Calculus III

W. Ted Mahavier

Lamar University
## Contents

**To the Student** iii  
- What is calculus? .................................................. iii  
- How this class works .............................................. iii  
- A common pitfall .................................................. iii  
- Board work .......................................................... iv  
- How to study ......................................................... iv  

1 Vectors and Lines 1  
2 Cross Product and Planes 6  
3 Limits and Derivatives 9  
4 Optimization and Lagrange Multipliers 19  
5 Integration 23  
6 Line Integrals, Flux, Divergence, Gauss’ and Green’s Theorem 32  
7 Practice Problems 40  
   - 7.1 Vectors, Lines, and Functions Drill .......................... 40  
   - 7.2 Cross Products, Planes, and Limits Drill ......................... 43  
   - 7.3 Domains, Graphing, and Derivatives Drill ......................... 45  
   - 7.4 Optimization and Lagrange Multipliers Drill ..................... 50  
   - 7.5 Integration Drill .............................................. 53  
   - 7.6 Line Integrals, Flux, Divergence, and Gauss’ Theorem Drill .... 58
To the Student

What is calculus?

The first semester of Calculus consisted of four main concepts: *limits, continuity, differentiation* and *integration*. Limits are required for defining each of continuity, differentiation, and integration. All four concepts are central to an understanding of applications in fields including biology, business, chemistry, economics, engineering, finance, and physics. The second semester extends your study of integration techniques and adds sequences and series. The third semester of calculus is a repeat of these concepts in higher dimensions.

In addition to mastering these concepts, I hope to impart in you the essence of the way a mathematician thinks of the world, an axiomatic way of viewing the world. And I hope to help you master the important skill of solving some difficult problems on your own, communicating these solutions to your peers, and responding to any questions your peers have.

How this class works

This class will be taught in a way that is (most likely) different from mathematics classes you have encountered in the past. Much of the class will be devoted to students working problems at the board and much of your grade will be determined by the amount of mathematics that you produce in this class. I use the word produce because it is my belief that the best way to learn mathematics is by doing mathematics.

Therefore, just as I learned to ride a bike by getting on and falling off, I expect that you will learn mathematics by attempting it and (occasionally) falling off! You will have a set of notes (provided by me) that you will turn into a book by working through the problems. If you are interested in watching someone else put mathematics at the board, working ten problems like it for homework, and then regurgitating this material on tests, then you are not in the correct class. Still, I urge you to seriously consider the value of becoming an independent thinker who tackles doing mathematics (and everything else in life) on your own, rather than waiting for someone else to show you how to do things.

A common pitfall

There are two ways in which students often approach my classes. The first is to say, “I’ll wait and see how this works and then see if I like it and put some problems up later in the semester after I catch on.” Think of it as a forty yard dash. Do you really want to wait and see how fast the other runners are? If you try every night to do the problems then either you will get a problem (YAY!) and be able to put it on the board with pride and satisfaction or you will struggle with the problem, learn a lot in your struggle, and then watch someone else put it on the board. When this person
puts it up you will be able to ask questions and help yourself and others understand it, as you say to yourself, “Ahhhh, now I see where I went wrong and now I can do this one and a few more for the next class.” If you do not try problems each night, then you will watch the student put the problem on the board, but perhaps will not quite catch all the details and then when you study for the tests or try the next problems you will have only a loose idea of how to tackle such problems. Basically, you have seen it only once in this case. The first student saw it once when s/he tackled it on his or her own, again when either s/he put it on the board or another student presented it, and then a third time when s/he studied for the next test or quiz. Hence the difference between these two approaches is the difference between participating and watching a movie. I hope that each of you will tackle this course with an attitude that you will learn this material and thus will both enjoy and benefit from the class.

**Board work**

Because the board work constitutes a reasonable amount of your grade, let’s put your mind at ease regarding this part of the class. First, by coming to class today you have a sixty percent grade on board work. Every problem you present pushes that grade a little higher. You may come see me any time for an indication of what I think your current level of participation will earn you at the end of the semester for this portion of the grade.

Here are some rules and guidelines associated with the board work. I will call for volunteers every day and will pick the person with the least presentations to present a given problem. You may inform me that you have a problem in advance (which I appreciate), but the problem still goes to the person with the least presentations on the day I call for a solution. Ties are broken either randomly (at the beginning) or by test grades (lower test grades taking priority). A student who has not gone to the board on a given day will be given precedence over a student who has gone to the board that day. To “present” a problem at the board means to have written the problem statement up, to have written a correct solution using complete mathematical sentences, and to have answered all students’ questions regarding the problem.

Since you will be communicating with other students on a regular basis, here are several guidelines that will help you. First, the whole class is on your side and wants to see you understand and present the problem correctly both for your sake and for their understanding. When you speak, don’t use the words “obvious,” “stupid,” or “trivial.” Don’t attack anyone personally or try to intimidate anyone. Don’t get mad or upset at anyone (and if you do, try to get over it quickly). Don’t be upset when you make a mistake – brush it off and learn from it. Don’t let anything go on the board that you don’t fully understand. Don’t say to yourself, “I’ll figure this out at home.” Don’t use concepts we have not defined. Don’t use or get examples or solutions from other sources without acknowledging it during the presentation. Don’t work together without acknowledging it at the board. Don’t try to put up a problem you have not written up.

Do prepare arguments in advance. Do be polite and respectful. Do learn from your mistakes. Do ask questions such as, “Can you tell me how you got the third line?” Do let people answer when they are asked a question. Do refer to earlier results and definitions by number when possible.

**How to study**

1. Read over your notes from class that day.

W. Ted Mahavier

www.jiblm.org
2. Make a list of questions to ask at the beginning of the next class.

3. Review the old and read the new definitions.

4. Work on several new problems.

5. Write up as many solutions as you can so that you can simply copy your well-written solutions onto the board the next day.
Chapter 1

Vectors and Lines

Welcome to calculus in three dimensions. The beauty of this material is how closely it parallels your first semester of calculus. After a brief introduction to the coordinate plane, you learned how to graph lines and parabolas. After a brief introduction to three-space, we will be graphing planes and paraboloids. Just as we defined continuity in terms of limits in Calculus I, we will define continuity of functions of several variables in terms of limits in this course. Lines were important as they allowed you to define tangent lines to functions and the derivative. Tangent planes to functions and surfaces will aid our definition of derivative. Once you understood the derivative, you used it to find maxima and minima of real valued functions and we will use the derivative of functions in three-space to find maxima and minima in our applications as well. Two of the most central ideas of your first calculus course were the chain rule and the fundamental theorem of calculus. We will extend your notion of the chain rule and the fundamental theorem in this course. Only near the end of the course will the work we do not have a parallel to your first course. At the end, we’ll cover Green’s and Gauss’ Theorems, which are necessary tools in physics and engineering, but which had no parallel in Calculus I.

Definition 1. $\mathbb{N}$ is the set of all Natural Numbers.

Definition 2. $\mathbb{R}$ is the set of all Real Numbers.

We will also use the notation, “$\in$,” to mean “is an element of.” Thus, “$x \in \mathbb{R}$” means “$x$ is an element of $\mathbb{R}$,” or “$x$ is a real number.” Similarly, $x, y \in \mathbb{R}$ means “$x$ and $y$ are real numbers.” In Calculus I and II, you lived in 2-space, or $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. Now you have graduated to 3-space!

Definition 3. Three dimensional space (Euclidean 3-space or $\mathbb{R}^3$), is the set of all ordered sequences of 3 real numbers. That is

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}.$$

Of course, there is no reason to stop with the number 3. More generally, $n$-space or $\mathbb{R}^n$ is the set of all ordered sequences of $n$ real numbers, but we will spend most of our time concerned with only $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, and occasionally, $\mathbb{R}^4$. Euclidean 4-space is handy since one might want to consider an object or shape in 3-space that is moving with respect to time, thus adding a 4th dimension.

We will write elements in $\mathbb{R}^3$ just as letters; hence, by $x \in \mathbb{R}^3$ we mean the element, $x = (x_1, x_2, x_3)$ where $x_1, x_2, x_3 \in \mathbb{R}$. The origin is the element, $o = (0, 0, 0)$.

Definition 4. If $x, y \in \mathbb{R}^3$ then $\overrightarrow{xy}$ is the directed line segment from $x$ to $y$. We abbreviate $\overrightarrow{ox}$ by $\overrightarrow{x}$. Directed line segments are referred to as vectors.
Physicists and mathematicians often speak of a vector’s magnitude and direction. Given a vector, \( \vec{x} \), by magnitude (or norm) we mean the distance between the point, \( x \), and the origin, \( o \). By direction we mean the direction determined by the directed line segment \( \vec{x} \) that has base at \( o \) and tip at the point, \( x \). When we say to “sketch the vector \( \vec{x} \)” we mean to draw the directed line segment from the origin to the point, \( x \).

We are making a distinction between points and vectors. Points are the actual elements of 3-space and vectors are directed line segments. The word scalar will be used to refer to real numbers (and later in your mathematical career as complex numbers or elements of any field). The word point may be used to mean a real number, an element of \( \mathbb{R}^2 \), an element of \( \mathbb{R}^3 \), etc.

Having carefully made clear the distinction between point and vector you will have to work hard to keep me honest; I tend to use the two more or less interchangeably.

**Definition 5.** If \( x, y \in \mathbb{R}^3 \), with \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \), and \( \alpha \in \mathbb{R} \) then:

- \( x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \) “Addition in \( \mathbb{R}^3 \)”
- \( \alpha x = (\alpha x_1, \alpha x_2, \alpha x_3) \) “Scalar Multiplication in \( \mathbb{R}^3 \)”

Now, to be precise, we should also define vector addition and scalar multiplication for vectors. Since the definition is identical (except for placing arrows above the \( x \) and \( y \)) we omit this. As you can see, always making a distinction between a point and a vector can be cumbersome.

**Problem 1.** Let \( \vec{x} = (1,2) \) and \( \vec{y} = (5,2) \) and sketch \( \vec{x}, \vec{y}, -\vec{x}, 2\vec{y} \).

Consider the two vectors, \( \vec{x} \vec{y} \) and \( \vec{y} - \vec{x} \). Both vectors have the same direction and the same magnitude. They are different because the vector \( \vec{y} - \vec{x} \) has its base at the origin and its tip at the point \( y - x \) while \( \vec{x} \vec{y} \) has its base at \( x \) and its tip at \( y \).

**Problem 2.** Sketch \( \vec{x} + \vec{y} \), \( \vec{x} - \vec{y} \) and \( \vec{x} \vec{y} \) where \( \vec{x} = (2,3) \) and \( \vec{y} = (4,2) \).

**Definition 6.** Let \( n \in \mathbb{N} \). A function from \( \mathbb{R} \) to \( \mathbb{R}^n \) is called a parametric curve.

We will be concerned primarily with vector valued functions where the range is \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). If \( n = 2 \) then such functions are also called planar curves and if \( n = 3 \) they are called space curves. We’ll put a vector symbol, \( \rightarrow \), over such functions to remind us that the range is not a real number but a point in \( \mathbb{R}^n \).

**Problem 3.** Let \( \vec{T}(t) = (2t, 3t) \). Sketch \( \vec{T} \) for all \( t \in \mathbb{R} \). If \( t \) represents time and \( \vec{T}(t) \) represents the position of a llama at time \( t \) then how fast is the llama traveling?

**Problem 4.** Let \( \vec{T}(t) = (1, 2) + (4, 5)t \). Sketch \( \vec{T} \) for all \( t \in \mathbb{R} \) and give the speed of a lemur whose position in the plane at time \( t \) seconds is given by \( \vec{T}(t) \).

**Problem 5.** What is the distance between the point (1,2,3) and the origin? What is the distance between (1,2,3) and (4,5,6)?

**Problem 6.** Sketch \( \vec{r}(t) = t(1,2,3) + (1-t)(3,4,-5) \) for all \( t \in \mathbb{R} \). Compute \( \vec{r}(0) \) and \( \vec{r}'(1) \).

**Problem 7.** What is the distance between \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \)?

**Problem 8.** Let \( x = (2, 5, -1) \) and \( y = (1, 2, 4) \).

1. Write an equation, \( \vec{T} \), for the line in \( \mathbb{R}^3 \) passing through \( x \) and \( y \) with \( \vec{T}(0) = x \) and \( \vec{T}(1) = y \).
2. Write an equation, $\mathbf{m}$, for the line in $\mathbb{R}^3$ passing through $x$ and $y$ so that $\mathbf{m}(0) = x$ and the speed of an object with position determined by the line is twice the speed of an object with position determined by the line in part 1 of this problem.

**Problem 9.** Find infinitely many parametric equations for the line passing through $(a,b,c)$ in the direction $(x,y,z)$.

**Problem 10.** Plot some points in order to graph $\mathbf{r}(t) = (\sin(t), \cos(t), t)$ for $t \in [0, 6\pi]$. How would you write the set representing the range of $\mathbf{r}$?

In Calculus I and II you studied functions from $\mathbb{R}$ to $\mathbb{R}$, for example $f(x) = x^3$, and functions from $\mathbb{R}$ to $\mathbb{R}^2$, for example $\mathbf{r}(t) = (\cos(t), \sin(t))$. Now we have added functions from $\mathbb{R}$ to $\mathbb{R}^3$ such as the examples in the previous few problems. We wish to add functions from $\mathbb{R}^2$ to $\mathbb{R}$ to our list next. Such functions are called **real-valued functions of several variables**.

**Definition 7.** A **real valued function of several variables** is a function $f$ from $\mathbb{R}^n \to \mathbb{R}$. We will primarily consider $n = 2$ and $n = 3$ in this course.

Here are examples of some functions we will encounter and their names.

1. real valued functions defined on $\mathbb{R}$
   \[ f : \mathbb{R} \to \mathbb{R}, \quad f(t) = t^2 + e^t \]

2. parametric curves (also called vector valued functions) defined on $\mathbb{R}$
   - (a) planar curves
     \[ \mathbf{r} : \mathbb{R} \to \mathbb{R}^2, \quad \mathbf{r}(t) = (e^{2t+1}, 3t + 1) \]
   - (b) space curves
     \[ \mathbf{r} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{r}(t) = (2t + 1, (3t + 1)^3, 4t + 1) \]

3. functions of several variables, or **multivariate** functions
   - (a) real valued functions of two variables
     \[ f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^2 + \sqrt{y} \]
   - (b) vector valued functions of three variables
     \[ f : \mathbb{R}^3 \to \mathbb{R}^2, \quad f(x, y, z) = (2xy, 3x + 4y^2) \]
   - (c) vector valued functions of several variables
     \[ f : \mathbb{R}^n \to \mathbb{R}^m \text{ where } m, n \geq 2 \]

**Problem 11.** Graph the set of all points $(x, y, z)$ in $\mathbb{R}^3$ that satisfy $x + y + z = 1$.

**Problem 12.** Graph the set of all points $(x, y, z)$ in $\mathbb{R}^3$ that satisfy $z = 4$.

**Definition 8.** If $\mathbf{x}$ is a vector in $\mathbb{R}^3$ then $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This is called the **magnitude** of $\mathbf{x}$.

We defined the norm on vectors, but the same definition is valid for **points** in $\mathbb{R}^3$.

**Theorem 1. Law of Cosines** Given any triangle with sides of lengths $a$, $b$, and $c$, and having an angle of measure $\alpha$ opposite the side of length $a$, the following equation holds: $a^2 = b^2 + c^2 - 2bc \cos(\alpha)$.

**Problem 13.** Sketch in $\mathbb{R}^2$ the vectors $(1,2)$ and $(3,5)$ and find the angle between these vectors by using the law of cosines.
Problem 14. Sketch in $\mathbb{R}^3$ the vectors $(1,2,3)$ and $(-2,1,0)$ and find the angle between these vectors by using the law of cosines.

Definition 9. If $\vec{x} = (x_1,x_2,x_3)$ and $\vec{y} = (y_1,y_2,y_3)$ then the dot product of $\vec{x}$ and $\vec{y}$ is defined by $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3$.

Again, we defined the dot product on vectors, but the same definition is valid for points in $\mathbb{R}^3$.

Definition 10. We say the vectors $\vec{x}$ and $\vec{y}$ are orthogonal if $\vec{x} \cdot \vec{y} = 0$.

Problem 15. Show that if $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^3$ then

$$|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 - 2\vec{x} \cdot \vec{y} + |\vec{y}|^2.$$

Problem 16. Find two vectors orthogonal to $\begin{pmatrix}1 \\ 2 \\ 3 \end{pmatrix}$. How many are there?

Problem 17. Find three vectors orthogonal to $\begin{pmatrix}1 \\ 2 \\ 3 \end{pmatrix}$. How many are there?

Problem 18. Use Theorem 1 + Problem 15 to show that if $\vec{x}, \vec{y} \in \mathbb{R}^3$ then $\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \theta$ where $\theta$ is the angle between $\vec{x}$ and $\vec{y}$.

Problem 19. Find two vectors orthogonal to both $\begin{pmatrix}1 \\ 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix}2 \\ -3 \\ 4 \end{pmatrix}$. Sketch all four vectors.

Problem 20. Show that if two non-zero vectors $\vec{x}$ and $\vec{y}$ are orthogonal then the angle between them is 90°. Hence any two orthogonal vectors are perpendicular vectors.

Problem 21. Given the vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$, find the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$ and diagonals $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$. The vertices of this parallelogram are the points: the origin, $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$.

Problem 22. Assume $\vec{u}, \vec{v} \in \mathbb{R}^3$. Find a vector $\vec{x} = (x,y,z)$ so that $\vec{x} \perp \vec{u}$ and $\vec{x} \perp \vec{v}$ and $x + y + z = 1$.

Problem 23. Prove or give a counter example to each of the following where $\vec{u}, \vec{v} \in \mathbb{R}^3$ and $c \in \mathbb{R}$:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u}(\vec{w} \cdot \vec{v}) = (\vec{u} \cdot \vec{w})(\vec{u} \cdot \vec{v})$
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$
5. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

Problem 24. Let $f(x,y) = x^2 + y^2$. Sketch the intersection of the graph of $f$ with the planes: $z = 0$; $z = 4$; $z = 9$; $y = 0$; $y = -1$; $y = 1$; $x = 0$; $x = -1$; $x = 1$. Now sketch all of these together in one 3-D graph.

In the previous problem, the intersection of the graph with $z = 0, z = 4$, and $z = 9$ are called level curves because each represents the path you would take if you walked around the graph always remaining at a certain height or level.

Recall composition of functions and the chain rule from Calculus I. We’ll extend these to the functions we study this semester.
Definition 11. If \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are differentiable functions then \((f \circ g)(t) = f(g(t))\) and \((f \circ g)'(t) = f'(g(t))g'(t)\).

Problem 25. Let \( g(x,y) = x^2 + y^3 \) and \( \mathbf{T}(t) = (0,1)t \). Compute \( g \circ \mathbf{T} \). Graph \( g \), \( \mathbf{T} \), and \( g \circ \mathbf{T} \).

Problem 26. Compute \((g \circ \mathbf{T})'(t)\) and \((g \circ \mathbf{T})'(2)\). What is the significance of this number with respect to your graphs from the previous problem?

Here is a reminder from Calculus II of the definition of arc length.

Definition 12. If \( f : \mathbb{R} \to \mathbb{R} \) is a function which is differentiable on \([a, b]\), then the arc length of \( f \) on \([a, b]\) is \( \int_a^b \sqrt{1 + [f'(x)]^2} \, dx \). If \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^2 \) is a vector valued function which is differentiable on \([a, b]\) so that \( \mathbf{r}'(t) = (x(t), y(t)) \) then the arc length of \( \mathbf{r} \) on \([a, b]\) is \( \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt \).

Problem 27. A man walks along a path on the surface \( f(x,y) = 4 - 2x^2 - 3y^2 \) from one point on the x-axis to a second point on the x-axis, always remaining directly above the x-axis. Graph the path and write an integral expression for the distance he walked and compute the distance he walked.

Problem 28. A lady walks along the surface from the previous problem staying exactly 3 units above the xy-plane. Write an integral expression for the distance she walks if she starts and stops at \((0, \frac{1}{\sqrt{3}}, 3)\) and never retraces her steps?

Problem 29. Redo Problem 25 and Problem 26 with \( \mathbf{T}(t) = (-1,1)t \).

Problem 30. Find the slope of the line tangent to \( f(x,y) = x^3 + 3y^2 \) at \((1,2,13)\) that lies above the line \( \mathbf{r}(t) = (1,2) + (1,1)t \).

Problem 31. Given \( \mathbf{a} = (4,3) \), \( \mathbf{b} = (1,-1) \), and \( \mathbf{c} = (6,-4) \), determine the angle between \( \mathbf{ba} \) and \( \mathbf{bc} \).

In the next problem the notation, \(| \cdot |\), is used for both the absolute value (on the left side of the equation) and the norm (on the right side of the equation). Is this bad notation? Consider the definition for the norm, that

\[ |(x_1,x_2)| = \sqrt{x_1^2 + x_2^2}. \]

Suppose we take the norm of a vector in \( \mathbb{R}^1 \), such as \((x_1)\). Then,

\[ |(x_1)| = \sqrt{x_1^2} = \text{ the absolute value of the number } x_1. \]

Thus, the absolute value of \( x \) is the norm of \( x \) so you have been studying norms since high school (elementary school?) without knowing it!

Problem 32. Show that if \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^2 \) then \( | \mathbf{u} \cdot \mathbf{v} | \leq | \mathbf{u} || \mathbf{v} | \). Can you show this for vectors in \( \mathbb{R}^3 \)?

The next result is known as the Triangle Inequality and it states essentially that the shortest distance between two points is the straight line. Look at a graph of \( \mathbf{u} \), \( \mathbf{v} \), \( u+v \), and \( u(u+v) \). If you travel from the origin, along the vector \( \mathbf{u} \) and then along the vector \( u(u+v) \) then you have traveled further than if you traveled along the vector \( u+v \).

Problem 33. Triangle Inequality Show that if each of \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^2 \) then \( | \mathbf{u} + \mathbf{v} | \leq | \mathbf{u} | + | \mathbf{v} | \).
Chapter 2

Cross Product and Planes

In Problem 22, you were given two vectors and asked to find a vector that was perpendicular to both. Because there are infinitely many vectors perpendicular to any two given vectors, we added another condition \((x_1 + x_2 + x_3 = 1)\) so that the answer would be unique. This problem was a warm-up for the definition of the cross product.

**Definition 13.** The **cross product** of two vectors is the vector that is perpendicular to both of them and has length that is the area of the parallelogram defined by the two vectors.

If \(\mathbf{u} = (u_1, u_2, u_3)\) and \(\mathbf{v} = (v_1, v_2, v_3)\) then the cross product of \(\mathbf{u}\) and \(\mathbf{v}\) may be computed by one of two methods:

**Method One**

\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)
\]

For the second method, you need a tool from linear algebra, determinants. If I have not done so already, ask me during class to show you how to compute the determinant of 2 by 2 and 3 by 3 matrices.

**Method Two** If we define \(\mathbf{i} = (1, 0, 0)\), \(\mathbf{j} = (0, 1, 0)\), and \(\mathbf{k} = (0, 0, 1)\) then we may compute the cross product as:

\[
\mathbf{u} \times \mathbf{v} = \text{det} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.
\]

**Problem 34.** Prove or give a counter example for each statement, assuming \(\mathbf{u} = (u_1, u_2, u_3)\), \(\mathbf{v} = (v_1, v_2, v_3)\), \(\mathbf{w} = (w_1, w_2, w_3)\) and \(k \in \mathbb{R}\).

1. \(k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})\)
2. \(\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})\)
3. \(\mathbf{u} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = 0\)
4. \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\)
5. \(k + (\mathbf{u} \times \mathbf{v}) = (k + \mathbf{u}) \times (k + \mathbf{v})\)
6. \(\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\)
7. \(\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w})\)
8. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w}$

One might think of planes in $\mathbb{R}^3$ as analogous to lines in $\mathbb{R}^2$. This is because a line is a one-dimensional object in $\mathbb{R}^2$ – that is it has dimension one less than the dimension of the space. In $\mathbb{R}^3$ a plane is two-dimensional – it has dimension one less than the dimension of the space.

**Definition 14.** Given $a, b, c \in \mathbb{R}$ where $a$ and $b$ are not both zero, the **line** determined by $a, b,$ and $c$ is the collection of all points $(x, y) \in \mathbb{R}^2$ satisfying $ax + by = c$.

Given this definition we can define a **plane** in the same manner.

**Definition 15.** Given $a, b, c, d \in \mathbb{R}$ where $a, b$ and $c$ are not all zero, the **plane** determined by $a, b, c$ and $d$ is the collection of all points $(x, y, z) \in \mathbb{R}^3$ satisfying $ax + by + cz = d$.

If algebraically we think of a plane as all $(x, y, z) \in \mathbb{R}^3$ satisfying $ax + by + cz = d$ where not all of $a, b$ and $c$ are zero, then geometrically we can think of a plane as uniquely determined by a vector and a point where the plane is perpendicular to the vector and contains the point. We need both because there are infinitely many planes perpendicular to a given vector, but knowing one point in the plane uniquely determines the plane.

**Problem 35.** Show that $(3, -2, 5)$ is perpendicular to the plane $3x - 2y + 5z = 7$ by choosing two points, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, in the plane and showing that $(3, -2, 5) \cdot x - y = 0$.

**Problem 36.** Show that $(a, b, c) \in \mathbb{R}^3$ is orthogonal to the plane $ax + by + cz = d$.

**Problem 37.** Determine whether these two planes are parallel.

1. $2x - 3y + \frac{5}{2}z = 9$
2. $x - \frac{3}{2}y + \frac{5}{4}z = 12$

**Problem 38.** Write in standard form ($ax+by+cz=d$) the equation of a plane $\perp$ to the first plane from previous problem and containing the point $(9, 2, 3)$.

**Problem 39.** Find all the planes parallel to the plane $x + y - z = 4$ and at a distance of one unit away from the plane. When does the “distance between two planes” make sense?

**Problem 40.** Find both angles between these two planes:

1. $2x - 3y + 4z = 10$
2. $4x + 3y - 6z = -4$

**Problem 41.** Find the equation of the plane containing $(2, 3, 4)$, $(1, 2, 3)$ and $(6, -2, 5)$.

For next two problems, there is a formula on the web or in some book, but it’s probably wrong because of a typographical error. Find a vector $\perp$ to both planes, determine the equation of a parametric line, $\vec{T}$, passing through both planes. Find the points where $\vec{T}$ intersects each plane. Find the distance between these points.

**Problem 42.** Find the distance between the two planes, $x + y + z = 1$ and $x + y + z = 2$.

**Problem 43.** Find the distance between the two planes, $3x - 4y + 5z = 9$ and $3x - 4y + 5z = 4$.

**Problem 44.** Find an equation for the distance between two planes, $ax + by + cz = e$ and $ax + by + cz = f$.
Problem 45. Find the equation of the plane containing the line \( \vec{I}(t) = (1 + 2t, -1 + 3t, 4 + t) \) and the point \((1, -1, 5)\).

The next problem asks for the intersection of two planes. What are all the possibilities for the intersection of any two planes? One possibility is a line. To find the equation of the line, there are a couple of ways we could think about this. First, we could find two points in the intersection and then find the equation of that line. Or we could observe that the intersection of two planes must be contained in each plane and thus is parallel to both planes. How can we find a vector that is parallel to both planes?

Problem 46. Find the intersection of the two planes \(3x - 2y + 6z = 1\) and \(3x - 4y + 5z = 1\).
Chapter 3

Limits and Derivatives

For the next two definitions, suppose that \(x, y, \text{ and } z : \mathbb{R} \to \mathbb{R}\) are differentiable functions.

**Definition 16.** The limit of the vector valued function of one variable \(\vec{f}(t) = (x(t), y(t), z(t))\), as \(t\) approaches \(a\) is defined by

\[
\lim_{t \to a} \vec{f}(t) = \left( \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)
\]

as long as each of these limits exists. If any one of the limits does not exist, then the limit of \(\vec{f}\) does not exist at \(a\).

**Definition 17.** The derivative of the vector valued function of one variable \(\vec{f}(t) = (x(t), y(t), z(t))\), is defined by

\[
\vec{f}'(t) = (x'(t), y'(t), z'(t))
\]

as long as each of the derivatives exists. If any one of the derivatives does not exist, then the derivative of \(\vec{f}\) does not exist.

**Problem 47.** Let \(\vec{f}(t) = (t^2 - 4, \sin(t), \frac{4t^3}{e^t})\). Compute \(\lim_{t \to 0} \vec{f}(t)\).

**Problem 48.** Compute \(\vec{f}'(0)\) and \(\vec{g}'(0)\) for \(\vec{f}\) from the previous problem.

We now wish to develop the rules for limits and derivatives that parallel the rules from calculus in one dimension. Of course, we know you have not forgotten any of these rules, so the ones we develop should look familiar! The good news is that the really hard work in proving these was done in the first semester of calculus and thus the work here is more notational than mathematical!

**Problem 49.** Let \(\vec{f}(t) = (t^2, t^3 - 1, \sqrt{t-1})\) and \(\vec{g}(t) = (2-t^2, t^3, \sqrt{t+1})\).

1. Compute \(\lim_{t \to 2} \vec{f}(t)\).
2. Compute \(\lim_{t \to 2} \vec{g}(t)\).
3. Compute \(\lim_{t \to 2} \vec{f}(t) + \lim_{t \to 2} \vec{g}(t)\).
4. Compute \(\lim_{t \to 2} [\vec{f}(t) + \vec{g}(t)]\).
5. What can you conjecture about \(\lim_{t \to a} [\vec{f}(t) + \vec{g}(t)]\) for arbitrary choices of \(a\), \(\vec{f}\), and \(\vec{g}\)?
Problem 50. State 5 rules for limits of the vector valued functions, $\mathbf{f}, \mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^3$ that parallel the limit rules from Calculus I and prove one of these conjectures. You may grab a book or look on the web to remind you of the rules from Calculus I.

Problem 51. Compute $\mathbf{f}'(2)$ and $\mathbf{g}'(2)$ where $\mathbf{f}$ and $\mathbf{g}$ are from Problem 49. Compute $(\mathbf{f} + \mathbf{g})'(2)$. What can you conjecture about $(\mathbf{f} + \mathbf{g})'(t)$ for arbitrary choices of $\mathbf{f}$ and $\mathbf{g}$?

Problem 52. State 5 rules for derivatives of vector valued functions that parallel the derivative rules from Calculus I. Prove one of these conjectures. You may grab a book or look on the net to remind you of the rules from Calculus I.

You may assume that which ever one you do not prove will end up on the next test. Yes, it is a well known fact that all calculus teachers can read students’ minds. How else would we always be able to schedule our tests on the same days as your physics tests?

Limits of Functions of Several Variables

Recall from Calculus I the various ways in which you computed limits. If possible, you substituted a value into the function. If not, perhaps you simplified the function via some algebra or computed a limit table or graphed the function. Or you might have applied L’Hopital’s Rule. Your instructor probably used the Squeeze Theorem to obtain the result that $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$. Recall that if the left hand limit equaled the right hand limit then the limit existed.

In three dimensions, the difficulty is that there are more paths to consider than merely left and right. For the limit to exist at a point $(a, b)$, we need that the limit as $(x, y)$ approaches $(a, b)$ exists regardless of the path we take as we approach $(a, b)$. We could approach $(a, b)$ along the x-axis for example, setting $y = 0$ and taking the limit as $x \to 0$. Or we could take the limit along the line, $y = x$. The limit exists if the limit as $(x, y) \to (a, b)$ along every possible path exists. We will see an example where the limit toward $(a, b)$ exists along every straight line, but does not exist along certain non-linear paths!

Definition 18. If $(a, b) \in \mathbb{R}^2$ and $L \in \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function, then we say that

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if $f(x, y)$ approaches $L$ as $(x, y)$ approaches $(a, b)$ along every possible path.

Problem 53. Sketch $f(x, y) = x^2 + y^2$, indicate the point $(2, 3, f(2, 3))$ and compute $\lim_{(x, y) \to (2, 3)} x^2 + y^2$.

Problem 54. Use any software to graph the function from the previous example near $(0, 0)$. Print and use a highlighter to mark the paths $x = 0, y = 0$, and $y = x$. If you use Maple, available in our lab, then the command to plot $f(x, y) = x^2 + y^2$ would be “plot3d($x^2 + y^2, x = -1..1, y = -1..2$);”.

Problem 55. Convert the previous problem to polar coordinates via the substitution $x = r \cos(\theta)$ and $y = r \sin(\theta)$ and then compute the limit as $r \to 0$.

Problem 56. Let $f(x, y) = \frac{x + y^2 + 2}{x - y + 2}$.

1. Graph $f$ using any software and state the domain.

2. Compute $\lim_{(x, y) \to (-1, 2)} f(x, y)$.  

W. Ted Mahavier

www.jiblm.org
Recall your Calculus I definition of continuity for functions of one variable.

**Definition 19.** A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( a \in \mathbb{R} \) if \( \lim_{x \to a} f(x) = f(a) \).

This definition says that for \( f \) to be continuous at \( a \) three things must happen. First, the function must be defined at \( a \). This means that \( a \) must be in the domain of the function so that \( f(a) \) is a number. Second, the limit of the function as we approach \( a \) must exist. And third, \( f(a) \) must equal the limit of \( f \) at \( a \). The same statement defines continuity for all functions.

**Definition 20.** A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous at \( a \in \mathbb{R}^n \) if \( \lim_{x \to a} f(x) = f(a) \).

**Problem 57.** Consider \( \lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2} \).

1. Compute this limit along the lines: \( x = 0, y = 0, y = x, \) and \( y = -x \).
2. Convert to polar coordinates and check the limit.
3. Graph using any software.
4. Why isn’t this function continuous at \((0,0)\)?
5. How can you modify \( f \) in such a way as to make it continuous at \((0,0)\)?

**Problem 58.** Compute \( \lim_{(x,y) \to (0,0)} \frac{xy + y^3}{x^2 + y^2} \) if it exists.

**Problem 59.** Determine whether the function \( f \) continuous at \((x,y) = (0,0)\) by considering the paths \( y = kx^2 \) for several choices of \( k \). \( f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \)

**Directional Derivatives**

Suppose we have the function \( f(x,y) = x^2 + y^2 \) and we are sitting on that function at some point \((a,b,f(a,b))\) (other than \((0,0,0)\)). Then there are many directions we can walk while remaining on the surface. Depending on the direction of our path, the rate of increase of our height, or slope of our path, may vary. Some paths will move us uphill and others downhill. Suppose while we sit at the point, \((a,b,f(a,b))\), we decide to walk in a direction that will not change the \( y \) coordinate, but only changes the \( x \) coordinate. Thus, we are walking on the surface and staying within the plane, \( y = b \). Walking in this way, we could go in one of two directions. Either we could go in the direction that increases \( x \) or decreases \( x \). Let’s go in the direction that increases \( x \). Now, consider the tangent line to the curve at this point that lies in the plane, \( y = b \). As we take our first step along the curve our rate of increase in height, \( z \), will be the same as the slope of that tangent line. This slope is the directional derivative of the function at the point \((a,b)\) in the \( x \) direction. If we had decided to fix \( x = a \) and walk in the direction that increases \( y \), then the slope of the line tangent to the function and in the plane \( x = a \) is the directional derivative of \( f \) at \((a,b)\) in the \( y \) direction.

The next definition formalizes this discussion and the problem immediately following it is an example that will make these notions of directional derivative precise!

**Definition 21.** If \( f : \mathbb{R}^2 \to \mathbb{R} \) is a function and \((a,b)\) is in the domain of \( f \) then the derivative of \( f \) in the \((1,0)\) direction at \((a,b)\) is the slope of the line tangent to \( f \) at the point \((a,b,f(a,b))\) and in the plane, \( y = b \).
Notation Suppose that \( f: \mathbb{R}^2 \to \mathbb{R} \) is a function as in the previous definition. There are many phrases and notations used to denote the derivative of \( f \) in the \((1,0)\) direction at \((a,b)\). For example,

- \( f_1 \) – the derivative of \( f \) with respect to the first variable
- \( f_x \) – the derivative of \( f \) with respect to \( x \)
- \( \frac{df}{dx} \) – the partial derivative of \( f \) with respect to \( x \)

Similarly, \( f_2 \), \( f_y \), and \( \frac{df}{dy} \) would denote the same concepts where the derivative was taken in the \((0,1)\) direction.

The derivative of \( f \) with respect to \( x \) at \((a,b)\) can be computed via the limit:

\[
f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}
\]

or, since \( y \) is being held constant and \( x \) is changing, one may just compute the derivative of \( f \) as if \( x \) is the variable and \( y \) is a constant.

**Problem 60.** Let \( f(x,y) = x^2 + y^2 \) and \((a,b) = (2, -3)\).

1. Sketch \( f \) and sketch the line tangent to \( f \) at the point \((2, -3, f(2, -3))\) that is in the plane, \( y = -3 \).
2. Let \( g(x) = f(x, -3) \) and compute \( g'(2) \).
3. What part of the graph of \( f \) is the graph of \( g \)?

**Problem 61.** Let \( f(x,y) = \frac{y}{x} \) and compute \( f_x(1,2) \) using the limit described following Definition 21.

**Problem 62.** Compute \( f_x \) and \( f_y \) for each function.

1. \( f(x,y) = x^3 - 4x^2 \)
2. \( f(x,y) = e^{xy^2} \)
3. \( f(x,y) = \frac{x^2}{\sin(xy)} \)
4. \( f(x,y) = e^{x^2y \sin(x-y)} \)

**Definition 22.** Just as in Calculus I, “second derivatives” are merely derivatives of the first derivatives. Thus \( f_{xx} = (f_x)_x \). I.e. \( f_{xx} \) is the derivative of \( f_x \) with respect to \( x \). Similarly, \( f_{xy} = (f_y)_y \), \( f_{yx} = (f_x)_y \) and \( f_{yy} = (f_y)_x \).

Other standard notations are:

\[
 f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}
\]

**Theorem 2. Clairaut’s Theorem** For any function, \( f \), whose second derivatives exist, we have \( f_{xy} = f_{yx} \).
Problem 63. Find \( f_{xx}, f_{xy}, f_{yx}, \) and \( f_{yy} \) for each function.

1. \( f(x, y) = e^{x^2 + y^2} \)
2. \( f(x, y) = \sin(xy + y^3) \)
3. \( f(x, y) = \sqrt{3x^2 - 2y^3} \)
4. \( f(x, y) = \cot \left( \frac{x}{y} \right) \)

The study of partial differential equations is the process of finding functions that satisfy some equation that has derivatives with respect to multiple variables. For example Laplace’s Equation is the equation \( u_{xx} + u_{yy} + u_{zz} = 0 \) and solutions give us information about the steady state of heat flow in a three dimensional object.

Problem 64. Which of these functions satisfy \( u_{xx}(x, y) + u_{yy}(x, y) = 0 \) for all \( (x, y) \in \mathbb{R}^2 \)?

1. \( u(x, y) = x^2 + y^2 \)
2. \( u(x, y) = x^2 - y^2 \)
3. \( u(x, y) = x^3 + 3xy^2 \)
4. \( u(x, y) = \ln(\sqrt{x^2 + y^2}) \)
5. \( u(x, y) = \sin(x) \cosh(y) + \cos(x) \sinh(y) \)
6. \( u(x, y) = e^{-x} \cos(y) - e^{-y} \cos(x) \)
7. Find a solution to this equation other than the one’s above.

In the last two problems, we studied functions of two variables and we defined derivatives in each direction, the \( x \) direction and the \( y \) direction. If \( f \) were a function of three variables, then there would be partial derivatives with respect to each of \( x, y, \) and \( z \). Let’s extend the notion of the partial derivatives with respect to \( x \) and \( y \) to functions with domain \( \mathbb{R}^n \) where \( n > 2 \). If \( f : \mathbb{R}^n \to \mathbb{R} \) then the domain of \( f \) is \( \mathbb{R}^n \) so there is a partial derivative of \( f \) with respect to the first variable, the second variable, and so on, up to the partial derivative of \( f \) with respect to the \( n^{th} \) variable. We use the notation, \( f_1, f_2, f_3, \ldots, f_n \) or \( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n} \) to denote the derivative of \( f \) with respect to each variable.

Problem 65. Let \( f(x_1, x_2, x_3, x_4, x_5) = x_3 \sqrt{(x_1)^3 + (x_2)^2 + x_4 e^{x_5 x_3}} \). Compute the five partial derivatives, \( f_1, f_2, \ldots, f_5 \).

This allows us to define the derivative for functions of \( n \) variables.

Definition 23. If \( f : \mathbb{R}^n \to \mathbb{R} \) is a function and each partial of \( f \) exists then the gradient of \( f \) is the function

\[
\nabla f : \mathbb{R}^n \to \mathbb{R}^n
\]

and is defined by

\[
\nabla f = (f_1, f_2, f_3, \ldots, f_n) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n} \right).
\]

Problem 66. For each function below, state the domain of the function, compute the gradient and state the domain of the gradient.
1. \( f(x, y) = x^3 y^2 - x^2 y^3 \)
2. \( g(x, y, z) = zx^3 - 3xyz + ln(x^2 y z^3) \)

For any function of two variables, \( f \), let’s assume once again that we are at a point on the function, \((a, b, f(a, b))\). We know that the slope of the line tangent to \( f \) at \((a, b, f(a, b))\) and above the line \( y = b \) is \( f_c(a, b) \). And we know that the slope of the line tangent to \( f \) at \((a, b, f(a, b))\) and above the line \( x = a \) is \( f_j(a, b) \). Now consider a a line in the \( xy \)-plane passing through \((a, b)\) in some direction \((c, d)\) that is not parallel to either the \( x \) axis or the \( y \) axis. What would the slope of the line tangent to \( f \) at \((a, b, f(a, b))\) and above this line be? From the point \((a, b)\) there are infinitely many directions that we might travel, not just the directions parallel to the \( x \) and \( y \) axes. We can define such a direction from \((a, b)\) by a vector, \((c, d)\). The slopes of the lines tangent to \( f \) at \((a, b, f(a, b))\) in the direction \((c, d)\) are called the directional derivatives of \( f \) at \((a, b)\) in the direction \((c, d)\). The partial derivatives of \( f \) with respect to \( x \) and \( y \) are your first examples of directional derivatives where the directions were \( \vec{i} = (1, 0) \) and \( \vec{j} = (0, 1) \). Here is the formal definition of the directional derivative.

**Definition 24.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. The directional derivative of \( f \) at \( \vec{u} \) in the direction \( \vec{v} \) is given by:

\[
D_{\vec{v}} f(\vec{u}) = \lim_{h \to 0} \frac{f(\vec{u} + h \vec{v}) - f(\vec{u})}{h},
\]

where \( \vec{v} \) must be a unit vector.

Of course, not every limit exists, so directional derivatives may exist in some directions but not others.

**Problem 67.** Using the definition just stated, compute the directional derivative of \( f(x, y) = 4x^2 + y \) at the point \( \vec{u} = (1, 2) \) in the direction \( \vec{v} \) for each \( \vec{v} \) defined below.

1. \( \vec{v} = (0, 1) \)
2. \( \vec{v} = (1, 0) \)
3. \( \vec{v} = (1, 1) \)
4. \( \vec{v} = (\sqrt{2}, \sqrt{2}) \)

**Problem 68.** Use Definition 24, Definition 21, and the discussion immediately following Definition 21 to show that if \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( \vec{i} = (1, 0) \), then \( D_{\vec{v}} f(\vec{v}) = f_x(\vec{v}) \).

Non-Definition: An analytical definition of \( f \) is differentiable at \( u \) is beyond the scope of this course, but a geometrical definition is not. In two dimensions (Calculus I), \( f \) was differentiable at \( a \) if there was a tangent line to \( f \) at \((a, f(a))\). In three dimensions (Calculus III), \( f \) is differentiable at \( \vec{u} \) if there is a tangent plane to \( f \) at \((u, f(u))\).

**Definition 25.** An \( \epsilon \)-neighborhood of \( \vec{u} \) is the set of all points with a distance from \( \vec{u} \) of less than \( \epsilon \). I.e. \( N_{\epsilon}(\vec{u}) = \{ \vec{v} : | \vec{u} - \vec{v} | < \epsilon \} \).

How will we be able to tell when a function is “nice,” that is, when a function has a derivative?
Theorem 3. If $\nabla f$ exists at $\vec{u}$ and at all points in some $\varepsilon$-neighborhood of $\vec{u}$ then $f$ is differentiable at $\vec{u}$.

Problem 69. Is $f(x,y) = y^3(x - \frac{1}{2})^2$ differentiable at $(1,2)$?

Problem 70. For each of the following problems, either prove it or give a counterexample by finding functions and variables for which it does not hold. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable. Assume $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Assume $\vec{x}, \vec{y}, \vec{x} + \vec{y}$ are in the domain of $f$ and $g$.

1. $\nabla f(\vec{x} + \vec{y}) = \nabla f(\vec{x}) + \nabla f(\vec{y})$
2. $\nabla (f + g)(\vec{x}) = \nabla f(\vec{x}) + \nabla g(\vec{x})$
3. $\nabla (c f)(\vec{x}) = c \nabla f(\vec{x})$
4. $\nabla f(c \vec{x}) = c \nabla f(\vec{x})$
5. $\nabla (f \cdot g)(\vec{x}) = \nabla f(\vec{x})g(\vec{x}) + f(\vec{x})\nabla g(\vec{x})$
6. $\nabla \left( \frac{f}{g} \right)(\vec{x}) = \frac{\nabla f(\vec{x})g(\vec{x}) - f(\vec{x})\nabla g(\vec{x})}{g^2}$

As in Calculus I, it is very nice to know when and where a function is continuous. The following theorem answers that question in both cases.

Theorem 4. From Calculus I, if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x$ then $f$ is continuous at $x$. In Calculus III, if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}$, then $f$ is continuous at $\vec{x}$.

Theorem 5. $D(\vec{v})f(\vec{u}) = \nabla f(\vec{u}) \cdot \vec{v}$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$ where $|\vec{v}| = 1$.

Problem 71. Redo Problem 67 using Theorem 5.

To date we have studied the derivatives of functions from $\mathbb{R}$ to $\mathbb{R}$, from $\mathbb{R}$ to $\mathbb{R}^2$, and from $\mathbb{R}^2$ to $\mathbb{R}$. Of course there is nothing special about $\mathbb{R}^2$ here. We might as well have studied $\mathbb{R}^n$ as all the derivatives would follow the same rules. Now let’s consider the derivative of a function, $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Definition 26. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is any vector valued function of two variables defined by $f(x,y) = (u(x,y), v(x,y))$ then the derivative of $f$ is given by

$$Df = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix}$$

Problem 72. Compute the derivative of $f(x,y) = (x^2 \sin(xy), \frac{e^y}{\tan(x)})$. 

W. Ted Mahavier
Table 3.1: Derivative Notation

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f : \mathbb{R} \to \mathbb{R} )</td>
<td>( f' : \mathbb{R} \to \mathbb{R} )</td>
</tr>
<tr>
<td>( f : \mathbb{R}^2 \to \mathbb{R} )</td>
<td>( \nabla f : \mathbb{R}^2 \to \mathbb{R}^2 )</td>
</tr>
<tr>
<td>( f : \mathbb{R}^2 \to \mathbb{R}^2 )</td>
<td>( Df : \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2) )</td>
</tr>
</tbody>
</table>

Because of the number of different domains and ranges of functions we are studying, we have several variations of the chain rule. Before we begin, I would like to take this opportunity to apologize for the number of notations used by mathematicians, physicists, and engineers for derivatives, partial derivatives, total derivatives, gradients, Laplacians, etc. There are a number of notations and all are convenient at one time or another. I attempt to adhere for the most part to the functional notation for derivatives (\( f_1, f_x, f' \), etc.), but Leibnitz notation, \( \frac{\partial f}{\partial x} \), is a convenient notation as well. Table 1 illustrates my preferred notations, where \( L(\mathbb{R}^2, \mathbb{R}^2) \) denotes the set of all 2x2 matrices.

The chain rule you learned in Calculus I, which is stated next, applies to each of the types of functions we just discussed; that is, given any two functions with domains so that their composition actually makes sense and so that they are differentiable at the appropriate places, we can compute their derivative using the same chain rule that you learned in Calculus I with one warning. When the domains and ranges of the functions change, the derivatives change. Thus, in the following statement, depending on the domain of \( f \), sometimes \( f' \) means the derivative of a parametric curve, but sometimes it means the gradient of \( f \), \( \nabla f \), and sometimes it means the matrix of derivatives of \( f \). The same holds for the derivative of \( g \). And finally, the symbol \( \cdot \) might mean multiplication or the dot product or matrix multiplication. You’ll know from context which one. The point is to realize that no matter how many different notational ways we have of writing the chain rule, it always boils down to this one.

**Theorem 6. Chain Rule** If \( f, g : \mathbb{R} \to \mathbb{R} \) are differentiable functions then

\[
(f \circ g)' = (f' \circ g) \cdot g'
\]

or with the independent variable displayed,

\[
(f \circ g)'(x) = f'(g(x)) \cdot g'(x).
\]

Next we state the chain rule for differentiating functions that are the composition of a function of two variables with a planar curve. Notice that this theorem is exactly the same as the original theorem, but restated for functions with different domains. Because \( f : \mathbb{R}^2 \to \mathbb{R} \) we replace the \( f' \) from the previous theorem with \( \nabla f \) and because \( \overrightarrow{g} : \mathbb{R} \to \mathbb{R}^2 \) we replace the \( g' \) in the previous theorem with \( \overrightarrow{g}' \). Thus the theorem still says (in English) that “the derivative of \( f \) composed with \( g \) is the (derivative of \( f \)) evaluated at \( g \) times the derivative of \( g \).”

**Theorem 7. Chain Rule** If \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( \overrightarrow{g} : \mathbb{R} \to \mathbb{R}^2 \) are both differentiable, then

\[
(f \circ \overrightarrow{g})' = (\nabla f) \circ \overrightarrow{g} \cdot \overrightarrow{g}'.
\]

Writing this with the independent variable \( t \) in place we could write:

\[
(f \circ \overrightarrow{g})'(t) = (\nabla f)(\overrightarrow{g}(t)) \cdot \overrightarrow{g}'(t).
\]

On the right hand side of the last line of the Chain Rule, we have the composition of \( \nabla f \) with \( g(t) \). Because we are multiplying vectors, \( \cdot \) represents dot product and not multiplication.
Problem 73. Let \( f(x,y) = x^2 - 3y^2 \) and \( \overrightarrow{g}(t) = (2,3) + (4,5)t \). Compute \( (f \circ \overrightarrow{g})' \) both by direct composition and by using Theorem 7.

Problem 74. Compute \( (w \circ \overrightarrow{g})' \) where \( w(x,y) = e^y \sin(y) - e^y \sin(x) \) and \( \overrightarrow{g}(t) = (3,2)t \). Write a complete sentence that says what line \( (w \circ \overrightarrow{g})'(-1) \) is the slope of.

Problem 75. Redo the following problems using this theorem and paying special attention to the use of unit vectors in both cases.

2. Problem 29.

Next we state the chain rule for differentiating functions that are the composition of a function of several variables with a function from the plane into the plane. Notice that this theorem is exactly the same as the original theorem, but restated for functions with different domains. Because \( f : \mathbb{R}^2 \to \mathbb{R} \) we replace the \( f' \) from the original theorem with \( \nabla f \) and because \( \overrightarrow{g} : \mathbb{R}^2 \to \mathbb{R}^2 \) we replace the \( g' \) in the previous theorem with \( D\overrightarrow{g} \). Thus the theorem still says that “the derivative of (\( f \) composed with \( g \)) is the (derivative of \( f \)) evaluated at \( g \) times the derivative of \( g \).” Because \( D\overrightarrow{g} \) is a matrix, the right hand side of this is now a vector times a matrix.

**Theorem 8. Chain Rule** If \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( \overrightarrow{g} : \mathbb{R}^2 \to \mathbb{R}^2 \) are both differentiable, then

\[
\nabla(f \circ \overrightarrow{g}) = \left( \nabla f \right) \circ \overrightarrow{g} \cdot D\overrightarrow{g}.
\]

Writing this with the independent variables displayed,

\[
\nabla\left( f(g(s,t)) \right) = \left( \nabla f \right) (g(s,t)) \cdot (D\overrightarrow{g})(s,t).
\]

Problem 76. Let \( f(x,y) = 2x^2 - y^2 \) and \( \overrightarrow{g}(s,t) = (2s+5t,3st) \). Compute \( \nabla(f \circ \overrightarrow{g}) \) in two ways. First, compute by composing and then taking the derivative. Second, apply Theorem 8.

Problem 77. Let \( w(x,y) = \ln(x+y) - \ln(x-y) \) and \( g(s,t) = (te^s,-e^t) \). Compute \( \nabla(w \circ \overrightarrow{g}) \) in two ways. First, compute by composing and then taking the derivative. Second, apply Theorem 8.

Problem 78. Prove or give a counterexample to the statement that \( f'(\overrightarrow{T}(t)) = (f(\overrightarrow{T}(t)))' \) for all differentiable functions \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( \overrightarrow{T} : \mathbb{R} \to \mathbb{R}^2 \).

The next problem tells us something important. If you are sitting on some function in three space and you are trying to decide what direction you should travel to go uphill at the steepest possible rate, then the gradient tells us this direction.

Problem 79. Use Problem 18 and Theorem 5 to show that \( D\overrightarrow{T}f(\overrightarrow{u}) \) is largest when \( \overrightarrow{v} = \nabla f(\overrightarrow{u}) \) and smallest when \( \overrightarrow{v} = -\nabla f(\overrightarrow{u}) \).

**Surfaces** Earlier we gave a list of the types of functions we have studied so far. Of course, there is only one definition for a function, so we are really talking about functions with different domains and ranges as was illustrated by the need for different chain rules for functions with different domains. In linear algebra, we see functions with domain the set of matrices (the determinant function) and \( T(f) = \int_0^1 f(x) \, dx \) is a function from the set of all continuous functions into the real numbers.
In earlier courses, you studied not only functions, but relations such as \( x^2 + y^2 = 1 \). What was the difference? Well, a function has a unique \( y \) for each \( x \) while a relation may have several \( y \) values for a given \( x \) value. In three dimensions a function will have a unique \( z \) for a given coordinate pair, \((x,y)\). When one has multiple \( z \) values for a given \((x,y)\) value, we call it a \textit{surface}. We have thus far studied mostly functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \) such as \( f(x,y) = x^2 + y^2 \) or \( f(x,y) = ye^{x^2} \), but surfaces are equally important. Surfaces are to functions in three-space as relations were to functions in two-space.

**Problem 80.** Sketch these functions in \( \mathbb{R}^3 \).

1. \( f(x,y) = |x| - |y| \)
2. \( h(x,y) = \sqrt{xy} \)
3. \( g(x,y) = \sin(x) \)
4. \( i(x,y) = 2x^2 - y^2 \)

**Problem 81.** Sketch these surfaces in \( \mathbb{R}^3 \).

1. \( x^2 + y^2 + z^2 = 1 \)
2. \( y^2 + z^2 = 4 \)
3. \( x^2 - y + z^2 = 0 \)
4. \( |y| = 1 \)

Surfaces may be expressed as \( F(x,y,z) = k \); For example, in the previous problem, we could rewrite these as

1. \( F(x,y,z) = 1 \) where \( F(x,y,z) = x^2 + y^2 + z^2 \).
2. \( F(x,y,z) = 4 \) where \( F(x,y,z) = y^2 + z^2 \).
3. \( F(x,y,z) = 0 \) where \( F(x,y,z) = x^2 - y + z^2 \).
4. \( F(x,y,z) = 1 \) where \( F(x,y,z) = |y| \).

**Tangent Planes to Functions**

**Problem 82.** Find the equation of the plane tangent to the function \( f(x,y) = 25 - x^2 - y^2 \) at the point \((3,1,15)\). Sketch the graph of the function and the plane.

**Problem 83.** Find the equation of the plane tangent to the function \( f(x,y) = \sqrt{x^2 + y^2} \) at the point \((3,4,5)\). Sketch the graph of the function and the plane.

**Tangent Planes to Surfaces**

**Definition 27.** If \( F(x,y,z) = k \) is a surface, then the \textit{tangent plane} to \( F \) at \( u = (x,y,z) \) is the plane passing through \( u \) with normal vector, \( \nabla F(u) \).

**Problem 84.** Find the equation of the tangent plane to the surface \( x^2 - 2y^2 - 3z^2 + xyz = 4 \) at the point \((3,-2,-1)\).

**Problem 85.** Find the equations of two lines perpendicular to the surface in the previous problem at the point \((3,-2,-1)\) on the surface.

**Problem 86.** Find the equations of two lines perpendicular to the surface \( z + 1 = ye^{x\cos(z)} \) at the point \( \vec{p} = (1,0,0) \).
Chapter 4

Optimization and Lagrange Multipliers

Probably the most applied concept in all of calculus is finding the maxima and minima of functions. In industry, these can be problems from engineering, such as trying to design an airplane wing that yields the maximum lift and stability while at the same time minimizing the drag coefficient. This way, we build a plane that flies easily while using less fuel. Since planes measure fuel consumption in gallons per second, a small change in wing design can result in considerable profit for the company (and a big raise for you). Have you noticed the addition of the upward turned tips at the ends of the airplane wings in recent years?

In the financial markets, mutual funds are sets of stocks. People may buy shares of the fund instead of buying shares of individual stocks. Mutual fund managers (and their clients) want to choose groups of stocks that will increase in value and make them (and their clients) rich. Thus, a fund manager wants to design a mutual fund that will maximize profits for the investors, but because investors fear volatility (large fluctuations in the value of their portfolios), the manager also wants to minimize the volatility of the mutual fund. This is an example of an optimization problem with what is called a constraint because you want to maximize profit but are constrained by the customers’ concerns about volatility.

Mathematicians have spent a considerable amount of time in industry working on both of these interesting problems that are representative of “real-world” applications. In mathematics, the difference between being able to understand or apply a formula to such a problem and the ability to derive or create your own formulas for the problems is the difference between working as an engineer, mutual fund manager, or biologist on a team (a wonderful job in it’s own right) and working in a think tank such as Bell Labs or Los Alamos or MSRI (the Mathematical Sciences Research Institute) where you are tackling the problems that no one can solve and creating the mathematics that will be implemented by the teams in industry.

In Calculus I you solved optimization or max-min problems by setting the derivative of a function to zero to tell you where the rate of change (or slope) of the function was zero. In Calculus III we do exactly the same thing. Except that instead of solving $f' = 0$ we are solving $\nabla f = 0$ and we are seeking a horizontal tangent plane instead of a horizontal tangent line. The procedures are very similar and both find the point in the domain of the functions where potential maxima and minima are attained.

**Definition 28.** A critical point of $f : \mathbb{R}^2 \to \mathbb{R}$ is any point $\vec{x} \in \mathbb{R}^2$ where $\nabla f(\vec{x})$ is zero or undefined.

Since $\nabla f = 0$ translates to $\nabla f(x, y) = (f_x(x, y), f_y(x, y)) = (0, 0)$ we are seeking points $(x, y)$ in the plane where both $f_x(x, y) = 0$ and $f_y(x, y) = 0$, or where at least one is undefined. Review the definitions for “local minimum, local maximum, and inflection point” for functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e. from Calculus I). You may look these up in a book or on the web.
Definition 29. An \( \varepsilon \)-neighborhood of the point \((s,t)\) in \(\mathbb{R}^2\) is the set of all points in \(\mathbb{R}^2\) that are a distance of less than \(\varepsilon\) away from \((s,t)\).

Definition 30. If \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) and \(x\) is in the domain of \(f\), then we say \((\overline{x}, f(\overline{x}))\) is a local minimum if \(f(\overline{x}) < f(\overline{y})\) for every \(\overline{y}\) in some \(\varepsilon\)-neighborhood of \(\overline{x}\).

Local maximum is defined similarly. A function may have infinitely many local minima and maxima.

Problem 87. Find the minimum of \(f(x,y) = x^2 - 2x + y^2 - 4y + 5\) in two ways.

1. Complete the square to write \(f\) as \(f(x,y) = (x - a)^2 + (y - b)^2\).
2. Set \(\nabla f(x,y) = 0\) and solve for \((x,y)\).

Problem 88. Let \(f(x,y) = 8y^3 + 12x^2 - 24xy\).

1. Find all critical points of \(f\).
2. Sketch \(f\) using any software to verify your answers.
3. Use any software to solve the whole problem. In other words, use software to compute your partial derivatives and solve for the roots of these partial derivatives.

Recall in Calculus I that if \(x\) were a critical point for \(f\) then \((x, f(x))\) could have been a minimum, maximum, or an inflection point for \(f\). Inflection points were critical points where the function \(f\) switched concavity (i.e. where the first derivative is zero and the second derivative switches signs). In Calculus III the analogous points are critical points that are neither maxima nor minima and we call these saddle points.

Definition 31. If \(f : \mathbb{R} \rightarrow \mathbb{R}^2\) is differentiable at \(x\) then we say \((\overline{x}, f(\overline{x}))\) is a saddle point if \(\nabla f(\overline{x}) = 0\) and no matter how small an \(\varepsilon > 0\) we choose, there are points \(\overline{y}\) and \(\overline{z}\) in the \(\varepsilon\)-neighborhood of \(\overline{x}\) so that \(f(\overline{y}) < f(\overline{x})\) and \(f(\overline{z}) > f(\overline{x})\).

This definition of a saddle point says that if we are at a saddle point and we decide to walk away from it, then there are paths away from the critical point along which \(f\) increases and paths away from the point along which \(f\) decreases. Given a critical point, how do we determine if it was a maximum, minimum, or saddle point? That is, how do we classify the critical points of \(f\)? How did we do it in Calculus I? We restate the Second Derivative Test which is a slick way to classify the critical points of the single-variable, real-valued functions from Calculus I. Notice how nicely it parallels the next theorem for classifying the multi-variable, real-valued functions of Calculus III.

Theorem 9. Second Derivative Test I If \(f : \mathbb{R} \rightarrow \mathbb{R}\) is differentiable and \(f'(x) = 0\) then

1. If \(f''(x) > 0\) then \((x, f(x))\) is a minimum.
2. If \(f''(x) < 0\) then \((x, f(x))\) is a maximum.
3. If \(f''\) switches signs at \(x\) then \((x, f(x))\) is an inflection point.

Here is a sweet theorem for classifying critical points of functions of two variables.

Theorem 10. Second Derivative Test II If \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) and \(\nabla f(\overline{u}) = 0\) and \(D = f_{xx}(\overline{u})f_{yy}(\overline{u}) - (f_{xy}(\overline{u}))^2\), then
1. \((\nabla f(\mathbf{u}'))\) is a local min if \(D > 0\) and \(f_{xx}(\mathbf{u}') > 0\).
2. \((\nabla f(\mathbf{u}'))\) is a local max if \(D > 0\) and \(f_{xx}(\mathbf{u}') < 0\).
3. \((\nabla f(\mathbf{u}'))\) is a saddle point if \(D < 0\).
4. No information if \(D = 0\).

**Problem 89.** Compute the critical points of \(f(x,y) = xy^2 - 6x^2 - 3y^2\) and classify these critical points as local maxima, local minima, or saddle points.

**Problem 90.** Compute and classify the critical points of \(f(x,y) = xy + \frac{2}{x} + \frac{4}{y}\).

**Problem 91.** Compute and classify the critical points of \(f(x,y) = e^{-(x^2+y^2-4y)}\).

The next problem reminds you of an important aspect of max/min problems from Calculus I. If you wanted the local maxima and minima of a function like \(f(x) = x^2\) on \([-1,3]\) then you checked not only the places where \(f'(x) = 0\) but also the values of \(f\) at the endpoints (boundary) of the interval. The same must be done for functions of several variables. When you applied this technique, you were applying the Extreme Value Theorem.

**Theorem 11.** Extreme Value Theorem for functions of one variable If \(f : [a,b] \to \mathbb{R}\) is a differentiable function then \(f\) has a maximum and a minimum on this interval.

The corresponding theorem for real-valued functions of two variables requires at least an intuitive idea of the notions of what it means for a set to be closed and bounded. A set \(S\) is bounded if there is a number \(M\) so that \(|\mathbf{x}| \leq M\) for all \(x \in S\). A set \(S\) is closed if it contains its boundary points. Think of the open and closed intervals. An open interval does not contain its end points, the points on the boundary of the set. A closed interval does contain its boundary points. The set of all points \((x,y)\) satisfying \(x^2 + y^2 \leq 9\) is closed, while the set of all points \((x,y)\) satisfying \(x^2 + y^2 < 9\) is not closed.

**Theorem 12.** Extreme Value Theorem for functions of two variables If \(M\) is a closed and bounded set and \(f : M \to \mathbb{R}\) is a differentiable function then \(f\) has a maximum and a minimum on \(M\).

**Definition 32.** A set \(S\) in \(\mathbb{R}^2\) is closed if it contains all its boundary points.

**Problem 92.** Let \(T(x,y) = 2x^2 + y^2 - y\) be the temperature at the point \((x,y)\) on the circular disk of radius 1 centered at \((0,0)\).

1. Find the critical points of \(T\).
2. Find the minimum and maximum of \(T\) over the circle (perimeter), \(x^2 + y^2 = 1\) by parameterizing the circle.
3. Find the minimum and maximum of \(T\) over the disk, \(x^2 + y^2 \leq 1\).

For the next problem, you will need to parameterize each of the four line segments that form the line and check the maximum and minimum of the function over not only the interior of the square, but also over each of the four lines.

**Problem 93.** Find the maximum and minimum of \(f(x,y) = 2x^2 - 3y^2 + 10\) over the square disk, \(S = \{(x,y)|0 \leq x \leq 3, 2 \leq y \leq 4\}\).

**Problem 94.** Find all maxima and minima of \(f(x,y) = x^2 - y^2 + 4y\) over the rectangle \(R = \{(x,y)|-1 \leq x \leq 1, -3 \leq y \leq 3\}\).
In Calculus I an \( n \)th degree polynomial will have at most \( n - 1 \) critical points. What about in Calculus III? Here is a 6th degree polynomial which has far more than 5 critical points. All of these may be found by hand. Grab some free graphing software from the web to help verify these.

**Problem 95.** Find all thirteen critical points of \( f(x,y) = x^3y^3 - x^3y - 3xy^3 + 3xy + 1 \).

**Problem 96.** Let \( f(x,y) = x^3 - y^3 \) and \( p = (2,4) \). Find the direction in which \( f \) increases the most rapidly.

**Problem 97.** Ted is riding his mountain bike and is at altitude (in feet) of \( A(x,y) = 5000e^{-(3x^2+y^2)/100} \). What is my slope of descent or ascent if I am riding in the direction \((-1,1)\) starting at the point, \((10,10,5000e^{-4})\)? In what direction should I travel to ascend the most rapidly? To descend the most rapidly?

**Problem 98.** Let \( T(x,y,z) = \frac{10}{x^2+y^2+z^2} \) and \( \mathbf{T}(t) = (t \cos(\pi t), t \sin(\pi t), t) \). If \( T(x,y,z) \) represents the temperature in space at the point \((x,y,z)\) and \( \mathbf{T}(t) \) represents Ted’s position at time \( t \), then compute \( (T \circ \mathbf{T})(3) \) and explain what this number represents in a complete sentence. To check your answer, compute it in two ways. First compose the functions and take the derivative. Second, use the chain rule.

We have tackled optimization problems before when we found maxima and minima of functions like \( f(x,y) = x^2 + x^2y + y^2 + 4 \) from Problem 87. We have also sought maxima and minima of curves (or paths) on surfaces when we found the maxima and minima of \( T(x,y) = 2x^2 + y^2 - y \) over the circle \( x^2 + y^2 + 1 \) as in Problem 92. Problem 92 is called a **constrained** optimization problem because we want a maxima or a minima of \( T \) subject to the constraint that it is above the unit circle. The method of Lagrange multipliers is a slick way to tackle constrained optimization problems.

**Theorem 13. Lagrange** Suppose that \( f,g : \mathbb{R}^2 \rightarrow \mathbb{R} \). To maximize (or minimize) the function \( f \) subject to the constraint \( g = 0 \) we solve the two equations,

1. \( \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \)
2. \( g(\mathbf{x}) = 0 \)

for \( \mathbf{x} \) and for \( \lambda \). The variable \( \lambda \) is called the Lagrange multiplier, and \( \mathbf{x} \) is the point at which \( f \) is maximized (or minimized).

**Problem 99.** Find the max and min of \( f(x,y) = x^2 + y^2 \) subject to \( xy = 3 \) in three ways. First use Lagrange multipliers by putting \( g(x,y) = xy - 3 \) so that the constraint is \( g(x,y) = 0 \). Second, substitute \( y = 3/x \) into the equation and solve. Third, sketch \( f \) and identify the portion of \( f \) that is above the equation, \( xy = 3 \).

**Problem 100.** Find any minima and maxima of \( f(x,y) = 4x^2 + y^2 - 4xy \) subject to \( x^2 + y^2 = 1 \). It may be helpful to eliminate \( \lambda \) first.

**Problem 101.** Find the minimum of \( f(x,y,z) = 3x + 2y + z \) subject to \( 9x^2 + 4y^2 - z = 0 \) via Lagrange multipliers.
Chapter 5

Integration

Calculus III continues to parallel Calculus I and here we are at integration. Let’s review integration in one variable before we tackle integration in several variables. By now you have been using antiderivatives (and hence the Fundamental Theorem of Calculus) to compute integrals for so long that it is worth remembering the original definition for the definite integral. If \( f : \mathbb{R} \to \mathbb{R} \) is a function, then we define \( \int_a^b f(x) \, dx \) as the limit of Riemann sums. If \( f \) is positive, then this is the limit of sums of areas of rectangles. Now we will define our integrals once again as limits of sums, but this time we will take limits of sums of volumes.

First, let’s formally restate the definition of the definite integral from Calculus I.

Definition 33. If \([a, b]\) is a closed interval then a partition \( P \) of \([a, b]\) is an ordered sequence \( a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \). The norm or mesh of \( P \) is \( \max \{ x_i - x_{i-1} : i = 1, 2, \ldots, n \} \) and is denoted \( \|P\| \).

Definition 34. The integral of \( f \) over \([a, b]\), denoted by \( \int_a^b f \) is defined (assuming the limit exists) as
\[
\int_a^b f = \lim_{\|P\| \to 0} \sum_{i=1}^n f(\hat{x}_i) \cdot (x_i - x_{i-1})
\]
where \( \hat{x}_i \in [x_{i-1}, x_i] \) and \( n \) is the number of divisions of the partition \( P \). If the limit does not exist, we say that \( f \) is not integrable.

Of course, this is the limit of the sum of the areas of a collection of rectangles where the width of the rectangles tends toward zero as the mesh of the partition tends toward zero. In Calculus III, we do the same except that we must partition both the x and y-axes and we are summing volumes over rectangles rather than sum areas over intervals.

Definition 35. Given any two sets, \( A \) and \( B \),
\[ A \times B = \{(a, b) : a \in A \text{ and } b \in B \}. \]

In the case where \( A \) and \( B \) are closed intervals these are simply rectangles in the plane,
\[ [a, b] \times [c, d] = \{(x, y) \mid x \in [a, b], y \in [c, d] \}. \]

Definition 36. If \( R \) is a rectangle, say \( R = [a, b] \times [c, d] \) then a partition, \( U \), of \( R \) is a partition of \([a, b]\), \( P = \{a = x_0 < x_1 < \ldots < x_n = b\} \) along with a partition of \([c, d]\), \( Q = \{c = y_0 < y_1 < y_2 < \ldots < y_m = d\} \). The norm or mesh of \( U \) is the largest width of any of the rectangles, \([x_{i-1}, x_i] \times [y_{j-1}, y_j]\) where \( i = 1, 2, \ldots n \) and \( j = 1, 2, \ldots m \).
Definition 37. If \( f \in C_R \) (i.e. \( f \) is continuous on the rectangle \( R \)) then

\[
\int_R f \, dA = \lim_{\|U\| \to 0} \sum_{i=1}^n \sum_{j=1}^m f(\hat{x}_i, \hat{y}_j)(x_i - x_{i-1})(y_j - y_{j-1})
\]

where \((\hat{x}_i, \hat{y}_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\).

Thankfully, in Calculus III we won’t have to compute any such limits because there is a theorem that states:

\[
\int_R f \, dA = \int_a^b \int_c^d f(x, y) \, dx \, dy
\]

which gives us a straightforward way to compute integrals using anti-derivatives by using the Fundamental Theorem of Calculus from Calculus I that stated:

Theorem 14. Fundamental Theorem of Calculus If \( f \) is continuous on \([a, b]\) and \( F \) is any anti-derivative of \( f \), then \( \int_a^b f = F(b) - F(a) \).

The next problem reminds us that in Calculus I we can integrate with respect to either \( x \) or \( y \) and obtain the same result. These techniques are especially handy in Calculus III.

Problem 102. Compute the area bounded by \( y = x^2 \), the \( x \)-axis, and \( x = 2 \) two ways. First, use an integral with respect to \( x \), \( \int_a^b \int_c^d f(x, y) \, dx \, dy \), then use an integral with respect to \( y \), \( \int_a^b \int_c^d f(x, y) \, dx \, dy \). Sketch a picture to help explain your endpoints of integration.

Most of the theorems that you proved in Calculus I have an analog in Calculus III. The following theorem provides a list of properties for double integrals that won’t surprise you. We won’t state them again, but they also hold for any integrable function \( f : \mathbb{R}^n \to \mathbb{R} \) where \( n > 2 \).

When we are integrating over a two dimensional region \( R \), we will write \( \int_R f \, dA \), where the \( A \) reminds us that we are integrating over a region with area. This will be a double integral. When we are integrating over a three dimensional region, we will write \( \int_R f \, dV \), where the \( V \) reminds us that we are integrating over a region with volume. This will be a triple integral.

Theorem 15. Suppose \( f \) is integrable on a closed and bounded rectangular region \( R \). Then

1. \( \int_R [f(x, y) + g(x, y)] \, dA = \int_R f(x, y) \, dA + \int_R g(x, y) \, dA \)

2. \( \int_R c[f(x, y)] \, dA = c \int_R f(x, y) \, dA \) where \( c \) is a real number

3. \( \int_R f(x, y) \, dA \geq \int_R g(x, y) \, dA \) if \( f(x, y) \geq g(x, y) \) for all \( (x, y) \in R \)

4. \( \int_R f(x, y) \, dA = \int_{R_1} f(x, y) \, dA + \int_{R_2} f(x, y) \, dA \), where \( R_1 \) and \( R_2 \) are rectangular regions such that \( R_1 \) and \( R_2 \) have no points in common except for points on parts of their boundaries and \( R = R_1 \cup R_2 \)

Problem 103. Compute \( \int_1^2 \int_0^3 2x + 3y \, dx \, dy \) and \( \int_0^3 \int_1^2 2x + 3y \, dx \, dy \). This is called “reversing the order of integration.” Sketch the solid that you found the volume of.

W. Ted Mahavier

www.jiblm.org
Problem 104. Compute and compare these three integrals:

1. \[ \int_{1}^{2} \int_{0}^{x-2} 2x + 3y \, dy \, dx \]
2. \[ \int_{0}^{x-2} \int_{1}^{2} 2x + 3y \, dx \, dy \]
3. \[ \int_{0}^{-1} \int_{1}^{y+2} 2x + 3y \, dx \, dy \]

As you can see from the previous two problems, when the limits of integration contain variables care must be taken in reversing the order of integration. When the limits of integration are numbers, we can reverse the order of the integrals and obtain the same result. This is known as Fubini’s Theorem.

Theorem 16. Fubini’s Theorem Suppose \( f \) is a continuous function of two variables, \( x \) and \( y \), defined on the rectangle \( R = [a, b] \times [c, d] \). Then

\[
\int_{R} f(x,y) \, dA = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) \, dy \right] \, dx = \int_{c}^{d} \left[ \int_{a}^{b} f(x,y) \, dx \right] \, dy.
\]

Problem 105. Evaluate \( \int_{B} 4x \ln(y)z \, dV \) over the box \( B = [0, 2] \times [1, 4] \times [-2, 5] \). Now change the order of integration and verify the result. How many choices are there for orders of integration?

Problem 106. Let \( f(x,y) = 4 - x^2 - y^2 \) over the region \( R = [0, 1] \times [0, 1] \). Compute \( \int_{0}^{1} \int_{0}^{1} f(x,y) \, dy \, dx \). Sketch the solid that this integral represents the volume of.

Problem 107. Let \( f(x,y) = 4 - x^2 - y^2 \). Write an integral for the volume of the solid that that is bounded by \( f \) and by the planes \( z = 0, x = 0, x = 2, y = 0, \) and \( y = 2 \). Compute this integral.

Problem 108. Compute \( \int_{0}^{1} \int_{0}^{1} \frac{y}{(xy+1)^2} \, dx \, dy \).

Problem 109. Compute \( \int_{0}^{\ln(3)} \int_{0}^{1} y e^{xy^2} \, dy \, dx \).

Problem 110. Compute \( \int_{R} \sin(x+y) \, dA \) where \( R = \left[ 0, \frac{\pi}{2} \right] \times \left[ 0, \frac{\pi}{2} \right] \).

Problem 111. Compute \( \int_{R} xy \sqrt{1+x^2} \, dA \) where \( R = [0, \sqrt{3}] \times [1, 2] \).

Thus far we have integrated primarily over domains that were rectangles, but we can also integrate over more general domains.

Problem 112. Let \( f(x,y) = 4x + 2y \).

1. Sketch the region in the xy-plane bounded by by \( x = 2, \) \( x = 4, \) \( y = -x, \) \( y = x^2 \).
2. Compute the volume of the solid below \( f \) and above this region.

Problem 113. Evaluate \( \int_{S} xy \, dA \) where \( S \) is the region bounded by \( y = x^2 \) and \( y = 1 \).
Problem 114. Evaluate \( \int_S \frac{2}{1 + x^2} \, dA \) over the region \( S \) determined by the triangle with vertices \((0,0), (2,2), \) and \((0,2)\) in the x-y plane. Sketch the solid that you found the volume of.

Problem 115. Fill in the blanks in order to change the order of integration.

\[
\int_0^1 \int_0^{1-x} f(x,y,z) \, dz \, dy = \int_0^2 \int_{-x}^{2-y} f(x,y,z) \, dx \, dz \, dy
\]

Problem 116. Sketch and compute the volume of the solid bounded by \( x^2 = 4y, \) \( z = 0, \) and \( 5y + 9z - 45 = 0. \) Write the integral both as \( \int \int \int \) \( dx \, dy \, dz \) and \( \int \int \int \) \( dy \, dx \, dz. \)

Problem 117. Fill in the blanks:

\[
\int_0^1 \int_{-y}^{y} f(x,y) \, dx \, dy = \int_{-x}^{x} \int_a^b f(x,y) \, dy \, dx
\]

\[
\int_0^2 \int_{2-y}^{2} f(x,y) \, dx \, dy = \int_{-x}^{x} \int_{-x}^{2} f(x,y) \, dy \, dx
\]

Problem 118. Compute \( \int_0^1 \int_{-x}^{x} e^{x+y} \, dy \, dx \) directly and by reversing the order of integration.

Coordinate Transformations

Loosely speaking, a coordinate transformation is a transformation from one coordinate system to another coordinate system and is also called change of variables. A cleverly chosen coordinate transformation can make a difficult integral easy. There are infinitely many, but the three most common are: the conversion from rectangular coordinates to polar coordinates (used in double integrals), from rectangular coordinates to spherical coordinates (used in triple integrals), and from rectangular coordinates to cylindrical coordinates (used in triple integrals).

In Calculus I when you did a trigonometric substitution, you did a change of variable. Written as a theorem, it would look like this.

**Theorem 17.** If \( f \) is an integrable function over \([a,b]\) and \( u : [a,b] \to [c,d] \) is a differentiable function then

\[
\int_a^b f(x) \, dx = \int_c^d f(u(t))u'(t) \, dt
\]

where we are making the substitution \( x = u(t) \) and \( u(c) = a \) and \( u(d) = b. \)

Work the following problem, using the given substitution, but keeping all the independent variables in tact, i.e. don’t toss out the “\( t \)” when you replace \( x \) by \( u(t). \)

**Problem 119.** Compute:

1. \( \int_3^4 \frac{1}{x^2 + 9} \, dx \) using the substitution \( u(t) = 3 \tan(t) \)

2. \( \int_3^4 \frac{1}{\sqrt{x^2 + 9}} \, dx \) using the substitution \( u(t) = 3 \tan(t) \)

For transformation of two variables the theorem is similar.
**Theorem 18.** If \( f \) is an integrable function over the domain \( B \) and \( x \) and \( y \) are differentiable functions transforming the region \( B \) to the region \( B' \), then
\[
\int_B \int f(x,y) \, dx \, dy = \int_{B'} \int f(x(u,v),y(u,v)) J(u,v) \, du \, dv
\]
where we are making the substitution \( x = x(u,v) \) and \( y = y(u,v) \) and
\[
J(u,v) = \det \begin{pmatrix}
x_u(u,v) & x_v(u,v) \\
y_u(u,v) & y_v(u,v)
\end{pmatrix} = x_u(u,v)y_v(u,v) - y_u(u,v)x_v(u,v)
\]

**Problem 120.** Integrate \( \int_0^4 \int_{y^2+1}^y y \, dx \, dy \). Now make the change of variable, \( u = \frac{2x-y}{2} \) and \( v = \frac{y}{2} \) and integrate the result.

**Polar Coordinates Refresher**

In polar coordinates, the point in the plane \( P = (x,y) \) is denoted by \( (r, \theta) \) where \( r \) is the signed distance from \( (0,0) \) to \( P \) and \( \theta \) is the angle between the vector \( \vec{P} \) and the positive x-axis. We restrict \( \theta \) to the interval \( [0, 2\pi] \). We say \( r \) is the signed distance to allow equations such as \( r = 4 \sin(\theta) \) where if \( \theta = \frac{11\pi}{6} \) then \( r = -2 \). The corresponding point would be \( (-2, \frac{5\pi}{6}) \) which is the same point as \( (2, \frac{5\pi}{6}) \). It follows that \( x, y, r, \) and \( \theta \) are related by the equations:

1. \( x = r \cos \theta \),
2. \( y = r \sin \theta \),
3. \( r = \sqrt{x^2 + y^2} \), and
4. \( \theta = \arctan \left( \frac{y}{x} \right) \).

**Problem 121.** Sketch each of the following pairs of polar functions on the same graphs.

1. \( r = 5 \cos(\theta) \) and \( \theta = 2\pi/3 \)
2. \( r = 2 \sin(\theta) \) and \( r = 2 \cos(\theta) \)
3. \( r = 2 + 2 \cos(\theta) \) and \( r = 1 \)

**Problem 122.** Compute \( \int_0^{\frac{\pi}{2}} \int_0^{\cos(\theta)} r^2 \sin(\theta) \, dr \, d\theta \).

**Problem 123.** Convert the previous integral to rectangular coordinates and recompute to verify your answer.

**Problem 124.** To make a change of variables from rectangular to polar coordinates, we let \( x \) and \( y \) be the functions, \( x(r, \theta) = r \cos \theta \) and \( y(r, \theta) = r \sin \theta \). Show that \( J(r, \theta) = r \) (from Theorem 18) by filling in the missing computation:
\[
J(r, \theta) = \det \begin{pmatrix}
x_r & x_\theta \\
y_r & y_\theta
\end{pmatrix} = \cdots = r.
\]
Using the previous problem and Theorem 18, we now see that when making a change of variable from rectangular to polar coordinates we have,

\[ \int_B f(x,y) \, dx \, dy = \int_{B'} f(r\cos(\theta), r\sin(\theta)) \, r \, dr \, d\theta \]

where \( B' \) is the region \( B \) represented in polar coordinates. The next problem demonstrates an intuitive geometric argument supporting this result.

**Problem 125.** Recall that the area of a sector of a circle with radius \( r \) spanning \( \theta \) radians is \( A = \frac{1}{2} r^2 \theta \). Let \( 0 < r_1 < r_2 \) and \( 0 < \theta_1 < \theta_2 < \frac{\pi}{2} \). Sketch the region in the first quadrant bounded by the two circles \( r = r_1 \), \( r = r_2 \), and the two lines \( \theta = \theta_1 \), and \( \theta = \theta_2 \). Show that the area of the bounded region is \( \left( \frac{r_1 + r_2}{2} \right)(r_2 - r_1)(\theta_2 - \theta_1) \).

**Definition 38.** Two circles are **concentric** if they share a common center and distinct radii.

**Definition 39.** An **annulus** is a region in the plane trapped between two concentric circles.

**Problem 126.** Let \( r_1 \) and \( r_2 \) be real numbers with \( 0 < r_1 < r_2 \). Let \( R \) be the portion of the annulus centered at the origin, between the two circles of radii \( r_1 \) and \( r_2 \), and above the x-axis. Convert \( \iint_R e^{x^2+y^2} \, dA \) over the region \( R \) to polar coordinates and compute. Write down the endpoints of integration in rectangular coordinates.

**Problem 127.** Convert to polar and compute \( \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r \, dr \, d\theta \).

**Problem 128.** Sketch the region bounded by \( r = 2 \) and \( r = 2(1 + \cos(\theta)) \) from \( \theta = 0 \) to \( \theta = \pi \). Compute \( \int_R y \, dA \) via polar coordinates.

**Problem 129.** Sketch \( \theta = \frac{\pi}{6} \) and \( r = 4\sin(\theta) \). Compute the area of the smaller of the two regions bounded by the curves in two ways:

1. First compute \( \int_R \int r \, dr \, d\theta \).

2. Check your answer by using the formula for the area inside a polar graph, \( \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta \).

**Problem 130.** Convert to polar and compute \( \int_1^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{-\frac{1}{2}} \, dy \, dx \).

**Problem 131.** Find the volume of the solid under \( z = 3xy \), above \( z = 0 \), and within \( x^2 + y^2 = 2x \).

**Problem 132.** Find the volume of the solid in the 1st octant (i.e. where \( x > 0, y > 0, z > 0 \)) under \( z = x^2 + y^2 \) and inside the surface \( x^2 + y^2 = 9 \).

**Cylindrical and Spherical Coordinates**

You already know how to represent points in the plane in two different ways, rectangular and polar coordinate systems. We wish to represent points in three-space in two new ways referred to as **cylindrical** and **spherical** coordinates.

**Definition 40.** The **Cylindrical Coordinate Representation** for a point \( P = (x,y,z) \) is denoted by \( (r, \theta, z) \) where \( (r, \theta) \) are the polar coordinates of the point \( (x, y) \) in the x-y-plane and \( z \) remains unchanged (i.e. \( z \) is the height of the point above or below the x-y-plane).
Definition 41. The Spherical Coordinate Representation for a point \( P = (x,y,z) \) is denoted by \((\rho, \phi, \theta)\) where \( \rho \) is the distance from the point to the origin, \( \phi \) is the angle between \( \overrightarrow{OP} \) and positive \( z \)-axis, and \( \theta \) is the angle between the \( x \)-axis and the projection of \( \overrightarrow{OP} \) onto the \( xy \)-plane. To avoid multiple representations of a single point in space we restrict \( \rho \geq 0 \), \( 0 \leq \phi \leq \pi \), and \( 0 \leq \theta \leq 2\pi \).

Problem 133. Sketch (or simply describe in words) each of the following regions in spherical coordinates.

1. the region between \( \rho = 2 \) and \( \rho = 3 \)
2. the region between \( \phi = \pi/2 \) and \( \pi/4 \)
3. the region below \( \rho = 10 \) and above \( \phi = \pi/3 \)

From Problem 124, we know that the Jacobian for polar coordinates is
\[
J(r, \theta) = \det \begin{pmatrix}
\cos(\theta) & -r \sin(\theta) \\
\sin(\theta) & r \cos(\theta)
\end{pmatrix} = r.
\]

The next two problems generate the Jacobian for cylindrical and spherical coordinate systems. Warning: If you haven’t had linear algebra yet, the next two problems require computing the determinant of \( 3 \times 3 \) matrices so you might skip them or look up Cramer’s Rule for computing determinants of matrices.

Problem 134. To make a change of variables from rectangular to cylindrical coordinates, we let \( x = x(r, \theta) = r \cos \theta, y = y(r, \theta) = r \sin \theta, \) and \( z = z. \) Show that \( J(r, \theta, z) = r \) (from Theorem 18) by filling in the missing computation:
\[
J(r, \theta, z) = \det \begin{pmatrix}
x_r & x_\theta & x_z \\
y_r & y_\theta & y_z \\
z_r & z_\theta & z_z
\end{pmatrix} = \cdots = r.
\]

Problem 135. Consider the solid trapped inside the cylinder \( x^2 + y^2 = 9 \), above the function \( f(x,y) = 4 - x^2 - y^2 \) and below the plane \( z = 10 \).

1. Write a triple integral in rectangular coordinates for the volume, but don’t compute.
2. Write a triple integral in cylindrical coordinates for the volume and compute it.
3. Write the volume as the volume of the cylinder minus the volume under the upside down paraboloid.

Problem 136. To make a change of variables from rectangular to spherical coordinates, we let \( x(\rho, \phi, \theta) = \rho \sin \phi \cos \theta, y(\rho, \phi, \theta) = \rho \sin \phi \sin \theta, \) and \( z(\rho, \phi, \theta) = \rho \cos \phi. \) Show that \( J(\rho, \phi, \theta) = \rho^2 \sin(\phi) \) (from Theorem 18) by filling in the missing computation:
\[
J(\rho, \phi, \theta) = \det \begin{pmatrix}
x_\rho & x_\phi & x_\theta \\
y_\rho & y_\phi & y_\theta \\
z_\rho & z_\phi & z_\theta
\end{pmatrix} = \cdots = \rho^2 \sin(\phi).
\]

Summarizing, when integrating with respect to polar, cylindrical, or spherical coordinates, we will always use the appropriate Jacobian as illustrated below.
1. Polar: \[
\int \int \ldots r \, dr \, d\theta, \quad \text{where} \quad x = r \cos(\theta), y = r \sin(\theta)
\]

2. Cylindrical: \[
\int \int \int \ldots r \, dr \, d\theta \, dz, \quad \text{where} \quad x = r \cos(\theta), y = r \sin(\theta), \& z = z
\]

3. Spherical: \[
\int \int \int \ldots \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta, \quad \text{where} \quad x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), \& z = \rho \cos(\phi)
\]

Note that the order of integration could change and you will find times when the choice of the order of integration transforms an apparently hard problem into an easy one.

**Problem 137.** Compute the volume of the region bounded by the two spheres \(\rho = 4\) and \(\rho = 6\) using spherical coordinates. Verify using the formula for the volume of a sphere, \(V = \frac{4\pi}{3}r^3\).

**Problem 138.** Sketch and find the volume that is bounded below by the cone \(\phi = \frac{\pi}{3}\) and bounded above by the sphere \(\rho = 6\).

**Problem 139.** Convert to spherical coordinates then evaluate:
\[
\int_{\pm 3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2-z^2}} (x^2 + y^2 + z^2)^{\frac{3}{2}} \, dy \, dz \, dx
\]

**Problem 140.** Convert to spherical coordinates and compute:
\[
\int e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dV \quad \text{where} \quad V \text{ is the region } x^2 + y^2 + z^2 \leq 1
\]

**Problem 141.** Find the volume of the ellipsoids:
1. \(\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} \leq 1 \text{ via the change of variables } x = 2a, y = 3b, z = 6c\)
2. \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \text{ via the change of variables } x = ua, y = vb, z = wc\)

**An Integration Application**

The **mass** of an object can be found by integrating the density function, \(\delta\), over the entire object. If \(\delta\) is the density function, then the **center of mass** is given by \((\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)\)

where

1. \(m = \text{mass} = \int \delta(x,y,z) \, dx \, dy \, dz\),
2. \(M_{xy} = \int \int z \, \delta(x,y,z) \, dx \, dy \, dz\),
3. \(M_{xz} = \int \int y \, \delta(x,y,z) \, dx \, dy \, dz\), and
4. \(M_{yz} = \int \int x \, \delta(x,y,z) \, dx \, dy \, dz\).
The numbers, $M_{xy}, M_{xz},$ and $M_{yz}$ are called the moments of the object with respect to $z, y,$ and $x$ respectively.

**Problem 142.** Find the center of the mass of the solid inside $x^2 + y^2 = 4,$ outside $x^2 + y^2 = 1,$ below $z = 12 - x^2 - y^2,$ and above $z = 0,$ assuming the constant density function $\rho(x, y, z) = k.$ Graph the solid and mark the center of mass to see if your answer makes sense!

**Problem 143.** Find the mass of the solid bounded by $z = 2 - \frac{1}{2}x^2, \ z = 0, \ y = x \ and \ y = 0,$ assuming the density is $\delta(x, y, z) = kz$ where $k > 0.$ Challenge: Write this integral with the various orders of integration, $dz \ dy \ dx, \ dy \ dx \ dz, \ dz \ dx \ dy,$ and $dx \ dz \ dy.$

**Problem 144.** Find the center of the mass of the solid from the previous problem.
Chapter 6

Line Integrals, Flux, Divergence, Gauss’ and Green’s Theorem

The phrases *scalar field* and *vector field* are new to us, but the concept is not. A scalar field is simply a function whose range consists of real numbers (*a real-valued function*) and a vector field is a function whose range consists of vectors (*a vector-valued function*).

**Definition 42.** Let \( n \) and \( m \) be integers greater than or equal to 2. A *scalar field* on \( \mathbb{R}^n \) is a function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \). A *vector field* on \( \mathbb{R}^n \) is a function \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \).

We now modify our definition for \( \nabla \). If \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) then we previously defined \( \nabla f = (f_x, f_y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \). We now redefine \( \nabla \) as an *operator* that acts on the set of differentiable functions and write

\[
\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).
\]

This encompasses our previous notation as we will write

\[
\nabla f = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)(f) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (f_x, f_y).
\]

We previously defined the dot product only between vectors (and points) in \( \mathbb{R}^n \) and now redefine that notation as well. If \( \vec{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and \( \vec{f}(x, y) = (p(x, y), q(x, y)) \) then we will write

\[
\nabla \cdot \vec{f} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot (p(x, y), q(x, y)) = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = p_x(x, y) + q_y(x, y).
\]

The next definition formalizes what we just wrote, stating it for \( \mathbb{R}^3 \).

**Definition 43.** If \( \vec{f}(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a differentiable vector field then the **divergence** of \( \vec{f} \) is the scalar field from \( \mathbb{R}^3 \rightarrow \mathbb{R} \) defined by: \( \nabla \cdot \vec{f} = p_x + q_y + r_z \)

And the next definition extends our definition of the cross product, \( \times \).

**Definition 44.** If \( \vec{f}(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) then the **curl** of \( \vec{f} \) denoted by \( \nabla \times \vec{f} \) is defined by the vector valued function

\[
\nabla \times \vec{f} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (p, q, r) = \left( \frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}, \frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}, \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \right)
\]

Notice that the divergence of \( \vec{f} \), denoted by \( \nabla \cdot \vec{f} \) is a scalar field while the curl of \( \vec{f} \), denoted by \( \nabla \times \vec{f} \), is a vector field.
Problem 145. Compute the divergence and curl of $\mathbf{f}: \mathbb{R}^3 \to \mathbb{R}^3$ given by

1. $\mathbf{f}(x, y, z) = (x^2yz, x^2 + y + \sqrt{z}, \pi x^2/yz)$
2. $\mathbf{f}(x, y, z) = (e^x, \ln(\text{xyz}), \sin(x^2y^3))$

Definition 45. Let $f: \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{g}: \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{h}: \mathbb{R}^3 \to \mathbb{R}^3$. Assume $f$, $\mathbf{g}$, $\mathbf{h}$ are differentiable. Then

1. $(f \mathbf{g})'(x, y, z) = f(x, y, z)\mathbf{g}(x, y, z)$
2. $(\mathbf{g} \cdot \mathbf{h})'(x, y, z) = \mathbf{g}(x, y, z) \cdot \mathbf{h}(x, y, z)$
3. $(\mathbf{g} \times \mathbf{h})'(x, y, z) = \mathbf{g}(x, y, z) \times \mathbf{h}(x, y, z)$

Problem 146. Let $f: \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{g}: \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{h}: \mathbb{R}^3 \to \mathbb{R}^3$. Prove or give a counter example:

1. $\nabla \cdot (\mathbf{g} + \mathbf{h}) = \nabla \cdot \mathbf{g} + \nabla \cdot \mathbf{h}$
2. $\nabla \cdot (f \mathbf{g}) = f(\nabla \cdot \mathbf{g})$
3. $\nabla \times (\mathbf{g} + \mathbf{h}) = \nabla \times \mathbf{g} + \nabla \times \mathbf{h}$
4. $\nabla \cdot (f \mathbf{g}) = f(\nabla \cdot \mathbf{g}) + \mathbf{g} \cdot \nabla f$
5. $\nabla \times (f \mathbf{g}) = f\nabla \mathbf{g} + \nabla f \times \mathbf{g}$
6. $\nabla \times (\mathbf{g} \cdot \mathbf{h}) = (\nabla \times \mathbf{g}) \cdot \mathbf{h} + \mathbf{g} \times (\nabla \cdot \mathbf{h})$

Arc Length and Line Integrals over Scalar Fields

Definition 12 gave the definitions for the arc length of functions and parametric curves.

Problem 147. Verify that the arc length of $f(x) = x^2$ from $(-1, 1)$ to $(1, 1)$ and the arc length of $\mathbf{f}(t) = (t, t^2)$ for $-1 \leq t \leq 1$ are equal by computing both.

Problem 148. For the fencing example worked in class, suppose we replace the base $c(t) = (t, g(t))$ with the curve $c(t) = (u(t), v(t))$ where $u$ and $v$ are some real-valued functions and we replace the height $f(t) = t^3$ with some real valued function $h$. What would the area of the wall be now? Your answer will be an integral in terms of $u, v,$ and $h$.

Problem 149. Two scalar line integrals:

1. A wall over $c(t) = (3 \sin(t), 3 \cos(t))$ from $t = 0$ to $t = \frac{\pi}{2}$ has height $h(x, y) = x^2y$. Graph the wall and determine its area.
2. A wall over $c(t) = (\sqrt{9 - t^2}, t)$ from $t = 0$ to $t = 3$ has height $h(x, y) = x^2y$. Determine its area.

What you have just been computing are called line integrals over scalar fields.

Definition 46. The line integral of a scalar field $g : \mathbb{R}^2 \to \mathbb{R}$ over the curve $c(t) = (x(t), y(t))$ is defined by $L = \int_c g(c'(t)) \cdot |c'(t)| \, dt$. 

W. Ted Mahavier

www.jiblm.org
The line integral over a scalar field is computed with respect to arc-length and we use the notation,
\[
\int_{c} g \, ds \quad \text{to mean} \quad \int_{c} g(\vec{c}(t)) \cdot |\vec{c}'(t)| \, dt.
\]

We may use line integrals to compute the mass of a curved piece of wire in a similar manner. Suppose we have a piece of wire in the plane that is bent into the shape of a curve, \( \vec{c} : \mathbb{R} \rightarrow \mathbb{R}^2 \). If \( \delta : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( \delta(x,y) \) represents the density of the wire at \((x,y)\), then the line integral of the density with respect to the arc length is the mass of the wire.

**Problem 150.** Recall from the last section of the last chapter that mass is the integral of density. Find the mass of a wire in the plane with density at \((x, y)\) of \( \delta(x, y) = 2xy\) and shape \( \vec{c}(t) = (3 \cos(t), 4 \sin(t)) \) from \( t = 0 \) to \( t = 2\pi \). Could such a wire exist?

**Problem 151.** Find the mass of the wire in three-space with density at \((x, y, z)\) of \( \delta(x, y, z) = 3z\) and in the shape \( \vec{c}(t) = (2 \cos(t), 2 \sin(t), 5t) \), \( t \in [0, 4\pi] \). Find its center of mass as well.

We now have many different types of integrals to compute. Here is a table of notations to help you determine which integral we are computing. Some of these notations we have already seen, some are coming soon.

- \( dx \) or \( dt \) means the usual, for example:
  \[
  \int_{1}^{2} x^2 \, dx = \int_{1}^{2} t^2 \, dt
  \]
- \( ds \) means we integrate with respect to arc length, for example:
  \[
  \int_{c} f \, ds = \int_{c} f(\vec{c}(t)) |\vec{c}'(t)| \, dt
  \]
- \( d\vec{c} \) means a line integral over a vector field along some curve, for example:
  \[
  \int_{c} \vec{f} \cdot d\vec{c} = \int_{c} \vec{f}(\vec{c}(t)) \cdot \frac{\vec{c}'(t)}{|\vec{c}'(t)|} \, ds = \int_{c} \vec{f}(\vec{c}(t)) \cdot \frac{\vec{c}'(t)}{|\vec{c}'(t)|} |\vec{c}'(t)| \, dt = \int_{c} \vec{f}(\vec{c}(t)) \cdot \vec{c}'(t) \, dt
  \]
- \( dA \) means we integrate over a two dimensional domain, for example the double integral:
  \[
  \int_{D} f \, dA = \int \int f(x,y) \, dx \, dy
  \]
- \( dV \) means we integrate over a three dimensional domain, for example the triple integral:
  \[
  \int_{D} f \, dV = \int \int \int f(x,y,z) \, dx \, dy \, dz
  \]

**Definition 47.** A vector field is said to be **conservative** if it is the gradient of some function. Conservative fields are also called **gradient** fields.

**Problem 152.** \( \vec{f}(x, y) = (-2xy + 2e^y, -x^2 + 2xe^y) \) is conservative since \( \vec{f} = \nabla g \) where \( g \) is given by \( g(x, y) = -x^2y + 2xe^y \).

**Problem 153.** Is \( \vec{f}(x, y) = (xy, x - y) \) conservative? If \( \vec{f} \) is conservative then there must be a function \( g \) so that \( \nabla g = \vec{f} \). Is there a function \( g \) so that \( g_x(x,y) = xy \) and \( g_y(x,y) = x - y \)? Why or why not?
Problem 154. Is \( \vec{f}(x, y) = (x^2 + y^2, 2xy) \) conservative?

Problem 155. Show that if \( \vec{f}(x, y) = (p(x, y), q(x, y)) \) and \( p_y(x, y) = q_x(x, y) \) then \( \vec{f} \) is a gradient field.

Problem 156. Is \( \vec{f}(x, y) = \left( \frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right) \) a gradient field?

The next two examples remind us that work may be computed by integrating force using two examples that you may well have seen in a previous calculus course.

Example 1. From physics we have Work = Force \( \times \) Distance. So lifting a 5 lb book 2 feet requires 10 foot – lbs of work. Suppose we have a spring that has force \( f \) proportional to the square of the distance it is stretched from equilibrium so that \( f(x) = kx^2 \) (where \( k > 0 \) is the spring’s coefficient). If \( 0 = x_0 \leq x_1 \leq \cdots \leq x_N = 4 \) then the work done to stretch it 4 units would be

\[
W = \lim_{N \to \infty} \sum_{i=1}^{N} kx_i^2 (x_{i+1} - x_i) \text{ where } x_i \in [x_i, x_{i+1}]
\]

\[
= \int_0^4 kx^2 \, dx \text{ by the definition of the integral.}
\]

Thus in one dimension, work is the integral of force: \( W = \int f(x) \, dx \).

Example 2. Suppose we have a cone full of water with radius 10 ft, height 15 ft, its point on the ground, and standing so that it holds the water. How much work is done in pumping all the water out of the cone? Hints: Force = Mass \( \times \) Acceleration (due to gravity) and Mass of one slice = area of slice \( \times \) Density of H\(_2\)O. Add (integrate) the amount or work done in moving each horizontal slice of water out of the tank. The density of water is 62.5 lbs/ft\(^3\) and acceleration due to gravity is 32 ft/sec\(^2\).

We now move to the study of vector (force) fields because we would like to be able to compute the total work done by an object as it moves through some field that acts on the object by either aiding or hindering its motion. As examples, think of a metal ball passing through a magnetic field, an electron passing through an electric field, a submarine passing through a fluid field, or a man passing through a gravitational field.

Suppose we have an object passing along a curve through a vector field. Let \( \vec{f} : \mathbb{R}^2 \to \mathbb{R}^2 \) be our (differentiable) vector field and \( c : \mathbb{R} \to \mathbb{R}^2 \) be our (differentiable) planar curve or path of the object. Therefore \( \vec{f} \) is of the form, \( \vec{f}(x, y) = (p(x, y), q(x, y)) \) for some \( p, q : \mathbb{R}^2 \to \mathbb{R} \) and \( c \) is of the form, \( c(t) = (a(t), b(t)) \) for some \( a, b : \mathbb{R} \to \mathbb{R} \). Thus we may compute the work done as the particle moves through the vector field by integrating over the field evaluated at the particle, dotted with the speed of the particle. This gives the component of the force that is acting in the direction of travel of the particle.

\[
W = \int c'(t) \cdot \vec{c}'(t) \, dt
\]

\[
= \int (p(a(t), b(t)), q(a(t), b(t))) \cdot (a'(t), b'(t)) \, dt
\]

\[
= \int p(a(t), b(t)) \, a'(t) + q(a(t), b(t)) \, b'(t) \, dt
\]

There are two additional ways in which vector line integrals are often written. First, the independent variable \( t \) is often omitted:

\[
W = \int c(\vec{f}(c(t)) \cdot \vec{c}'(t) \, dt = \int c(\vec{f}) \cdot d\vec{c}.
\]
Second, if we \( f(x, y) = (p(x, y), q(x, y)) \) and \( \overrightarrow{c}(t) = (x(t), y(t)) \) then
\[
W = \int f(\overrightarrow{c}) \cdot d\overrightarrow{c} = \int (p(x,y),q(x,y)) \cdot (x'(t),y'(t)) \, dt \\
= \int p(x(t),y(t)) \cdot x'(t) + q(x(t),y(t)) \cdot y'(t) \, dt \\
= \int p \, dx + q \, dy
\]

**Problem 157.** Suppose we have a particle traveling through the force field, \( \overrightarrow{f}(x, y) = (xy, 2x-y) \).

1. Compute the work done as the particle travels through the force field \( f \) along the curve \( \overrightarrow{c}(t) = (t, t^2) \) from \( t = 0 \) to \( t = 1 \).
2. Compute the work done as the particle travels through the force field \( f \) along the curve \( \overrightarrow{c}(t) = (2t, 4t^2) \) from \( t = 0 \) to \( t = \frac{1}{2} \).

**What you have just computed is called a line integral over a vector field.**

**Definition 48.** The line integral of the vector field \( \overrightarrow{f} : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( \overrightarrow{f}(x, y) = (p(x,y),q(x,y)) \) over the curve \( \overrightarrow{c} : \mathbb{R} \to \mathbb{R}^2 \) given by \( \overrightarrow{c}(t) = (x(t),y(t)) \) is defined by
\[
L = \int \overrightarrow{f}(\overrightarrow{c}) \cdot d\overrightarrow{c} = \int \left( p(x(t),y(t)), q(x(t),y(t)) \right) \cdot (x'(t),y'(t)) \, dt
\]

**Problem 158.** Compute the line integral where \( \overrightarrow{c} \) is the rectangle with corners at \((0, 0), (1, 0), (1, 2), (0, 2)\) and force field \( \overrightarrow{f}(x, y) = (4x+y, x+2y) \). Start your path at the origin and work counter-clockwise around the rectangle. You will need to compute four integrals along four parameterized lines.

You have been using the Fundamental Theorem of Calculus for three semesters now. This powerful result in one dimension is all that is needed to prove the version we need for three dimensions.

**Theorem 19. The Fundamental Theorem of Calculus** If \( F \) is any anti-derivative of \( f \) then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

**Theorem 20. The Fundamental Theorem of Calculus for Line Integrals** If \( \overrightarrow{f} \) is a gradient field and \( g \) is any anti-derivative of \( \overrightarrow{f} \) (i.e. \( \nabla g = \overrightarrow{f} \)) then
\[
\int_{\overrightarrow{c}} \overrightarrow{f}(\overrightarrow{c}) \cdot d\overrightarrow{c} = g(\overrightarrow{c}(b)) - g(\overrightarrow{c}(a)).
\]

**Problem 159. Prove Theorem 20.**

**Problem 160. Compute the line integral from Problem 158 using Theorem 20.**

The next two theorems are important results associated with line integrals. We don’t assert all the necessary hypothesis – curves are smooth, functions are assumed to be integrable, etc.

Theorem 21 states that if we reverse the path along which we compute a line integral, then we change the sign of the result. Thinking of the line integral as work, this makes sense, because the work done by traveling one direction along a path should be the opposite of the work done traveling the other way. When we have a curve \( \overrightarrow{c} \) and we write \( -\overrightarrow{c} \) we mean the same set of points in the plane, but we simply reverse the direction. If we have the curve \( \overrightarrow{c}(t) = (x(t), y(t)) \) from \( t = a \) to \( t = b \) then \( -\overrightarrow{c} \) is simply the same curve where \( t \) goes from \( t = b \) to \( t = a \).
Theorem 21. If $\vec{c} : \mathbb{R} \to \mathbb{R}^2$ is a parametric curve and $\vec{f}$ is a vector field then
\[ \int_{c_1} \vec{f}(\vec{c}) \cdot d\vec{c} = -\int_{c_2} \vec{f}(\vec{c}) \cdot d\vec{c}. \]

Problem 161. Show that $\int_{c_1} \vec{f}(\vec{c}) \cdot d\vec{c} = -\int_{c_2} \vec{f}(\vec{c}) \cdot d\vec{c}$ where $\vec{f}(x, y) = (xy, y-x)$, $\vec{c}(t) = (t^2, t)$, $t \in [0, 2]$. Remember that since $c$ is the path from $(0, 0)$ to $(4, 2)$, then $-c$ is the same path but from $(4, 2)$ to $(0, 0)$.

Theorem 22 states that line integrals over conservative fields are independent of path. Suppose you and I start at the same point at the bottom of a mountain and we walk to the top following different paths. Did we do the same amount of work? Yes. Because gravity is a conservative (gradient) field it does not matter what path we follow as long as we both start at the same place and end at the same place. By the same logic, if we start at a point on the mountain, walk around, and return to the same spot, then the total work is zero.

Theorem 22. If $c_1^2$ and $c_2^2 : \mathbb{R} \to \mathbb{R}^2$ are two (distinct) paths beginning at the point, $(x_1, y_1)$ and ending at the point $(x_2, y_2)$ in the plane and $\vec{f}$ is a gradient field then
\[ \int_{c_1} \vec{f}(\vec{c}) \cdot d\vec{c} = \int_{c_2} \vec{f}(\vec{c}) \cdot d\vec{c}. \]

Problem 162. Let $f$ be the vector field $\vec{f}(x, y) = (3x^2y+x, x^3)$ and $c$ be the planar curve $\vec{c}(t) = (\cos(t), \sin(t))$ from $t = 0$ to $t = \pi$.

1. Write out and simplify, but don’t compute, the line integral of $f$ over $\vec{c}$.
2. Compute this line integral by choosing the simpler path $p(t) = (-t, 0)$ from $t = -1$ to $t = 1$ and applying Theorem 22.
3. Recompute this line integral by finding a function $g$ so that $\nabla g = (3x^2y+x, x^3)$ and applying Theorem 20.

Problem 163. Let $\vec{f}(x, y) = (x-y, x+y)$.

1. Compute $\int_{c_1} \vec{f}(\vec{c}) \cdot d\vec{c}$ where $\vec{c}_1(t) = (t, t^2)$, $t \in [0, 1]$.
2. Compute $\int_{c_2} \vec{f}(\vec{c}) \cdot d\vec{c}$ where $\vec{c}_2(t) = (\sin(t), \sin^2(t))$, $t \in [2\pi, \frac{5\pi}{2}]$.
3. Explain your answer.

When we compute line integrals, we often integrate along a curve that is the boundary of some region. Here are a few of the buzz words about curves that we will use.

Definition 49. Let $\vec{c} : [a, b] \to \mathbb{R}^2$. We say that $\vec{c}$ is a simple closed curve if $\vec{c}$ starts and ends at the same point (i.e. $\vec{c}(a) = \vec{c}(b)$) and never crosses itself.

When we are integrating around a simple closed curve $\vec{c}$ with respect to the arc-length of the curve, we will use the notation $\int_{\vec{c}} \cdots ds$.

Definition 50. Let $\vec{c} : [a, b] \to \mathbb{R}^2$ be a simple closed curve on $[a, b]$. We say that $c$ is positively oriented if as $t$ increases from $a$ to $b$ and we traverse the curve $\vec{c}$ then the enclosed region is on our left.
As we will show in a forthcoming lecture, Gauss’ Divergence Theorem is equivalent to Green’s Theorem. They are simply the same theorem stated in two different ways. Gauss’ Divergence Theorem says that if we have a certain simple closed curve representing the boundary of a region over which we have a vector field (a flow), then the flow across boundary (the flux) must equal the integral of the divergence of the fluid (or the electricity or whatever) over the region enclosed by the boundary.

For both Green’s and Gauss’ Theorems, when integrating along the boundary of the region the simple closed curve must be positively oriented. That is, you must integrate in a counter-clockwise direction so that the region is on your left as you travel along the parametric curve.

**Theorem 23. Gauss’ Divergence Theorem for the Plane** If \( F \) is a vector field and \( c \) is a simple closed curve and \( n \) is the unit normal to \( c \) then

\[
\oint_{c} F \cdot n \, ds = \int_{D} \nabla \cdot F \, dA
\]

**Problem 164.** Verify Gauss’ Divergence Theorem by computing both integrals (the flux integral and the divergence integral) with \( f(x, y) = -4(x, y) \) and \( \vec{c}(t) = (\cos(t), \sin(t)) \) assuming \( t \in [0, 2\pi] \).

**Problem 165.** Verify Gauss’ Divergence Theorem for \( f(x, y) = (xy, 2x - y) \) over the region \( D \) in the first quadrant bounded by \( y = 0, x = 0 \) and the line \( y = 1 - x \).

**Problem 166.** Verify Gauss’ Divergence Theorem for the flow \( f(x, y) = (0, y) \) over the rectangular region, \( \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\} \).

**Theorem 24. Green’s Theorem** If \( \vec{f} : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}, \vec{c} : \mathbb{R} \rightarrow \mathbb{R}^{2} \) is a vector field and \( \vec{c} : \mathbb{R} \rightarrow \mathbb{R}^{2} \) is a positively oriented simple closed curve so that \( \vec{c}(t) = (x(t), y(t)) \) and \( D \) is the region enclosed by \( \vec{c} \) then

\[
\oint_{c} \vec{f}(\vec{c}) \cdot d\vec{c} = \int_{D} q_{x}(x, y) - p_{y}(x, y) \, dA
\]

**Problem 167.** Verify Green’s Theorem for the flow \( f(x, y) = (-y^{2}, xy) \) over the region bounded by \( x = 0, y = 0, x = 3, \) and \( y = 3 \).

**Problem 168.** Verify Green’s Theorem for the vector field \( f(x, y) = (-y, x) \) over the region bounded by the curve \( \frac{x^{2}}{4} + \frac{y^{2}}{9} = 1 \).

**Problem 169.** Verify Green’s Theorem for the vector field \( f(x, y) = (xy, 3x) \) over the region formed by the three points, \((-1, 0), (1, 0), \) and \((0, 4)\).

**Problem 170.** Find the area of the ellipse \( \frac{x^{2}}{a^{2}} + \frac{y^{2}}{b^{2}} = 1 \) via the formula \( A = \frac{1}{2} \int x \, dy - y \, dx \) and the parametrization \( \vec{c}(t) = (a \cos(t), b \sin(t)) \), \( t \in [0, 2\pi] \).

Gauss’ Divergence Theorem is valid in higher dimensions as well, although it is often the case that integrating over certain parts of the boundary is challenging.

**Theorem 25. Gauss’ Divergence Theorem for Three Dimensions** If \( S \) is a solid in three dimensional space, we write

\[
\int_{\partial S} f \cdot n \, dS = \int_{S} \nabla \cdot f \, dV
\]

where \( \partial S \) denotes the boundary of \( S \), \( n \) denotes the unit outward normal and \( dV \) indicates that we are integrating over the entire volume of the solid.
Problem 171. Verify Gauss’ Divergence Theorem for the vector field \( f(x,y,z) = (x^2 y, 2xz, yz^3) \) over the three dimensional box, \( 0 \leq x \leq 1, 0 \leq y \leq 2, \) and \( 0 \leq z \leq 3. \)

Problem 172. Let \( S \) be the cylinder of radius 2 with base centered at the origin and height 3. Let \( f(x,y,z) = (x^3 + \tan(yz), y^3 - e^{xz}, 3z + x^3). \) Use the divergence theorem to compute the flux across the side of the cylinder.

Problem 173. Let \( S \) be the solid bounded by \( 2x + 2y + z = 6 \) in the first octant. Let \( f(x,y,z) = (x, y^2, z). \) Sketch the solid and verify Gauss’ Divergence Theorem.

Congratulations, for some of you this note constitutes having independently worked through three semesters of Calculus, which is quite an accomplishment.
Chapter 7

Practice Problems

This chapter consists of practice problems for students to work after we have covered a topic. Section 7.X has practice problems with solutions for Chapter X.

7.1 Vectors, Lines, and Functions Drill

Vectors

1. Let \( \vec{x} = (-1, 3) \) and \( \vec{y} = (4, 1) \) and graph \( \vec{x}, \vec{y}, \vec{xy}, \) and \( \vec{x} - \vec{y} \).
2. Find the norm of each vector in the previous problem.
3. Graph \( \vec{(1, -2, 1)} \) and \( \vec{(-2, 4, 3)} \) and find the angle between them.

Lines

1. Sketch the vectors \( x = (1, 3, 5) \) and \( y = (2, 4, -3) \) and the line through these points.
2. Find a parametric equation, \( \vec{l} \), for the line in the previous problem so that \( \vec{l}(0) = x \) and \( \vec{l}(1) = y \).
3. Plot \( \vec{l}(t) = (2, -1, 4) + (1, 0, 0)t \). Assuming this represents the position of an object, compute the speed of the object.
4. Find an equation for the position of the object that has: the same speed as the object in the previous problem, the same position at time 0, and is travelling in the opposite direction.

Functions

1. Sketch a graph of \( f(x, y) = 2x^2 + 3y^2 \).
2. Sketch the intersection of \( f \) from the previous problem with the plane, \( z = 4 \).
3. Sketch the function \( g(x, y) = x^3 + y^2 \). Be sure and label a few points.
4. Compute the composition \( g \circ \vec{l} \) where \( g \) is from the previous problem and \( \vec{l}(t) = (2, -1) + (1, 0)t \).
5. Compute \( (g \circ l)'(3) \) and indicate its meaning.
6. Sketch \( l(t) = (0, 1) + (1, 0)t \) for \( t \geq 0 \).
7. Sketch $c(t) = (3 \cos(t), 3 \sin(t))$ for $0 \leq t \leq 2\pi$.

8. Sketch $e(t) = (4 \cos(t), 2 \sin(t))$ for $0 \leq t \leq 2\pi$.

9. Graph $\vec{r}(t) = (\cos(t), 2t, \sin(t))$ for $t \in [0, 6\pi]$; if $\vec{r}(t)$ is the position of an object at time $t$ then show that the speed of the object is constant.
Vectors, Lines, and Functions Solutions

Vectors
1. a parallelogram with diagonals
2. \(|x| = \sqrt{10}\) and \(|y| = \sqrt{17}\)
3. approximately 122 degrees

Lines
1.
2. \(\vec{l}(t) = (1, 3, 5) + (1, 1, -8)t\) (\(\infty\) possible solns)
3. \(||\vec{l}'(t)|| = 1\)
4. \(\vec{p}(t) = (2, -1, 4) - (1, 0, 0)t\) (\(\infty\) possible solns)

Functions
1. a squished paraboloid
2. an ellipse
3. a water slide
4. \((t + 2)^3 + 1\)
5. 75, the slope of the line (in three-space) that lies above the line \(\vec{l}\) and is tangent to the graph of \(g\) at the point \((5, -1, 126)\)
6. \(l =\) line
7. \(c =\) circle, radius = 3
8. \(e =\) ellipse
9. \(s =\) spiral; a spring wrapped around the y-axis in three-space; speed is \(\sqrt{5}\)
7.2 Cross Products, Planes, and Limits Drill

Cross Products

1. Compute the cross product of \( \vec{u} = (3, 2, -1) \) and \( \vec{v} = (2, \pi, -3) \) and state the significance of the resulting vector.

2. Find two vectors of unit length orthogonal to both \( \vec{a} = (4, -3, 2) \) and \( \vec{b} = (2, 5, -3) \).

Planes

1. Find the equation of a plane parallel to \( 3x + 2z = 4 - y \) and containing \((2, 4, -2)\).

2. Find the equation of the line that represents the intersection of \( 3z - 4x + 2y = 4 \) and \( 2z - x = 6 - y \).

3. Find an equation for the plane containing \((1, 2, 3), (2, 3, 4), \) and \((-1, 1, 1)\). Find another equation for the same line.

4. Find intersection of \( 2x - 3y + 5z = 4 \) and \( 2x + 5y - 10z = 5 \).

Limits

1. Consider the function defined by \( f(x, y) = \begin{cases} \frac{x^2}{x^2 + y} & x^2 + y \neq 0 \\ 0 & (x, y) = (0, 0) \end{cases} \)

Does the limit exist as \((x, y) \to (0, 0)\) along \( y = kx \) exist for every \( k \in \mathbb{R} \)? Does the limit exist as \((x, y) \to (0, 0)\) along \( y = kx^2 \) exist for every \( k \in \mathbb{R} \)? Is the function \( f \) continuous at \((x, y) = (0, 0)\)?
Cross Products, Planes, and Limits Solutions

Cross Products

1. \((\pi - 6, 7, 3\pi - 4)\)
2. \(\pm \frac{1}{\sqrt{933}}(-1, 16, 26)\)

Planes

1. \(3x + y + 2z = 6\)
2. \((4, 10, 0) + (-2, -10, 4)t\) (infinitely many other ways to write this one line; check yours by substituting it into each plane)
3. \(z - x = 2\)

Limits

1. No, for a chosen value of \(k\), the limit along the path \(y = kx^2\) would be \(\frac{1}{1+k}\). Therefore the limit along different parabolic paths (different values of \(k\)) would yield different results.
7.3 Domains, Graphing, and Derivatives Drill

Domains of Functions

1. Find the domain of \( f(x, y) = \ln(x^2 + y^2 - 1) \).
2. Find the domain of \( g(x, y) = \tan^{-1}\left(\frac{x}{y}\right) \).
3. Find the domain of \( h(x, y) = \frac{4}{|x| - |y|} \).
4. Find the domain of \( k(x, y, z) = \frac{2xy}{z^2 + z - 1} \).
5. Find the domain of \( \ell(x, y, z) = \frac{xz}{\sqrt{1 - y^2}} \).
6. Determine the domain over which \( f(x, y) = \ln(x^3y^4) \) is continuous.
7. Determine the domain over which \( g(x, y) = \frac{x}{|x||y|} \) is continuous.
8. Determine the domain over which \( h(x, y, z) = \frac{x^2 - 1}{z\sqrt{y^2 - 1}} \) is continuous.

Graphing Functions

1. Graph \( f(x, y) = 1 - x^2 + y^2 \).
2. Graph \( g(x, y) = x + y \).
3. Graph \( h(x, y) = \frac{x^2}{9} + \frac{y^2}{4} \).
4. Graph \( k(x, y) = |x| - |y| \).
5. Graph \( r(x, y) = \sin(x) \).

Partial Derivatives and Gradients

1. Let \( g(x, y) = x^3 - 4\log_7 x^2 + \sin^{-1}(xy) \) and compute \( g_x \).
2. Let \( g(x, y) = \frac{x^2}{\sin(xy)} \) and compute \( g_y \).
3. Let \( g(x, y) = e^{x^2} \) and compute \( \nabla g \).
4. Let \( g(x, y) = x \ln(y) - 4xy + x \) and compute \( g_x(1, 1) \) and \( g_y(1, 1) \).
5. Let \( g(x, y) = 5x^2y \sin(x - y) \) and compute \( g_y(\pi, 2\pi) \).
6. Let \( h(x, y, z) = \sqrt{2xz - 5y} + \cos^3(z \sin(x)) \) and compute \( \nabla h \).
7. Let \( h(x, y, z) = z^y \) and compute \( h_x(2, 3, 4) \).
8. Find \( f_{xx}, f_{zy}, \) and \( f_{zxy} \) for \( f(x, y, z) = \frac{y}{x} - \sin(zy) - 3z^3 \).
### Directional Derivatives

Remember: Directions vectors should be unit vectors.

1. Using the definition of directional derivative, compute the derivative of \( f(x, y) = x - y^2 \) at \((1, 2)\) in the direction, \((1, 1)\).

2. Find \( D_\vec{u} f(p) \) for \( f(x, y) = e^{xy} + 2x^2y^3 \); \( p = (3, 1) \); \( \vec{u} \) is a unit vector parallel to \( \vec{v} = (-5, 12) \).

3. Find \( D_\vec{u} f(p) \) for \( f(x, y) = e^{xy} + \ln(xy) \); \( p = (1, 2) \); \( \vec{u} \) is a unit vector which makes an angle \(-\frac{\pi}{3}\) from the positive \( x \) axis.

4. Find \( D_\vec{u} f(p) \) for \( f(x, y, z) = x^2 y - 4y^2 z + xyz^2 \); \( p = (-2, 1, -1) \); \( \vec{u} \) is a unit vector parallel to the vector \( \overrightarrow{AB} \) where \( A = (2, 1, -5) \) and \( B = (-2, 4, 3) \).

### Derivatives

For each of the following functions, compute the indicated “derivative” of the function. Because of the differing domains of the functions, the derivative could be a function (a partial derivative), a vector of functions (a gradient), or a matrix of functions (the ‘total’ derivative)!

1. \( f(x, y) = x^2y^3 \)

2. \( g(x, y) = \left( \frac{x^2}{y} - 3xy, \sin\left(\frac{x}{y}\right) \right) \)

3. \( h(x, y) = x^2 - e^{xy\sqrt{z}} + \sinh(yz) \)

4. \( r(s, t, u) = (st, s^2tu, \sqrt{stu}, \ln(st^2u)) \)

### Chain Rule

1. Let \( f(x, y) = 2x^2 - 7y \) and \( \vec{g}(t) = (\sin(t), \cos(t)) \). Compute the derivative of \( f \circ g \) in two ways. First compose \( f \) and \( g \) and take the derivative. Second, apply the chain rule. Verify that your solutions are the same.

2. Let \( f(x, y) = 4x^3y + e^{3y} + \frac{2}{x}, x(t) = t^2, y(t) = 4t - 3, \) and \( g(t) = (x(t), y(t)) \). Compute \( (f \circ g)'(-1) \).

3. For each of the following problems, find \( \frac{dg}{dt} \) and evaluate at the given value of \( t \).
   (a) \( g(x, y) = 3xy + e^x y^2 \) where \( x(t) = 4t^2 + t, y(t) = 6 + 5t, \) and \( t = 0 \)
   (b) \( g(x, y, z) = x^3y + xz + \frac{x}{y-z} \) where \( x(t) = t^2 + 3, y(t) = 4t - t^2, z(t) = \cos(t - 3\pi), \) and \( t = 0 \)

4. Let \( f(x, y) = xy\ln(x) \) and \( \vec{g}(s, t) = (2st, t - s^3) \). State the domain \( f, g, \) and \( f \circ g \). Compute the gradient of \( (f \circ g) \).

5. For each of the following problems, find \( f_s \) and \( f_t \) (i.e. \( \frac{\partial f}{\partial s} \) and \( \frac{\partial f}{\partial t} \)).
   (a) \( f(x, y) = 2x - y^2, x(s, t) = s\cos(t), \) and \( y(s, t) = (s + t)e^t \)
(b) \( f(x,y,z) = (x + 2y + 3z)^4 \) and \( x(s,t) = s + t, \, y(s,t) = s - t, \, z(s,t) = st \)

6. For each of the following problems, find \( \frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \text{ and } \frac{\partial g}{\partial w} \) (i.e. \( g_u, g_v, \text{ and } g_w \)).

(a) \( g(x,y) = (x+y)\ln(xy) \), \( x(u,v,w) = u + v - 3w \), and \( y(u,v,w) = uv + 3w \)

(b) \( g(x,y,z) = yz + xz + xy \), \( x(u,v,w) = u + v - 3vw \), \( y(u,v,w) = v + w + 4uvw \), and \( z(u,v,w) = u + w - 5uv \)

7. For each of the following three problems, find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at the indicated point.

(a) \( z^3 - xy + 2yz + y^3 - 3 = 0; (1, 1, 1) \)

(b) \( \frac{1}{x^2} + \frac{1}{y^3} + \frac{1}{z^2} = \frac{49}{36}; (-1, 2, 3) \)

(c) \( ye^{xy} \cos(2xz) = 1; (\pi, 1, 4) \)

Tangent Lines and Planes

1. Find the (shortest) distance from the point \((0, 1, 0)\) to the plane \(x + 2y + 3z = 4\).

2. Find the equation of the tangent plane to the function \(z = x^2 - 4y^2\) at \((3, 1, 5)\).

3. For each of the following problems, find an equation of the tangent plane and equations of the normal line to the surface at the indicated point.

(a) \( z + 1 = xe^y \cos(z); (1, 0, 0) \)

(b) \( x^2 + 2y^2 + 3z^2 = 6; (1, -1, 1) \)

4. Find the tangent plane approximation of \( h(x,y) = x + x \ln(xy) \) when \( x = e \) and \( y = 1 \). Use this approximation to estimate \( h(2.7, 1.05) \).

5. Find the tangent line to the level curve of \( g(x,y) = e^{xy} + 3x^2 \sqrt{y} \) at \( p = (-2, 0) \).
Domains, Graphing, and Derivative Solutions

Domains of Functions

1. \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \} \)
2. \( \{(x, y) \in \mathbb{R}^2 : y \neq 0 \} \)
3. \( \{(x, y) \in \mathbb{R}^2 : |x| \neq |y| \} \)
4. all of \( \mathbb{R}^3 \) except where \( z \) equals the Pisot numbers
5. \( \{(x, y) \in \mathbb{R}^2 : |y| < 1 \} \)

Graphing Functions

1. a saddle centered at \((0, 0, 1)\)
2. a plane
3. squished paraboloid
4. planar saddle with a ‘point’ at \((0, 0, 0)\)
5. a three dimensional sine wave

Partial Derivatives and Gradients

1. \( g_x(x, y) = 3x^2 - \frac{8}{\ln(7)} + \frac{y}{\sqrt{1-(xy)^2}} \)
2. \( g_y(x, y) = \frac{-x^3 \cos(xy)}{\sin^2(xy)} \)
3. \( \nabla g(x, y) = (y^2 e^{xy^2}, 2xy e^{xy^2}) \)
4. \( g_x(1, 1) = g_y(1, 1) = -3 \)
5. \( g_y(\pi, 2\pi) = 2\pi 5\pi^2 \)
6. \( \nabla h(x, y, z) = (\frac{z}{\sqrt{2xz-3y}} - 3z \cos(x) \cos^2(z \sin(x)) \sin(z \sin(x)), \frac{5}{\sqrt{2xz-3y}} - 3 \sin(x) \cos^2(z \sin(x)) \sin(z \sin(x))) \)
7. \( h_z(2, 3, 4) \approx 17,035 \)
8. \( f_{xx} = \frac{12y}{x^2}, f_{xy} = -\cos(zy) + zysin(zy), f_{xzy} = 0 \)
Directional Derivatives

Remember: Directions vectors should be unit vectors.

1. \[ \frac{-3}{\sqrt{2}} \]
2. \[ \frac{1}{13}(588 + 31e^3) \]
3. \[ (1 - \frac{\sqrt{3}}{2})(e^2 + \frac{1}{2}) \]
4. \[ \frac{42}{\sqrt{89}} \]

Derivatives

1. \[ \nabla f(x,y) = (2xy^3, 3x^2y^2) \]
2. \[ Dg(x,y) = \begin{pmatrix} \frac{2x}{y} - 3y & -x^2 - 3x \\ \frac{1}{y} \cos\left(\frac{x}{y}\right) & -\frac{x}{y^2} \cos\left(\frac{x}{y}\right) \end{pmatrix} \]
3. \[ \nabla h(x,y) = (2x - y \sqrt{z}e^{xy\sqrt{z}}, -x \sqrt{z}e^{xy\sqrt{z}} + z \cosh(yz)) \]
4. \[ Dr(s,t,u) = \begin{pmatrix} t & s & 0 \\ 2stu & s^2u & s^2t \\ \frac{2\sqrt{stu}}{st^u} & \frac{s^2u}{st^u} & \frac{s^2t}{st^u} \end{pmatrix} \]

Chain Rule

1. \( (f \circ g)'(t) = \sin(t)(4 \cos(t) + 7) \)
2. \( (f \circ g)'(-1) \approx 188 \)
3. (a) 84
4. \[ \nabla (f \circ g) = ((2t^2 - 8s^3t)\ln(2st) + (2st^2 - 2s^4t)\frac{1}{s}, (4st - 2s^4)\ln(2st) + (2st^2 - 2s^4t)\frac{1}{s}) \]
5. (a) \( f_s(s,t) = 2 \cos(t) - 2(s + t)e^{2t} \) and \( f_t(s,t) = -2s \sin(t) - 2(s + t)e^{2t} - 2(s + t)^2e^{2t} \)

Tangent Lines and Planes

1. \[ \frac{\sqrt{14}}{7} \]
2. \[ 6x - 8y - z = 5 \]
3. (a) No solutions yet!
7.4 Optimization and Lagrange Multipliers Drill

Critical Points

1. For each of the following functions, find the critical points and determine if they are maxima, minima, or saddle points.
   (a) \( f(x, y) = 1 - x^2 - y^2 \)
   (b) \( g(x, y) = e^{-x} \sin y \)
   (c) \( F(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5 \)
   (d) \( g(x, y) = x^2 + xy + y^2 \)
   (e) \( z = 8x^3 - 24xy + y^3 \)

2. For each of the following functions, find the absolute extrema of the function on the given closed and bounded set \( R \) in \( \mathbb{R}^2 \).
   (a) \( f(x, y) = 2x^2 - y^2; \ R = \{ (x, y) : x^2 + y^2 \leq 1 \} \)
   (b) \( g(x, y) = x^2 + 3y^2 - 4x + 2y - 3; \ R = \{ (x, y) : 0 \leq x \leq 3, -3 \leq y \leq 0 \} \)

3. Find the direction at which the maximum rate of change of \( g(x, y) = \ln(xy) - 3x + 2y \) at \( p = (3, 2) \) will occur and find the maximum rate of change.

4. Find the direction in which the function \( f(x, y) = x^3 - y^5 \) increases the fastest at the point \( (2, 4) \).

5. For each of the following four problems, find all critical points of \( f \) and classify these critical points as relative maxima, relative minima, or saddle points using the second derivative test whenever possible.
   (a) \( f(x, y) = xy^2 - 6x^2 - 3y^2 \)
   (b) \( f(x, y) = \frac{9x}{x^2 + y^2 + 1} \)
   (c) \( g(x, y) = x^2 + y^3 + \frac{768}{x + y} \)

Optimization and Lagrange Multipliers

1. Find all extrema of \( f(x, y) = 1 - x^2 - y^2 \) subject to \( x + y = 1, x \geq 0 \), and \( y \geq 0 \).
2. Find all extrema of \( f(x, y) = 1 - x^2 - y^2 \) subject to \( x + y \leq 1, x \geq 0 \), and \( y \geq 0 \).
3. Find the absolute extrema of the function \( g(x, y) = 2 \sin(x) + 5 \cos(y) \) on the rectangular region \( R = \{ (x, y) : 0 \leq x \leq 2, 0 \leq y \leq 5 \} \).
4. Find the minimum value of \( z = x^2 + y^2 \) subject to \( x + y = 24 \).
5. Find the extreme values of \( f(x, y) = 2x^2 + y^2 - y \) subject to \( x^2 + y^2 = 4 \) using Lagrange multipliers.
6. Find three positive numbers whose sum is 123 such that their product is as large as possible.
7. A container in \( \mathbb{R}^3 \) has the shape of a cube with each edge length 1. A (triangular) plate is placed in the container so that it intersects the cube in the plane \( x + y + z = 1 \). If the container is heated so that the temperature at each point is given by \( T(x, y, z) = 4 - 2x^2 - y^2 - z^2 \) in hundreds of degrees. What are the hottest and coldest points on the plate?
8. A company has three production plants, each manufacturing the same product. If plant A produces $x$ units at the cost of $x^2 + 2,000$, plant B produces $y$ units at the cost of $2y^2 + 3,000$, and plant C produces $z$ units at the cost of $z^2 + 4,000$. If there is an order for 11,000 units to be filled, determine how the production should be arranged among these three plants so that the total production cost can be minimized.
Optimization and Lagrange Multipliers Solutions

Critical Points

1. (a) \((0, 0, 1)\) is a max
   (b) none, why?
   (c) \((0, 1, 4)\) is a min
   (d) \((0, 0, 0)\) is a min

2. (a) \((\pm 1, 0, 2)\) are local minima and \((0, \pm 1, -1)\) are local maxima
   (b) all points to consider should be: \((2, -1/3, 0, 0, 3, 0), (3, -3), (0, -3), (2, 0), (0, -1/3), (3, -1/3), \) and \((2, -3): (2, -1/3, -22/3)\) is a min; \((0, -3, 18)\) is a max

3. \((-8/3, 5/2)\) and \(\sqrt{481}/6\)

4. \((3, -320)\)
   (a) \((3, \pm 6, -54)\) are saddles and \((0, 0, 0)\) are all maxima
   (b) \((0, 1, 9/2)\) (D is long – use Maple!)
   (c) icky algebra – use Maple!

Optimization and Lagrange Multipliers

1. \((1/2, 1/2, 1/2)\) is the max, each of \((1, 0, 0)\) and \((0, 1, 0)\) is a min

2. \((0, 0, 1)\) is the max, each of \((1, 0, 0)\) and \((0, 1, 0)\) is a min

3. there is one saddle on the interior, but no extrema on the interior; the extrema occur on the boundaries at \((0, 0), (\pi/2, 5), (2, 0), \) and \((\pi/2, 0)\)

4. \((12, 12, 288)\)

5. \((0, \pm 2)\) and \((\pm \sqrt{15}/2, 1/2)\)

6. \(x = 41, y = 41, z = 41\)
7.5 Integration Drill

Basics

1. Evaluate \( \int x^3y - 3x \, dx \) and \( \int^3 y^3 - 3x \, dy \)

2. Evaluate \( \int xye^{xy} \, dx \) and \( \int^1_{-1} xye^{xy} \, dy \)

3. Compute the area of the region bounded by the parabola \( y = x^2 - 2 \) and the line \( y = x \) by first integrating with respect to \( x \) and then integrating with respect to \( y \).

4. Compute the area bounded by \( y = x^3 \) and \( y = 5x \) in four ways. (a) Single integral with respect to \( x \), (b) single integral w.r.t. \( y \), (c) double integral, \( dx \, dy \), and (d) double integral, \( dy \, dx \).

Regions

1. Sketch the region \( \phi = \pi/6 \) in spherical coordinates.

2. Sketch the region bounded by \( r = 1 \) and \( r = 2\sin(\theta) \) in polar coordinates.

3. Sketch the region bounded by \( x^2 + y^2 \leq 9 \) and write in polar coordinates.

4. Sketch the region between \( x^2 + y^2 = 25 \), \( x^2 + y^2 = 4 \), and \( x \geq 0 \) and write in polar coordinates.

5. Find the area of the region \( D \) bounded by \( y = \cos(x) \) and \( y = \sin(x) \) on the interval \([0, \pi/4]\).

Integration

1. Compute \( \int^2_0 \int^3_0 x^2 y^5 - 3xy^5 \, dy \, dx \) and \( \int^3_0 \int^2_0 x^2 y^5 - 3xy^5 \, dx \, dy \). Are they equal? What theorem is this an example of?

2. Evaluate \( \int^1_0 \int^y_0 e^{x^2} \, dx \, dy \).

3. Compute \( \int_R x^2 e^{xy} \, dA \) where \( R = \{(x,y) : 0 \leq x \leq 3, 0 \leq y \leq 2\} \).

4. Compute \( \int^3_0 \int^2_0 \sqrt{2x+y} \, dy \, dx \).

5. Compute \( \int_R \frac{\ln(\sqrt{y})}{xy} \, dA \) where \( R = \{(x,y) : 1 \leq x \leq 4, 1 \leq y \leq e\} \).

6. Evaluate \( \int_D 160xy^3 \, dA \) where \( D \) is the region bounded by \( y = x^2 \) and \( y = \sqrt{x} \).

7. Sketch and find the volume of the solid bounded above by the plane \( z = y \) and below in the \( xy \)-plane by the part of the disk \( x^2 + y^2 \leq 1 \).

8. Sketch the region, \( D \), that is bounded by \( x = y^2 \) and \( x = 3 - 2y^2 \) and evaluate \( \int_D (y^2 - x) \, dA \)

9. Compute \( \int^2_0 \int^\sin(x)_0 y\cos(x) \, dy \, dx \).
10. Determine the endpoints of integration for $\int \int_S e^{xy} \, dA$ where $S$ is the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{9}$. Don’t integrate.

11. Determine the endpoints of integration for $\int \int_S dA$ where $S$ is bounded by a the $x = y^2 + 4y$ and $x = 3y + 2$.

12. Determine the endpoints of integration for $\int \int_S 2x \, dA$ where $S$ is the region bounded by $yx^2 = 1, y = x, x = 2,$ and $y = 0$.

13. Evaluate $\int \int_D dA$ where $D$ is the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

**Polar Coordinate Integration**

1. Sketch the region $D$ between $r = \cos \left( \frac{\theta}{2} \right)$ and $x^2 + y^2 = 1$ with $0 \leq \theta \leq \pi$. Evaluate $\int_D 1 \, dA$.

2. Find the volume of the solid bounded by the cone $\phi = \frac{\pi}{6}$ and the sphere $\rho = 4$.

3. Consider $\int \int_R x^2 + y^2 + 1 \, dA$ where $R = \{(x,y) : x \geq 0, 9 \leq x^2 + y^2 \leq 16\}$. Write the integral in both rectangular and polar coordinates. Compute each to verify your answer.

**Coordinate Transformations**

1. Find the Jacobian for the transformation: $x = u^2 + v^2 + w, y = uv - v,$ and $z = \ln(w) - \frac{v}{u}$

2. Find the Jacobian for the transformation: $x = r \cos \theta, y = r \sin \theta,$ and $z = z$.

3. Find the Jacobian for the transformation: $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta,$ and $z = \rho \cos \phi$.

4. Evaluate $\int_0^4 \int_{\sqrt{y}/2}^{(y/2)+1} \frac{2x - y}{2} \, dx \, dy$ using $u = \frac{2x - y}{2}$ and $v = \frac{y}{2}$.

5. Use the transformation $x = \frac{u}{v}$ and $y = v$ to rewrite (but not evaluate) the double integral $\iint \sqrt{xy^3} \, dx \, dy$ over the region in the plane bounded by the $x$-axis, the $y$-axis, and the lines $y = -2x + 2$ and $x + y = 7$.

6. Compute $\int_0^1 \int_0^{y^2} (1 - y) \sin\left(\frac{x}{y}\right) \, dx \, dy$ using $u = \frac{x}{y}$ and $v = 1 - y$.

7. Write an integral in rectangular coordinates that represents the area enclosed by the ellipse $\frac{x^2}{16} + \frac{y^2}{49} = 1$. Now, compute this integral by using the transformation $x = 4u$ and $y = 7v$.

8. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation $x = au, y = bv,$ and $z = cw$.

9. Evaluate the triple integral $\int_0^6 \int_0^8 \int_\frac{x^2+y^4}{2} 2x - y \, dx \, dy \, dz$ using the transformation $u = \frac{2x - y}{2}, v = \frac{y}{2},$ and $w = \frac{z}{3}$.
Triple Integrals, Cylindrical, and Spherical Coordinates

1. Sketch and find the volume of the solid formed by \( f(x,y) = 4x + 2y \) above the region in the \( xy\)-plane bounded by \( x = 2, x = 4, y = -x, y = x^2 \).

2. Fill in the blanks:
   \[
   \int_0^1 \int_0^{1-x^2} \int_0^{1-x-2y} f(x,y,z) \, dz \, dy \, dx = \int_0^1 \int_0^{1-x^2} f(x,y,z) \, dy \, dz \, dx \\
   = \int_0^1 \int_0^{1-x^2} f(x,y,z) \, dx \, dy \, dz
   \]

3. \[
   \int_0^6 \int_0^{12-2x} \int_0^{12-2x-3y} f(x,y,z) \, dz \, dy \, dx = \int_0^6 \int_0^{12-2x} f(x,y,z) \, dy \, dz \, dx
   \]

4. \[
   \int_0^1 \int_0^{\sqrt{1-36x^2}} \int_0^{\sqrt{1-36x^2-z^2}} f(x,y,z) \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-36x^2}} f(x,y,z) \, dx \, dz \, dy
   \]

5. Find the volume of the solid bounded by \( x^2 + y^2 + z = 8 \) and \( z = 4 \).

6. Evaluate \[
   \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z \sqrt{4-x^2-y^2} \, dz \, dy \, dx
   \]
   by converting to (a) cylindrical coordinates; (b) spherical coordinates.

Application - Center of Mass

1. Suppose \( \delta(x,y) = x + y \) is the density function of a thin sheet of material bounded by the curve \( x^2 = 4y \) and \( x + y = 8 \). Find its total mass. Find its first moments. Find its center of mass. Find its second moments.
Integration Solutions

Basics

1. \( \frac{1}{4}x^4y - \frac{3}{2}x^2 \) and \( 4x^3 - 6x \)
2. \( e^{xy}(x - \frac{1}{2}) \) and \( e^x(1 - \frac{1}{x}) + e^{-x}(1 + \frac{1}{x}) \)
3. 9/2
4. 25/4

Regions

1. it’s a cone
2. it’s the intersection of two circles
3. \( r \leq 3 \)
4. \( 2 \leq r \leq 5 \) and \( \theta \in [-\pi/2, \pi/2] \)
5. \( \sqrt{2} - 1 \)

Integration

1. \( -705 \), Fubini’s Theorem
2. \( \frac{1}{2}(e - 1) \)
3. \( -\frac{1}{4}(17 - 5e^6) \)
4. \( \approx 11.62 \)
5. \( \approx .346 \)
6. 6
7. \( \frac{2}{3} \)
8. \( \frac{24}{5} \)
9. \( \frac{1}{5}\sin^3(2) \)
10. \( \int_0^{81} \int_{\frac{1}{4}}^{\sqrt{x}} e^{xy} dA \)
11. 4.5
12. \( \frac{2}{3} + 2\ln(2) \)
13. \( 6\pi \)

Polar Coordinate Integration

1. \( \frac{\pi}{4} \)
2. \(-\frac{64(\sqrt{3} - 2)}{3}\pi\)

3. \(\frac{189\pi}{4}\)

Coordinate Transformations

1. \(-2u(\frac{1}{w} - 1) - 2v + \frac{1}{u}(2v^2 - w) + v/u^2\)

2. \(r\)

3. \(\rho^2\sin(\phi)\)

4. 2

5. \(\int_0^7 \int_{7v-v^2}^{v-v^2/2} \sqrt{u}udv\)

6. 

7. area of ellipse is \(\pi ab\) in this case \(28\pi\) (just like circle)

8. volume of ellipsoid is \(4/3\pi abc\) (just like sphere!)

Triple Integrals, Cylindrical, and Spherical Coordinates

1. \(2472/5\)

2. \(\int_0^1 \int_0^{1-x} \int_0^{\frac{1-x-z}{z}} f(x,y,z) dy dz dx\)

3. 

4. 

5. \(8\pi\)
7.6 Line Integrals, Flux, Divergence, and Gauss’ Theorem Drill

Note: In this section, we write the functions and integrals using many different notations. If you are unsure about the meaning of a notation, please ask!

Vector Fields, Curl, and Divergence

1. Sketch the vector field, \( \vec{f}(x, y) = x^2 \vec{i} + j \).
2. Sketch the vector field, \( \vec{g}(x, y) = (x, -y) \).
3. Sketch the vector field, \( \vec{h}(x, y, z) = y \vec{j} \).
4. Compute the divergence and curl of the vector field, \( \vec{f}(x, y, z) = (y^2 z, x^3 + z + y, \cos(xyz)) \).
5. Find \( g \) satisfying \( \nabla g = F \) if it exists. \( F(x, y) = (ye^{xy} + 2x) \vec{i} + (xe^{xy} - 2y) \vec{j} \).
6. Find \( g \) satisfying \( \nabla g = F \) if it exists. \( F(x, y) = (e^x \sin(y), e^x \cos(y)) \).
7. Is \( \vec{f}(x, y) = (2xy, x^2) \) a conservative vector field? If so, find a potential for it.
8. Is \( \vec{h}(x, y) = (\cos(x), \sin(x)) \) a conservative vector field? If so, find a potential for it.
9. Is \( \vec{g}(x, y, z) = (e^x \sin z + yz) \vec{i} + (xz + y) \vec{j} + (e^x \cos z + xy + z^2) \vec{k} \) a conservative vector field? If so, find a potential for it.
10. Is \( \vec{g}(x, y) = 2x \vec{i} + y \vec{j} \) a conservative vector field? If so, find a potential for it.
11. Show that \( \vec{F}(x, y, z) = x \vec{i} + y \vec{j} + 2z \vec{k} \) is conservative and find a function \( f \) such that \( \vec{F} = \nabla f \).

Line Integrals over Scalar Fields

1. Let \( f(x, y) = x + y \) and \( \vec{c} \) be the unit circle in \( \mathbb{R}^2 \). Evaluate \( \int_{\vec{c}} f \, ds \). Recall that \( ds \) means to evaluate with respect to the arc length.
2. Evaluate \( \int_{\vec{c}} \sqrt{xy + 2y + 2} \, ds \) with \( \vec{c} \) the line segment from \((0, 1)\) to \((0, -1)\).
3. Evaluate \( \int_{\vec{c}} (x - y + z - 2) \, ds \) where \( \vec{c} \) is the line segment from \((0, 1, 1)\) to \((1, 0, 1)\).

Line Integrals over Vector Fields

1. Compute \( \int_{\vec{c}} \vec{f} \cdot dr \) where \( \vec{f}(x, y) = (y, x^2) \) and \( \vec{c}(t) = (4 - t, 4t - t^2) \) for \( 0 \leq t \leq 3 \).
2. Compute \( \int_{\vec{c}} \vec{f} \cdot dr \) where \( \vec{f}(x, y) = (y, x^2) \) and \( \vec{c}(t) = (t, 4t - t^2) \) for \( 1 \leq t \leq 4 \).
3. Compute \( \int_{\vec{c}} \vec{F} \cdot dr \) where \( \vec{F}(x, y) = (-\frac{1}{2}x, -\frac{1}{2}y, \frac{1}{4}) \) and \( \vec{c}(t) = (\cos(t), \sin(t), t), 1 \leq t \leq 4 \).

Divergence Theorem and Green’s Theorem

1. Verify the divergence theorem for the flow \( f(x, y) = (0, y) \) over the circle, \( x^2 + y^2 = 5 \).
2. Let \( f(x, y) = (u(x, y), v(x, y)) = (-x^2, xy) \) and \( \vec{c} = \{(x, y) : x^2 + y^2 = 9 \} \) and \( D \) be the region bounded by \( \vec{c} \). Verify Green’s Theorem by evaluating both \( \int_{\vec{c}} f(\vec{x})d\vec{x} \) and \( \int_D v_x(x, y) - u_y(x, y)dA \).

W. Ted Mahavier
www.jiblm.org
3. Verify Green’s Theorem where \( f(x,y) = (4xy, y^2) \) and \( \mathbf{c} \) is the curve \( y = x^3 \) from the \((0,0)\) to \((2,8)\) and the line segment from \((2,8)\) to \((0,0)\).

4. If you want more practice on verifying Green’s and Gauss’ theorems, then note that each problem that asks you to verify Gauss’ theorem could have asked you to verify Green’s theorem and vice-versa. You won’t need solutions because you are computing both sides of the equation and they must be equal if all your integration is correct.

**Divergence Theorem in Three Dimensions**

1. Verify the divergence theorem for \( f(x,y,z) = (xy, z, x+y) \) over the region in the first octant bounded by \( y = 4, z = 4 - x, z = 0, y = 0, \) and \( x = 0. \)

2. Verify the divergence theorem for \( f(x,y,z) = (2x, -2y, z^2) \) over the region \([0,3] \times [0,3] \times [0,3]. \)
Line Integrals, Flux, Divergence, and Gauss’ Theorem Solutions

Vector Fields, Curl, and Divergence

1. no sketch
2. no sketch
3. no sketch
4. no solution
5. \( g(x, y) = e^{xy} + x^2 - y^2 \)
6. \( g(x, y) = e^x \sin(y) \)
7. yes, \( g(x, y) = x^2y \)
8. yes, \( g(x, y) = y \sin(x) \)
9. yes, \( g(x, y, z) = e^x \sin(z) + xyz + \frac{1}{2}y^2 + \frac{1}{2}z^3 \)
10. no solution
11. no solution

Line Integrals over Scalar Fields

1. 0
2. 8/3
3. \(-\sqrt{2}\)

Line Integrals over Vector Fields

1. 69/2
2. \(-69/2\)
3. 3\(\pi\)/4

Divergence Theorem and Green’s Theorem

1. If both sides are equal, you probably got it right!
2. 81\(\pi\)/2
3. \(-256/15\)
4. no solution

Divergence Theorem in Three Dimensions

1. no solution
2. no solution