in the course.)

The following are some of my course policies, taken from the syllabus I hand out on the first day.

**Math 537: Mathematical Modeling**

“The royal road to knowledge, it is easy to express:
to err, and err, and err again,
but less, and less, and less.”

This is a class about mathematical modeling. It will be about using mathematics to model situations in the real world in order to understand it and to make recommendations.
and predictions. It will also be about clearly communicating your findings to others. We will try to understand situations from many different mathematical perspectives, including numerically, graphically, algebraically, and verbally (the “rule of four”).

This course will be made up of a sequence of modeling problems. You will work on these problems, usually in groups, during our class meetings. After you have had some time to work on a problem, you will turn in a written report on it, and we will discuss it as a class. You will then typically have one more week to revise your work and will then turn in a final draft.

Homework assignments will typically be due electronically on Blackboard on Sunday nights at midnight so that I can have time to look them over before our next class meeting time.

Technology: This will be a technology intensive course. We will be using a variety of spreadsheet programs to model problems, and word processing programs to write them up. We will be using Geogebra, Excel, and the Google Docs spreadsheet as spreadsheet programs. GeoGebra is a free program and can be downloaded from the link on the course webpage.

Group Work: We will often work in groups in this course. Whenever a group hands in a written assignment, they are required to put on the paper the names of those who participated fully, and only those names. Your name on the assignment certifies that you participated equally in the project. It is dishonest to turn in work that is not solely and equitably the creation of the team members. You are not required to include on the report the name of someone who started but did not finish, or who did not contribute their share. Groups will be expected to find time to work together on the group problems outside of our class meeting time. This is a three credit course, so you should expect to be spending several hours a week working on this class outside of our official meeting time.

Outside Sources: The central aim of this course is to give you experience developing your own mathematical models. You therefore should not consult outside sources for information about ways that other people have constructed models for the same situations. However, you may wish to look for external data to compare your models to; this is acceptable and is encouraged. So, it is okay to look at external sources for data about situations you are modeling, but not okay to look for solutions to the problems we are working on.

Electronic Discussion Board: An electronic discussion board for this class has been set up on Blackboard. This is a great forum for continuing class discussions outside of our synchronous meeting time. Participation on this discussion board will count as part of your class participation grade. You should try to have on average at least one substantive post on the discussion board per week. If you’d like to, you can subscribe to the discussion board forum on Blackboard so that you get emailed whenever someone posts something.

Homework: Most homework for this class will be written reports. In writing them, you should imagine that you are writing a report as a consultant for a peer: someone who has roughly the same mathematical background that you do (such as another teacher at your school), but who has not yet thought carefully about the problem you are working on. Most reports will be submitted twice: once as a rough draft that I will make comments on, and once as a final report that will receive a grade. Homework submission for this class will be online, through the course Blackboard site.

Final Exam: This class will have a take-home final exam.

Course Objectives:

• Gain experience with modeling as an open-ended process including investigation, analysis, and communication;
• Explore connections to the K-12 curriculum, especially algebra and data analysis;
• Explore modeling related to current events and quantitative literacy;
• Gain experience with the Rule of Four, connecting graphical, algebraic, numerical, and verbal descriptions of problems.

**Plans may change:** Please understand that this syllabus represents my plans for the course, but some things may change depending on how the course is going. Please don’t hesitate to let me know if you have ideas about changes that might improve the class, or if you think the class is too hard, too easy, too fast, or too slow.
Chapter 1

Day 1: A Simple Population Model

Consider the following simple population model: Rabbits reproduce very quickly. Let’s assume that an adult pair of rabbits gives birth to one new pair of baby rabbits every month, a male and a female. We’ll also assume that it takes one month for a baby rabbit to become an adult, and one month’s gestation time between the time a pair of rabbits becomes adult and the time their first pair of offspring are born. For the purposes of this model, we assume that rabbits never die.

We start with a single pair of newborn rabbits at \( t = 0 \) months.

1. Make a chart in GeoGebra showing how many rabbit pairs are alive after \( n \) months. One column should label the number of months that have elapsed, starting with \( t = 0 \) months, and another column should show the number of pairs of rabbits alive at that time. It may be useful to include columns showing the number of babies and adults at any given time.

2. Make a graph of the number of rabbit pairs alive after \( n \) months. The easiest way to do this is to add a new column to your spreadsheet containing the coordinate pairs you would like to graph.

3. When modeling a situation, we often have data points that we would like to fit some kind of curve to in order to make predictions about what will happen in the future. Try to come up with the best function that you can find to match your data points.

4. How could we modify our model to make it more realistic?

Turn in a written report about what you did. In writing your report, you may find it helpful to note that spreadsheets from GeoGebra can be easily copied into Excel or Word. If you want to create a table in Word, first copy your spreadsheet, then select the data and goto “Insert table” in the Table menu.

Teaching Notes:

In this first problem, we try to fit a curve to the Fibonacci sequence. This problem is likely to bring up some basic questions:

- How can we decide what type of function to try to fit to some given data?

- In particular, what characterizes exponential data, and how can we fit an exponential curve to our data?
What ways do we have to measure our error, and to measure how good a fit our curve is?

If they have previously seen the exponential regression feature of their calculators or of a spreadsheet program, students may try to just use that to fit a curve. In this case, I explain to them that they are going to have to write up an explanation of what’s going on here, and the better they can understand what they are doing, the better their explanation is likely to be. Using a built-in feature of their calculator is likely to leave students without any sense of where the numbers the calculator found came from. One possible method for fitting a curve by hand is to look at the limit of the ratios of successive terms, and then to pick a starting value to fit the data with an exponential curve using that limit (which is the golden ratio). Another possible strategy is to graph the natural log of the data and fit a straight line to it.

If you mention to students that one way to improve an initial model is to then look at the error data and try to fit a curve to the error data as well, some enterprising student may notice that, in this case, if you fit an exponential curve to this data, the error data is exponential as well. Once you notice this, you can find an almost exact fit for this data as the sum of two exponential functions working purely inductively.

I do not expect students to find an exact deductive solution to this problem unless they have seen it before. However, it is possible to find such a solution. Any such solution must fit the Fibonacci recurrence relation \( f(n + 1) = f(n) + f(n - 1) \). If \( f(x) \) is an exponential function of the form \( ab^x \), then this recurrence relation becomes \( ab^{n+1} = ab^n + ab^{n-1} \), which simplifies to \( b^2 - b - 1 = 0 \). If we call the two roots of this quadratic equation \( \phi_1 \) and \( \phi_2 \), then the functions \( \phi_1^x \) and \( \phi_2^x \) will both fit the Fibonacci recurrence relation; thus any linear combination of these two functions will, as well. So, to find an exact solution to this problem, we just have to find a linear combination that fits the initial condition \( f(0) = f(1) = 1 \). This should give us the same solution that was found inductively.
Chapter 2

Some financial models

Model the following situations. In each case, you should start out by building an appropriate spreadsheet model, but then also attempt to find a mathematical description of the situation if you can.

1. Money in a savings account accrues a fixed percentage interest each year. (This is usually called the APR, the Annual Percentage Rate.) Create a spreadsheet to show how much money will be in the account after $n$ years.

2. Actually, though, interest in a savings account is normally compounded monthly. Create a model in which interest is compounded monthly. How does the effective annual interest rate compare with the official APR?

3. Create a model in which interest is compounded $n$ times a year. Can you describe what happens as $n$ gets big?

4. The Colorado Mega Millions Lottery offers winners a choice between 26 equal annual payments or else a single lump sum payment of some percentage of the won amount. According to the official lottery rules, “[t]he cash option prize shall be determined by dividing the Grand Prize amount that would be paid over 26 annual installments by a rate established by the Mega Millions Finance Committee prior to each drawing divided by the number of total jackpot winners.” The Mega Millions Finance committee has asked you as a consultant to write a report to help them identify what rate would be a fair equivalent. As you are considering this problem, it will probably be helpful to consider that either the recipient or the lottery board can put whatever money they have into a savings account. So, one approach to this problem would be to figure out how much money a recipient would have after receiving all 26 installments if they put each one immediately into a bank account and didn’t spend any of the money, and to compare this with a lump sum that is likewise immediately put into a bank account.

5. The most common type of loan is one in which a fixed amount of money, called the principal, is lent out at a fixed annual rate and paid back over a fixed period of time in equal installments. Create a model to figure out the size of each installment given information about the other variables. Using either the information from an actual loan that you have or else from a typical loan, use your model to compute what percentage of the money paid back to the bank is interest, and what percentage is repayment of the principal. You could do this for a mortgage, a car loan, a student loan, or some other kind of loan.
**Teaching Notes:**

This set of problems can easily be divided up into two assignments depending on the timing of your class. Problem 5 is closely related to Problem 4, so it could be assigned as a follow-up to Problem 4 once Problem 4 has been discussed.

Note that students are expected to develop a spreadsheet model and then a mathematical description for all of these problems.

Questions 1–3 are designed to lead students to discovering the constant $e$ in a meaningful context. Students should be able to come up with explicit formulas for Questions 1 and 2 along with deductive justifications for the formulas. Students who are comfortable with proofs by induction may want to give such proofs here.

Question 4 develops what is known as the present value of an annuity. In order to find a formula here, students will need to know how to sum a geometric series. The specific answer students get will depend on the interest rate they pick for the bank account.

Question 5 is really just another version of question 4, in which the student is playing the role of the bank. Students are not likely to see it this way, but this problem can be solved directly using the same kinds of methods used to solve question 4. Most students probably have not thought about how loans work before, and may not realize that interest is paid each month on the outstanding balance of the loan.

A useful tool for solving problems 4 and 5 empirically is the Goal Seek function in Excel, which causes the computer to figure out how to set one cell in a spreadsheet in order to cause another cell to equal a predetermined value.
Chapter 3

The temperature in Ft. Collins

The United States Historical Climatology Network provides climate data from across the United States. Go to their website at

http://cdiac.ornl.gov/epubs/ndp/ushcn/ushcn.html

and download historical data about average monthly temperature in Ft. Collins.

Create a formula to model the average monthly temperature in Ft. Collins. How can you measure if your formula is a good match?

Teaching Notes: Most students start by trying to fit a sine curve to the average temperature for each month. This requires them to figure out how to compute these average temperatures using their spreadsheets, and brings up the question of what each constant does in the generic sine function formula \( f(x) = A \sin(Bx + C) + D \). After they have done this, they need to find some way of measuring how good a fit their function is, which leads to a discussion of possible ways to measure the error: absolute vs. signed error, average error vs. total error, squared error vs. unsquared error. Looking at the way the error changes over time should lead to the observation that the temperates are slowly rising over time; this can be added to the model by adding a linear term to our function. Students may not realize that they have measured a local aspect of global warming even after they have done this, so that is something you probably want to bring up if no one mentions it.

The best fitting solutions that my students came up with had an average absolute error of about 2.65 degrees.

While we were working on this assignment, I also assigned the discussion group question about how to find ways to make numbers meaningful. This tied nicely into our discussion of ways to understand the meaning of all of the numbers that arose in the functions that students came up with.

This assignment is not that closely related to the other assignments, so if you had to leave one out, this might be a good choice. However, I like having an assignment that uses a large real data set that is closely tied to an area of current national and international concern (global warming).
Chapter 4

More Population Growth

4.1 Part 1

1. One simple model of population growth is one in which a fixed percentage of all animals alive at one stage reproduce and add their offspring to the next stage. Model this situation.

2. A somewhat more sophisticated model, originally studied in the mid-nineteenth century, assumes that there is a maximum size of the population we are studying, called its carrying capacity. We then define the current saturation \( s \) to be the fraction of the carrying capacity that the current population represents. In this model, we assume that the percentage of the current population that reproduces at any stage is directly proportional to \( 1 - s \).

Build a model of population growth under these conditions. How does it compare to the simpler model given in question 1?

3. Data about the world population from 1950 until the present provided by the United Nations can be found at

   http://esa.un.org/unpd/wpp/unpp/panel_population.htm

   How do our two models fit with this data? Can you determine which model would be better to use to predict future population numbers?

Teaching Notes: Question 1 gives rise to the same exponential model that we have already seen when looking at simple interest. Question 2 gives rise to a more complicated model known as a logistic population growth model. Note that this is a discrete logistic model, and so it is slightly different from the continuous logistic model that some students may have previously encountered in a calculus or differential equations class. Students will not be able to solve the recursive equation developed in Question 2 to find an explicit formula describing the current population at any given stage \( n \). Students should try to understand why that is, and what is different here from the exponential case. We will eventually find an explicit formula for a continuous version of this model in Section 5.2.

Once the students have built these two models, question 3 asks them to fit both of them to actual world population data to see how they match up.
4.2 Part 2

On the UN webpage, the tab labeled “detailed indicators” gives more detailed demographic detail about the world’s population. Use some of this data to create a more detailed model of how the world population changes. Your model should divide the population into different age groups such as children, adults, and retirees, and use the number of people in each age group at one stage to predict the number of people in each age group at the next stage. For example, you might assume that a fixed percentage of the children at one stage become old enough to be considered adults at the next stage. Compare your model to the actual data from the UN webpage.

Teaching Notes: Part two asks students to build a model that divides the population up into three categories—children, adults, and retirees—and then to model how these categories influence one another over time. We can write down linear equations that model how each category at one stage depends on the three categories at the previous stage. For example, letting $C_n$, $A_n$, and $R_n$ represent the children, adults, and retirees alive at stage $n$, we might have

$$C_n = \alpha_{CC} C_{n-1} + \alpha_{CA} A_{n-1} + \alpha_{CR} R_{n-1},$$

where $\alpha_{CA}$ would represent the birth rate (the percentage of adults of child-bearing age that have children during the time between two stages); $\alpha_{CR}$ would represent the percentage of retirees having children, which would presumably be zero; and $\alpha_{CC}$ would represent the percentage of children at one stage that were still children at the next stage. If we write down similar equations for $A_n$ and $R_n$, we can combine them all in the matrix equation

$$\begin{bmatrix} C_n \\ A_n \\ R_n \end{bmatrix} = \begin{bmatrix} \alpha_{CC} & \alpha_{CA} & \alpha_{CR} \\ \alpha_{AC} & \alpha_{AA} & \alpha_{AR} \\ \alpha_{RC} & \alpha_{RA} & \alpha_{RR} \end{bmatrix} \begin{bmatrix} C_{n-1} \\ A_{n-1} \\ R_{n-1} \end{bmatrix}.$$ 

If we let $T$ denote the matrix in this equation, and let $\vec{P}_n = (C_n, A_n, R_n)$, then we can write this more simply as $\vec{P}_n = T \vec{P}_{n-1}$. Note that this is directly analogous to the recursion we would write for exponential growth, and so, just as in the case of exponential growth, we obtain the equation $\vec{P}_n = T^n \vec{P}_0$.

A big part of this assignment is finding appropriate values for the various constants in $T$ from the data on the given webpage. Many of the constants can be estimated without looking at the data, of course. For example, $\alpha_{CR}$ and $\alpha_{AR}$ should be zero, since the number of children and adults at one stage does not depend on the number of retirees at the previous stage. Likewise, $\alpha_{RC}$ should be zero, since the number of retirees at one stage doesn’t depend on the previous number of children. As another example, if we classify people aged 0–14 as children, and let each stage last five years (since this corresponds to our given data), then we would expect $\alpha_{CC}$, the percentage of children that were children in the previous stage, to be about 2/3. Likewise, we would expect $\alpha_{AC}$, the percentage of children who became adults, to be about 1/3. We can estimate $\alpha_{AA}$ and $\alpha_{RA}$ similarly. (If we want, we can also include an estimated death rate in these numbers.) $\alpha_{RR}$ and $\alpha_{CA}$, which represent the death rate among the retirees and the birth rate among the adults of child-bearing age, are harder to estimate, but we could probably come up with something using an estimated average lifespan and an average number of offspring during that lifespan.

All of these numbers, however, can be found more accurately by computing them directly from the UN data. Note that the “Population by five-year age group and sex” data is most useful here.
Chapter 5

Continuous Population Growth

Most of the situations we have modeled so far have divided time up into discrete time periods. This makes sense for many real-world situations, such as in animal populations that have one mating season per year. However, in many other real-world situations, some kind of change is happening continuously. For example, human populations are always growing. We can often model situations where change is happening continuously by using differential equations.

For example, rather than assuming that new births in a population are a fixed percentage of the population from one generation to the next, we can assume that the rate at which the population continuously increases is directly proportional to the current population. In the language of differential equations, we can express this by writing

\[
\frac{dP}{dt} = kP.
\]

In order to build a model of this situation, we can start with a given population at time \( t = 0 \) years and then estimate the population at some time in the future using the known slopes of the curve.

5.1 Part 1

1. Build a spreadsheet model of this situation where the new population is computed at one year intervals. That is, use the population and slope at time \( t = 0 \) to predict the population at time \( t = 1 \). Then, use the population and slope at time \( t = 1 \) to predict the population at time \( t = 2 \), and so on. The proportionality constant \( k \) and the initial population should be parameters in your model.

2. In order to get a more accurate model, we can compute more intermediate values for \( P \). Compute monthly values for \( P \). How does this change our estimate of \( P \) after one year?

3. Now, add another parameter \( n \) to your model, and compute \( n \) values per year. To find the real values of our function \( P \), we can take the limit as \( n \) goes to infinity. Try to find a equation for \( P \).

4. How does this problem relate to other problems we have done so far?
5. If we measure our population in units corresponding to our starting population, then we get the initial condition \( P(0) = 1 \). Explore the relationship between \( P(1) \) and \( k \) in this case.

6. One way to help understand this problem is through the use of a slope field. (You may have studied slope fields in a calculus class; they are especially emphasized in the AP Calculus curriculum.) A slope field for a differential equation is found by plotting a line segment at each point on a grid with the slope given by the differential equation. This grid can then be used to visualize the family of curves satisfying the differential equation.

Use GeoGebra to create a slope field for this differential equation with slopes shown for all points with integer coordinates between 0 and 10. What does the slope field tell us about the possible solutions to this differential equation?

7. In Calculus, you learned how to solve differential equations like this. Solve this differential equation using techniques from Calculus, and compare your results to the work you have done so far.

### 5.2 Part 2

In the previous questions, we have extended our analysis of exponential population growth to the case where the growth happens continuously rather than in discrete generations. Now, develop a similar analysis of the logistic population growth described in Section 4.1, Problem 2. First, you will need to develop a differential equation describing logistic growth. Then, plot approximations of a solution for this equation using time intervals of a year and of a month. Next, plot a slope field for your differential equation. Finally, solve the differential equation using methods from calculus.

**Teaching Notes:** In previous projects, we have developed the number \( e \) as the amount of money in a bank account after one year if we start with $1 and the bank account pays 100% interest compounded continuously. In this project, we connect this to the description of \( e \) as the base of the exponential function whose value at any point gives the slope at that point. This project also allows us to see how the discrete models we have been using are connected to continuous models, and it allows us to finally find an explicit function to model logistic growth, which we couldn’t find in the discrete case.

Students should be able to build a slope field from scratch using GeoGebra. This is much more memorable than using a pre-built slope field application found on the web. For part two, the carrying capacity should be chosen to be less than 10 so that the difference between the slope field in part 1 and the slope field in part 2 can be seen.

I usually have students work through Part 1 in groups, but then do Part 2 on their own to show that they absorbed what was done in Part 1.

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Chapter 6

Iterating the logistic equation

We have seen that in many modeling situations, it is helpful to iterate a function. In order to get a better understanding of iteration in general, we are going to spend some time exploring what happens when we iterate the function \( f(x) = 2x(1-x) \) for different starting values in the interval \([0, 1]\). More generally, we can examine what happens when we iterate the function \( f_k(x) = kx(1-x) \) for different values of \( k \). This equation is called the logistic equation because it represents the change in a population under logistic growth with carrying capacity 1. If \( x \) is the saturation of some population, then this equation says that the survival rate between generations is directly proportional to one minus the saturation. This differs from the logistic growth we have previously studied, in which the logistic equation gave the change in the population, rather than the new population.

Explore what happens when we iterate the logistic equation. You should develop and explore whatever questions seem interesting to you. Write a report describing your findings.

6.1 Possible questions about iteration of the logistic equation

If you are having trouble deciding what to look at, here are some possible questions to look at and things to try. Note that there are too many questions here for you to look at all of them, and your group may be interested in some other questions entirely, and that is fine. This is just a list of possibilities to get you started.

1. If \( x < 0 \), what happens to \( x \) under iteration? Likewise, what happens if \( x > 1 \)? Can you prove this will always happen?

2. If \( 0 \leq x \leq 1 \), for which values of \( k \) is \( x \) guaranteed to stay between 0 and 1 under iteration?

3. Where are the fixed points of these functions? How do they depend on \( k \)? A fixed point is a value where \( f(x) = x \). You could look at this algebraically or graphically.

4. Try graphing \( f_k(x) \) along with some of its iterates and the line \( y = x \). What does this tell you? (The iterates of \( f_k(x) \) are \( f_k(f_k(x)), f_k(f_k(f_k(x))), \) etc.)

5. Try drawing cobweb diagrams for these functions. A cobweb diagram traces a path on the graph of a function showing what happens to a given point \( x_0 \) when the function is repeatedly applied to that point. To draw a cobweb diagram, a straight line is
Iterating the logistic equation

traced from \((x_0, x_0)\) to \((x_0, f(x_0))\), and then from there to \((f(x_0), f(x_0))\), and then on to \((f(x_0), f(f(x_0)))\), etc. It is possible to use either Geogebra or Excel to produce cobweb diagrams.

6. Try making a table of what the function tends to for different values of \(k\) and different starting values of \(x\). For which values of \(k\) is there an attracting fixed point? For which values of \(k\) is there an attracting pair of numbers that alternates? (We call this an attracting 2-cycle.) Can you find cycles of more than 2 numbers?

7. What makes a fixed point attract or repel? What makes it more or less attractive? Look, for example, at values of \(k\) near 2 and near 2.9. What makes a cycle attracting? Try looking, for example, at \(k = 3.2\).

8. If we start with \(k = 2\) and slowly increase \(k\), how does the behavior of the function change? Are there specific points where it changes dramatically?

9. Try drawing a “Family portrait” showing what values the function tends towards: For different values of \(k\) between 0 and 4, iterate a lot of times (say, 40), then plot the point \((k, y)\) where \(y\) is each of the next ten iterates.

Teaching Notes: This question is clearly much more open-ended than the previous questions, and also more abstract. It deals with discrete dynamical systems, which is an ideal topic for students to explore on their own using spreadsheets, and allows students to develop conjectures and then prove them themselves. While this project isn’t explicitly tied to a real-world modeling problem here, the group question on the take-home final, in Section A.2, is designed to connect the work done here with discrete dynamical systems to the work we have previously done with population growth.

One possible source for further information about discrete dynamical systems is the book *An Introduction to Chaotic Dynamical Systems* by Robert Devaney [1], which explicitly discusses iteration of the logistic equation at some length.
Appendix A

Take Home Final

In completing this exam, you may consult your notes and handouts from this class, but you may not consult any other sources or discuss the problems with anyone other than me (and your group members, in the case of the group problem).

Your final copy will not be accepted if it does not include the following honor pledge:

“I pledge on my honor that this examination represents my own work in accordance with University and class rules; I have not consulted any outside sources, nor have I given or received help in completing this exam.”

A.1 Individual Work: Modeling the spread of an infectious disease

The Centers for Disease Control and Prevention are responsible for tracking the spread of infectious diseases though the U.S. population. They have asked you, as a consultant, to model the spread of a strain of flu through a fixed population of people. In building your model, you should assume that the population can be divided into three categories of people: susceptible people, who have never been infected; infected people; and recovered people, who have had this strain of flu and are now immune. Your model should track how many people are in each of these categories each week. You can assume that each person encounters a fixed percentage $p$ of the total population each week, and that each time an infected person encounters a susceptible person, there is a fixed probability $k$ that they will pass on the flu to them. Historical data from the CDC indicates that, on average, a person is infected for one week, and the total number of people infected tends to peak during the eighth week of an outbreak.

Since the CDC is tasked with limiting the number of people exposed to the flu, they would like you to also develop a model for what happens when a certain number of people are vaccinated for the flu each week. According to your models, is it more effective to try to vaccinate more people, or to limit contact with infected people?

Submit your spreadsheet model along with a written report to the CDC explaining your model and your conclusions.

Teaching Notes:

This problem is similar to the problem about modeling population growth using different age groups in Section 4.2. As usual, students will have to choose appropriate constants
in building their models, and will probably have to play with them a bit in order to get the outbreak to peak in the eighth week.

A.2 Group Work: Controlling animal populations

A recent article in the New York Times discusses plans to reduce the number of resident Canada geese in New York State. The US Fish and Wildlife service is tasked with measuring and regulating the US populations of fish and wildlife. They have asked you, as consultants, to write a report examining the result of several possible policies with respect to hunting and fishing. For different animals and fish, they have the option of allowing a fixed number of animals or a certain percentage of the current population to be killed by hunters each year. We can also model the growth of the population using either an exponential model, or else by using a logistic model. This gives rise to four possible scenarios: exponential growth with constant harvesting, exponential growth with proportional harvesting, logistic growth with constant harvesting, and logistic growth with proportional harvesting. Compare these four scenarios. In particular, can you describe the possible long term behaviors of each? It may be helpful to use your new-found knowledge of iteration here.

Submit any spreadsheet models you develop along with a written report to the US Fish and Wildlife Service explaining your conclusions.

A.2.1 Agencies Plan to Reduce Canada Geese Population in New York State by Two-Thirds

This article written by Isolde Raftery was published in the New York Times on July 23, 2010.

Officials plan to reduce the number of Canada geese in New York State by two-thirds, eventually trimming the population to 85,000 from 250,000, according to a report prepared by several city, state and federal agencies.

The reduction is part of a larger plan that also calls for the near halving of the Canada geese population in 17 Atlantic states, to 650,000 from 1.1 million. The New York Times obtained a copy of the report.

In New York City, the report says, the current goose population of 20,000 to 25,000 is “five times the amount that most people would find socially acceptable,” suggesting the number would be reduced to about 4,000.

A high-level official of the United States Department of Agriculture who is familiar with the proposal called it a “one-of-a-kind plan.”

“New York is leading the way,” he said, speaking on condition of anonymity because he was not authorized to speak to the press. Plans for other areas, he said, “do not include all the scientific background.”

The document contained specifics of goose removal in the New York area: “The captured geese are placed alive in commercial turkey crates. The geese would be brought to a secure location and euthanized with methods approved by the American Veterinary Medical Association. Euthanized geese would be buried.”

The plan emerged from five months of meetings that followed the crash-landing of US Airways Flight 1549 in the Hudson River after geese flew into its engines and disabled them in January 2009. The plan was completed in summer 2009 but not made public.
Glenn Phillips, executive director of New York City Audubon, called the plan “a little extreme.”

“It’s clear that some action needs to be taken,” Mr. Phillips said. “It’s not clear that there are really five times as many Canada geese as there should be.”

He mentioned a study written last year by the wildlife services division of the Agriculture Department that says goose killings near La Guardia Airport reduced bird hits by 80 percent.

The report also said killing geese that live farther from the airport might not limit situations like the US Airways one.

Maureen Wren, a spokeswoman for the State Department of Environmental Conservation, which oversees management of Canada geese in the state, emphasized that the proposed reduction was a long-term goal. Statewide, she said, the goose population will also continue to be managed through hunting and treating fertilized eggs.

In years ahead, however, the state could ask the federal government to extend hunting seasons or increase the number of geese that hunters may shoot.

Susan Russell, a former vice president of Friends of Animals, who lives in New Jersey, was shocked when she learned on Friday of the proposal to remove 170,000 Canada geese.

“There’s something about the Canada goose they’re extremely loyal, they’re emotional, they’re garrulous and they’re funny, they’re brave,” Ms. Russell said. “I observe this year after year.”

Government officials who attended the meetings that led to the plan peppered wildlife specialists about other options, the Agriculture Department official said. The questioners included officials from the New York City Parks Department, the National Park Service and Mayor Michael R. Bloomberg’s office, he said.

“They wanted to know facts; they wanted to know why we can’t harass them and make them go away,” he said. “But when you do that, it’s just making problems for someone else. People with money push the geese onto people without the money. That’s usually what happens when you get rid of harassment problems.”

In the meetings, the officials learned that there had been 78 Canada goose strikes over 10 years at local airports and that those strikes caused more than $2.2 million in aircraft damage. And they were reminded that 24 people were killed in 1995 when an Air Force surveillance plane struck Canada geese in Alaska.

It is unknown how many geese have been killed so far under the plan.

The first steps outlined in the plan went into effect last summer, when 1,235 geese in the city were gassed to death. The total for this summer remains unknown, though nearly 400 were killed after being rounded up in Prospect Park this month.

“Prospect Park that was a bad situation for the geese,” the Agriculture Department official said. “They were being fed doughnuts and bread by people. That’s wrong.

“I know people had strong emotional bonds to the animals, and there’s no way for me to explain to most of them that the Canada geese don’t need the people” to feed them.

Next year, officials will count how many geese return to the parks to determine if the killings had an effect.

But the count has already begun, at least informally. People spotted 28 geese waddling along the lake shore in Prospect Park on Thursday.

Teaching Notes: This problem can be done without reference to the included article [3]; however, this article gives a point of reference, and provides one particular population
that students can try to model if they so choose.

These four models are best understood using the iterative techniques developed in Chapter 6, and, because of this, they make a great final problem that ties together everything that has been done in the class. Note that the functions that we are iterating are the functions that give the population at one stage as a function of the population at the previous stage.

Exponential growth with proportional harvesting \( f(x) = x + kx - dx = k_2x \) just gives another exponential model with a different proportionality constant. This exponential model has a fixed point at zero; if the proportionality constant is greater than one, it is a repelling fixed point, and if it is less than one, it is an attracting fixed point. This means that if the death rate is greater than the birth rate, the population will grow without bound, while if it is less than the birth rate, it will decrease to zero.

Exponential growth with fixed harvesting \( f(x) = x + kx - h \) gives a model with a single fixed point which occurs when the number of animals harvested exactly matches the number of new animals born. This fixed point is repelling, which means that if we start with more animals than this, the number of animals will grow without bound, while if we have few than this, the number will decrease to zero.

Logistic growth with fixed harvesting \( f(x) = x + kx(1 - x/c) - h \) gives rise to a model that can have two, one, or zero fixed points, depending on the number of animals harvested at each stage. If zero animals are harvested, then we just have logistic growth. In this case we have a repelling fixed point at the origin, and an attracting fixed point at the carrying capacity. As we increase \( h \), the two fixed points move closer together, until finally, when \( h = \frac{.25k}{c} \), there is a single fixed point at \( .5c \) which is attracting from above but repelling from below. (This is because the greatest growth rate in the logistic model occurs when the population is \( .5c \).) If \( h \) is bigger than \( .5c \), there is no fixed point and the population declines to zero. Thus, in this model, we can cause the population to tend to any value between \( .5c \) and \( c \) by picking an appropriate harvesting rate.

Finally, logistic growth with proportional harvesting \( f(x) = x + kx(1 - x/c) - dx \) gives us a model that has two fixed points as long as \( d < k \). In this case, there is a repelling fixed point at zero, and an attracting fixed point above zero. This is the only one of the four cases in which we can make any value we want an attracting fixed point by picking an appropriate constant for \( d \), so this is the only case in which we can always cause our population to reach a desired size using a consistent course of action.
Appendix B

Sample Schedule

This is the schedule followed by my online classes that met once a week. Each assignment was introduced in one week’s class; a rough draft was due one week later, and the final draft was due two weeks later, at which time we would have a final discussion about the assignment and then start on the next assignment. Thus, each assignment took two weeks to complete. When this class was offered in a face-to-face format, we met twice a week, and so there was more flexibility to spend a little more or a little less time on each assignment. However, we still spent approximately two weeks on each one.

Sample Class Schedule

<table>
<thead>
<tr>
<th>weeks</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td>Assignment 1: A Simple Population Model</td>
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<tr>
<td>3–4</td>
<td>Assignment 2: Some Financial Models</td>
</tr>
<tr>
<td>5–6</td>
<td>Assignment 3: The Temperature in Ft. Collins</td>
</tr>
<tr>
<td>7–8</td>
<td>Assignment 4: More Population growth, Part 1</td>
</tr>
<tr>
<td>9–10</td>
<td>Assignment 4: More Population growth, Part 2</td>
</tr>
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<td>11–12</td>
<td>Assignment 5: Continuous Models</td>
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<td>15–16</td>
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</tr>
</tbody>
</table>
Appendix C

Discussion Group Prompts

The following are the discussion prompts that I assigned throughout the semester for students to write about on our class discussion board on BlackBoard. During weeks when there was no assigned discussion prompt, students were expected to discuss questions that came up as they were working on their class projects.

- Introduce yourself to your classmates in a post on the course discussion board. Your introduction should include whatever you think would be useful for me and your classmates to know about you, and should include some interesting fact about yourself that you think most other people in the class don’t already know about you. You should also let us know what your previous experience with modeling has been—have you ever taken a mathematical modeling class before? If so, what was it like? Have you ever done anything that you would consider to be mathematical modeling with your students?

- Find a number from the real world that is hard to understand, and find a way to explain it so that it is meaningful. Post your explanation in the discussion board.

- Post to the discussion group on the topic of how you might be able to adapt some aspect of this course to your own classroom.

- Pick some part of the NCTM standards—this could be a content standard, or a process standard, or one of the bulletpointed substandards under one of these standards. Discuss it in relation to our class and in relation to your own class.

- Look at the webpage for the High School Mathematical Contest in Modeling, at http://www.comap.com/highschool/contests/. In particular, look at some of the old questions. Discuss some aspect(s) of some old question(s).
Bibliography

