Calculus III

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To the Instructor

The Course

These materials constitute a one-semester course in multivariate calculus intended as the third semester of a three-semester course. The notes begin with the definition of Euclidean three space and conclude with the Divergence Theorem in three dimensions. In addition to problems I penned, material in this course comes from a variety of sources: the multitude of calculus books I have taught from over the years, Dale Daniel of Lamar University, Shing So of the University of Central Missouri, and courses I took, including one taught by my father, William S. Mahavier. While my memory is not perfect, I believe that many of the practice problems came from Dr. Daniel and Dr. So. The notes have been used at Lamar University six times over a period of nine years with an average class size of approximately thirty-five students who meet for four fifty-minute periods each week. The large class size and full syllabus prevent me from using a Moore Method approach where students present problems one at a time and work through a self-contained set of notes. To maximize the odds for success and to significantly increase the rate of coverage, I make three concessions that cause me to label this course a problem-based course or active-learning course rather than a Moore Method course.

1. I allow students to use other resources (web, books, tutoring lab) to help them understand the concepts.
2. I provide practice problems with answers that the students are expected to work after we have covered a given topic.
3. I give lectures, on average once a week. In some iterations, I have lectured every Monday to assure steady progress.

A few semesters, I have not allowed students to look at other sources. While the top performing students do no worse, many students struggle. I believe this could easily be overcome if the class size were smaller, the syllabus not quite so full, or if we had an extra meeting each week. Most semesters, I have offered one extra problem session per week addressing the practice problems. Approximately ten to twenty percent of the class attended and benefitted (based on evaluations and grades).

In the instructor’s version of these notes, I have recorded discussions about the mini-lectures that I gave during the most recent iteration. A recurring theme of these lectures is my attempt to unify calculus by constantly tying Calculus III back to Calculus I and foreshadowing that many of these same themes are valid in more general spaces (think Hilbert spaces). While such discussions vary with each class based on student questions, these are representative of what I cover to introduce topics. There is no way to include all the discussions that result from student questions even though responses to student questions about other students’ presentations constitute the majority of my class interaction. What I’ve offered here are only the major presentations that I make during the semester. I’ve denoted by
(I) – the ones that were used to introduce concepts, and

(R) – the ones that were in response to student work or student questions.

While these mini lectures and examples appear in the notes at the approximate places where I presented them this semester, that may not be the optimal timing for presentation of this material in a different semester. Such lectures are presented in response to questions from students, at a time when students are stalled, or just-in-time to prepare students for success on upcoming problems. The number and length of these lectures depend on the strength and progress of the class. If no student has a problem to present or very few have a problem, then I will present. If we are about to embark on another section that I feel needs an introduction, then I will present. I have taught such strong classes that I never presented except in response to students’ questions. I have also taught classes where I carefully timed when to introduce new material to expedite progress and assure that we completed the requisite material.

The reader will note that the lectures are shorter and less frequent at the beginning of the course than at the end. There are two reasons for this. First, it is important early on that the students buy in to the concept of taking responsibility for presenting the material. If I present very much, they will seek to maximize my lectures in order to minimize their work and optimize their grades. Second, the material gets tougher near the end and I doubt anyone’s ability to create notes that enable students to “discover” the concepts of Lagrange Multipliers, Green’s Theorem, and Gauss’ Divergence Theorem in the time allotted. I’ve used the bullet symbol, •, to indicate the end of each such discussion when it was not obvious.

Omitted from these notes are two intertwined themes. First, I regularly discuss my experiences from industry and how much more important it was to be able to solve one hard problem on my own than it was to be able to solve ten problems just like the one my boss showed me. In fact, the latter never happened. Independence of thought and creative solutions are highly valued in industry. And I repeat my single most important goal for them – I want them to solve some problems on their own and overcome the belief that to learn they must have someone show them first. Such a transition can create a major change in life, allowing one to tackle almost anything independently without fear.

The vast majority of class time goes to students putting up problems. After a few minutes of introducing concepts on the first day, I pose problems at the board and students present solutions. From that day forward, I begin each class by calling out problem numbers and choosing from the class the students with the least number of presentations. I break ties by test grades or by my estimate of the tied students’ abilities, allowing apparently weaker students preference. For the first few class periods, I give students who have not taken a course from me preference over those who have. I allow multiple problems to go on the board at once, between four and twelve on any given day, fully utilizing white boards at the front and rear of the classroom. As problems are put on the board, I circulate and answer questions at the desks. Once all students have completed writing their solutions, I encourage them to explain their solutions to the class. If they are uncomfortable doing so, I will go over the problem asking them questions about their work. Most problems in the notes are there to illustrate or foreshadow an important point, so after almost every problem I spend a few minutes discussing the purpose of this particular problem and often foreshadow important concepts with additional examples, pictures, or ideas. If time is short after the problems are written on the board (perhaps I lectured a bit too long or perhaps I underestimated how long it would take to put these problems up), I may review and discuss these problems myself. This is a very difficult judgement to make because there are times when I struggle to make sense of the student’s work and s/he will sit me down and explain the solutions more eloquently than I! On the other hand, there are also times when a student’s explanation is so lacking, or approach so obscure, that I need
to work a clarifying example in order to bring the class along and ensure future success. When correcting, expanding on, or showing alternate methods I am overly complementary of some aspect of the student’s presentation as my goal is never to demean, belittle, or show a “better” approach. After each class, I post on my website each problem number that was presented and the student who presented it so that all students know what has been presented and what has not.

Another difficult judgement call is how much to foreshadow upcoming material via mini lectures and when to give them. If no student has a presentation, then clearly it is time for me to present. Sometimes only a very few top students have something to present and they already have a presentation grade of A. When this happens, I may choose to lecture to bring the rest of the class along. This is a delicate decision. On the one hand, other students with less presentations will have these problems the next day, so these talented students have been “cheated” out of a presentation. On the other hand, I record in my grade book that they had these problems but were not allowed to present them. Since I am typically already recruiting such students as majors, they know I have confidence in them. In the worst case scenario, these bright students have to do more work to assure they maintain their lead in presentations which is hardly a bad thing. In truth, they have not lost any “points.” On occasion, I tell the class that I’m not sure if they need a lecture or if student presentations are the best choice and I let them vote. Surprisingly, the vote is almost always near unanimous for one of the two choices: mini lecture or problems.

A slightly challenging symptom of allowing students to look at other resources is that sometimes students will use tools that we have not yet developed. For example they might look up and apply the formula for the cross product or the formula for the distance between two planes. When this happens, I first thank the student for showing us a quick and correct solution using a formula and note that this receives full credit. Then I use this as an opportunity to reinforce the axiomatic nature of mathematics by giving an on-the-spot mini lecture where I accomplish three things. I rework the problem from first principles and I state that I’m glad we now have these formulas, but we must derive them before we can trust them. I tell them that, jokingly, were we simply engineers, we would accept anything that was fed to us, but as mathematicians we must validate such formulae before we use them. I do this with tongue-in-cheek, but they understand that I am looking at this with the perspective of a mathematician who wants to deeply understand all that I use. I’ll also tell which problems are aimed at deriving them and give guidance on how we might do so. In an optimal situation, the student who looked up the formula will dig in and derive the formula. If not, another student will. I’ll always praise both students because now we have the formula and we know that it is valid.

Problem Sets

Chapter 9 consists of nothing except practice problems with solutions. Over the years students have always asked for more practice, so I have provided these in response to these requests. Any student may ask at the beginning of class about any of these problems and I’ll work them out. My class meets four days a week with an optional problem session on the fifth day. During these sessions I’ll answer questions, revisit anything that has already gone on the board or help them with the practice problems. I’m careful not to do examples or discuss problems in the problem sequence that have not been presented since I don’t want students who are unable to come to these sessions to be at a disadvantage.
Grading

Students receive fifty percent of their grade from their presentations and fifty percent of their grade from the average of their test grades. I give a comprehensive final examination and two or three term tests. My first approximation for a student’s presentation grade is $70 + (2 \text{ points per fully correct presentation})$. Thus a student with fifteen perfect presentations (one per week) would earn approximately 100 points. In other classes, I’ve used $60 + (3 \text{ points per fully correct presentation})$. After my rough computation, I adjust the presentation grade based on the quality, number, and difficulty of the presentations. Of course the number of presentations per student is highly dependent on class size, so the number of points per presentation may need to be adjusted from one semester to the next. I tell students at the beginning of the semester to try for one presentation per week. When I record student presentations after each class, I record in my grade book a grade of 1-4: $1 = \text{wrong}$, $2 = \text{half right}$, $3 = \text{mostly right}$, $4 = \text{completely right}$ and $4^* = \text{something exceptional was done here}$. This allows me to see how many of the problems presented were, in my opinion, perfect. It also allows me to recognize which students are doing proofs or unique approaches to problems. I also tell them that this grade must be subjective, since the student who presents five beautifully illustrated proofs deserves at least as good a grade as the student who presents ten elementary problems. This semester I gave elementary weekly quizzes to keep the class on track and counted these as a third of the overall grade.

Goals

When teaching calculus, no two teachers will share exactly the same goals. I hope to train the students to read mathematics carefully and critically. I want to teach them to work on problems on their own until they solve them rather than seeking out other materials and I emphasize this, even as I allow them to go to other sources because of the size of the classes. I want them to see that there are really only a few basic concepts in calculus, the limiting processes that recur in continuity, differentiability, integration, arc-length, and Gauss’ theorem. I want them to see that calculus in multiple dimensions really is a parallel to calculus in one dimension. For these reasons, I try to write tersely and without giving long-winded “intuitive” introductions and I write without graphics. I believe it is better to define “annulus” carefully and make the students interpret the English than to draw a picture and say “this is an annulus.” The latter is more efficient but does not train critical reading skills. I want to see them communicate clearly with one another in front of the class in both their questions and in their responses. I want them to write correct mathematics on the board. The amount of pedagogical trickery that I use to support these goals is too long to list here, but it all rests firmly on always focusing on my confidence that they will succeed and responding to everything, even negative comments, in a positive light.

If a student raises a concern about the teaching method, I explain how much more work it is to me to teach this way. I discuss how much easier it would be to lecture, giving monkey-see, monkey-do homework and that I sacrifice my own time and energy to better them by developing my own notes, not using a book, and giving my office hours freely. In the beginning of the semester when students make mistakes at the board, I correct them, adding something like, “These corrections don’t count against a presentation grade because this was a good effort, yet we want to make sure that we see what types of mistakes I might count off for on an exam.” Later in the semester when students ask about how I will issue presentation grades, I encourage them to drop by my office any time to discuss their grades. I also honestly admit that the grades are subjective by giving them realistic scenarios. “Suppose I gave two points per problem and you presented ten presentations, every one of which was a proof of a difficult theorem or a derivation of a difficult formula that

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we used for the rest of the semester. Another student presented fifteen problems, each the easiest from each section. Would you want the second student to receive an A while you received a B? So, we agree that while all of us, myself included, would like to give a point value to every presentation, it’s simply impossible due to the number of factors. I’ll read a few student comments from evaluations to convince them that I really am on their side and previous students’ quotes help more than anything I could say.

**Student Comments**

The course and notes that I present here are a culmination of changes based on nine years of student comments. Over such a long run, I have a few favorites that result from the evaluations I pass out each semester. Certain responses recur. In response to “I believe the most effective part of this course was...”

1. “Noticing that sometimes professors seem superfluous.”
2. “The board work is, in my opinion, the most effective. When I work a problem, I have an idea of how to attempt it. Then I see a student put a problem on the board, and it helps cement the concept.”
3. “Working the problem on the board, and the practice problems in the notes.”
4. “Participating in board work; it was really helpful. That lets me know my fellow classmates do not seclude me.”
5. “Board work and independence.”
6. “If you want to truly learn the material, take his class. It requires doing work on your own, but in my opinion, it is the best way to learn.”
7. “Learning to communicate problems to colleagues, and think independently.”
8. “The boardwork, students explain things better than teachers.”
9. “The most effective part of the course was also the part I hated the most. Having to prepare problems and present them, I hated it but it was helpful.”

In response to, ”I would like my professor to know that...”:

1. “This is the first math course I have ever taken where I was able to learn the subject by reading only the supplied materials and working through them on my own.”
2. “He did a great job and made me comprehend the math as opposed to just knowing formulas to solve problems.”

Full evaluations are available to any potential instructor upon request.

**Caveats**

Calculus I, II, and III are, in my opinion, the hardest mathematics courses to teach. I don’t recommend them as a starting point for a novice interested in IBL teaching. In *The Moore Method: A*
Pathway to Learner-Centered Instruction in the chapter on grading, I advocate a modification of the Moore Method that works well.

I have used many other authors’ notes for various courses and never once felt that they suited my needs in exactly the form I received them. However, they often provided a good base from which to modify and create my own notes. I encourage you not to take these as finished, but rather to modify them to suit your own needs.

Finally, certain concessions are made when teaching via IBL. The time for discovery means that students obtain a deeper understanding of the material they see. And it means that not every application can be addressed. There are obvious omissions, such as the minimal treatment of velocity and acceleration for parametric curves. I make no apology for sacrificing a multitude of applications for the sake of developing my students’ independence of thought and ability to learn without excessive examples. The feedback from my students documenting their successes as they move on to graduate school, enter industry, or enter the public school system constantly validates the following quote by Edwin Moise.

I believe Moore's work proves something of broad significance... that sheer knowledge does not play the crucial role in mathematical development that most people suppose. The amount of knowledge that a small class can acquire, struggling at every stage to produce its own proofs, is quite small. The resulting ignorance ought to be a hopeless handicap but it isn't. The only way I can see to resolve this paradox is to conclude that mathematics is capable of being learned as an activity and that knowledge which is acquired in this way has a power which is out of all proportion to its quantity. “Activity and motivation in mathematics,” American Mathematical Monthly, Vol. 72, No. 4, April 1965, pp. 407-412.

While this class is not taught as a “proofs” class, the students do struggle mightily with what the instructor could demonstrate in short order. Yet the struggle itself is what produces the strong student who, when inevitably lacking in knowledge on some topic, will be able to fill in any gaps. Who among us graduated knowing all that we needed to know?
To the Student

What is calculus?

The first semester of Calculus consisted of four main concepts: limits, continuity, differentiation and integration. Limits are required for defining each of continuity, differentiation, and integration. All four concepts are central to an understanding of applications in fields including biology, business, chemistry, economics, engineering, finance, and physics. The second semester extends your study of integration techniques and adds sequences and series. The third semester of calculus is a repeat of these concepts in higher dimensions.

In addition to mastering these concepts, I hope to impart in you the essence of the way a mathematician thinks of the world, an axiomatic way of viewing the world. And I hope to help you master the important skill of solving some difficult problems on your own, communicating these solutions to your peers, and responding to any questions your peers have.

How this class works

This class will be taught in a way that is (most likely) different from mathematics classes you have encountered in the past. Much of the class will be devoted to students working problems at the board and much of your grade will be determined by the amount of mathematics that you produce in this class. I use the word produce because it is my belief that the best way to learn mathematics is by doing mathematics.

Therefore, just as I learned to ride a bike by getting on and falling off, I expect that you will learn mathematics by attempting it and (occasionally) falling off! You will have a set of notes (provided by me) that you will turn into a book by working through the problems. If you are interested in watching someone else put mathematics at the board, working ten problems like it for homework, and then regurgitating this material on tests, then you are not in the correct class. Still, I urge you to seriously consider the value of becoming an independent thinker who tackles doing mathematics (and everything else in life) on your own, rather than waiting for someone else to show you how to do things.

A common pitfall

There are two ways in which students often approach my classes. The first is to say, “I’ll wait and see how this works and then see if I like it and put some problems up later in the semester after I catch on.” Think of it as a forty yard dash. Do you really want to wait and see how fast the other runners are? If you try every night to do the problems then either you will get a problem (YAY!) and be able to put it on the board with pride and satisfaction or you will struggle with the problem, learn a lot in your struggle, and then watch someone else put it on the board. When this person
puts it up you will be able to ask questions and help yourself and others understand it, as you say to yourself, “Ahhhh, now I see where I went wrong and now I can do this one and a few more for the next class.” If you do not try problems each night, then you will watch the student put the problem on the board, but perhaps will not quite catch all the details and then when you study for the tests or try the next problems you will have only a loose idea of how to tackle such problems. Basically, you have seen it only once in this case. The first student saw it once when s/he tackled it on his or her own, again when either s/he put it on the board or another student presented it, and then a third time when s/he studied for the next test or quiz. Hence the difference between these two approaches is the difference between participating and watching a movie. I hope that each of you will tackle this course with an attitude that you will learn this material and thus will both enjoy and benefit from the class.

**Board work**

Because the board work constitutes a reasonable amount of your grade, let’s put your mind at ease regarding this part of the class. First, by coming to class today you have a sixty percent grade on board work. Every problem you present pushes that grade a little higher. You may come see me any time for an indication of what I think your current level of participation will earn you at the end of the semester for this portion of the grade.

Here are some rules and guidelines associated with the board work. I will call for volunteers every day and will pick the person with the least presentations to present a given problem. You may inform me that you have a problem in advance (which I appreciate), but the problem still goes to the person with the least presentations on the day I call for a solution. Ties are broken either randomly (at the beginning) or by test grades (lower test grades taking priority). A student who has not gone to the board on a given day will be given precedence over a student who has gone to the board that day. To “present” a problem at the board means to have written the problem statement up, to have written a correct solution using complete mathematical sentences, and to have answered all students’ questions regarding the problem.

Since you will be communicating with other students on a regular basis, here are several guidelines that will help you. First, the whole class is on your side and wants to see you understand and present the problem correctly both for your sake and for their understanding. When you speak, don’t use the words “obvious,” “stupid,” or “trivial.” Don’t attack anyone personally or try to intimidate anyone. Don’t get mad or upset at anyone (and if you do, try to get over it quickly). Don’t be upset when you make a mistake – brush it off and learn from it. Don’t let anything go on the board that you don’t fully understand. Don’t say to yourself, “I’ll figure this out at home.” Don’t use concepts we have not defined. Don’t use or get examples or solutions from other sources without acknowledging it during the presentation. Don’t work together without acknowledging it at the board. Don’t try to put up a problem you have not written up.

Do prepare arguments in advance. Do be polite and respectful. Do learn from your mistakes. Do ask questions such as, “Can you tell me how you got the third line?” Do let people answer when they are asked a question. Do refer to earlier results and definitions by number when possible.

**How to study**

1. Read over your notes from class that day.

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2. Make a list of questions to ask at the beginning of the next class.
3. Review the old and read the new definitions.
4. Work on several new problems.
5. Write up as many solutions as you can so that you can simply copy your well-written solutions onto the board the next day.
Chapter 1

Vectors and Lines

Welcome to calculus in three dimensions. The beauty of this material is how closely it parallels your first semester of calculus. After a brief introduction to the coordinate plane, you learned how to graph lines and parabolas. After a brief introduction to three-space, we will be graphing planes and paraboloids. Just as we defined continuity in terms of limits in Calculus I, we will define continuity of functions of several variables in terms of limits in this course. Lines were important as they allowed you to define tangent lines to functions and the derivative. Tangent planes to functions and surfaces will aid our definition of derivative. Once you understood the derivative, you used it to find maxima and minima of real valued functions and we will use the derivative of functions in three-space to find maxima and minima in our applications as well. Two of the most central ideas of your first calculus course were the chain rule and the fundamental theorem of calculus. We will extend your notion of the chain rule and the fundamental theorem in this course. Only near the end of the course will the work we do not have a parallel to your first course. At the end, we’ll cover Green’s and Gauss’ Theorems, which are necessary tools in physics and engineering, but which had no parallel in Calculus I.

Instructor’s Note (I) The first day may well be the most important of the semester. For that reason, we don’t start with my syllabus, grading, or other mundane details. Rather, we start by doing mathematics with the main goal to place the class at ease and send a few students to the board in a relaxed environment. We define vector addition, scalar multiplication, points in \( \mathbb{R}^3 \) and vectors in \( \mathbb{R}^3 \), just as they are defined below. We then give a parametric equation for a circle and discuss how we might write \( y = x^2 \) (or any other function) as a parametric curve in one or more ways. Then we give two points in the plane, ask the students to create a parametric equation for the line between two points and have one or more students place their (hopefully different) solutions on the board. We discuss whether they are correct and how different equations may produce the same set of points. Once this is done, I have them convert the parameterized lines to slope-intercept form. This takes the entire first period and gets several students to the board in a relaxed setting, which is my primary goal for the first day. I send them home to read the notes and start working the problems for presentation on the next day, promising that I’ll post a syllabus soon. Because we use a web site for all communication with the class, that is where I’ll post the course notes, syllabus, and other details. You can probably see examples of my classes right now at www.mathnerds.org/ted.

Definition 1. \( \mathcal{N} \) is the set of all Natural Numbers.

Definition 2. \( \mathbb{R} \) is the set of all Real Numbers.

We will also use the notation, \( \in \), to mean “is an element of.” Thus, \( x \in \mathbb{R} \) means “\( x \) is an element of \( \mathbb{R} \),” or “\( x \) is a real number.” Similarly, \( x, y \in \mathbb{R} \) means “\( x \) and \( y \) are real numbers.” In
Calculus I and II, you lived in 2-space, or $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$. Now you have graduated to 3-space!

**Definition 3.** Three dimensional space (Euclidean 3-space or $\mathbb{R}^3$), is the set of all ordered sequences of 3 real numbers. That is

$$\mathbb{R}^3 = \{(x_1,x_2,x_3) : x_1,x_2,x_3 \in \mathbb{R}\}.$$ 

Of course, there is no reason to stop with the number 3. More generally, $n$-space or $\mathbb{R}^n$ is the set of all ordered sequences of $n$ real numbers, but we will spend most of our time concerned with only $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and occasionally, $\mathbb{R}^4$. Euclidean 4-space is handy since one might want to consider an object or shape in 3-space that is moving with respect to time, thus adding a 4th dimension.

We will write elements in $\mathbb{R}^3$ just as letters; hence, by $x \in \mathbb{R}^3$ we mean the element, $x = (x_1,x_2,x_3)$ where $x_1,x_2,x_3 \in \mathbb{R}$. The **origin** is the element, $o = (0,0,0)$.

**Definition 4.** If $x,y \in \mathbb{R}^3$ then $\overrightarrow{xy}$ is the directed line segment from $x$ to $y$. We abbreviate $\overrightarrow{ox}$ by $\vec{x}$. Directed line segments are referred to as vectors.

Physicists and mathematicians often speak of a vector’s **magnitude** and **direction**. Given a vector, $\vec{x}$, by magnitude (or norm) we mean the distance between the point, $x$, and the origin, $o$. By direction we mean the direction determined by the directed line segment $\vec{x}$ that has base at $o$ and tip at the point, $x$. When we say to “sketch the vector $\vec{x}$” we mean to draw the directed line segment from the origin to the point, $x$.

We are making a distinction between points and vectors. Points are the actual elements of 3-space and vectors are directed line segments. The word **scalar** will be used to refer to real numbers (and later in your mathematical career as complex numbers or elements of any field). The word, **point** may be used to mean a real number, an element of $\mathbb{R}$, an element of $\mathbb{R}^3$, etc.

Having carefully made clear the distinction between point and vector you will have to work hard to keep me honest; I tend to use the two more or less interchangeably.

**Definition 5.** If $x,y \in \mathbb{R}^3$, with $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$, and $\alpha \in \mathbb{R}$ then:

- $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ “Addition in $\mathbb{R}^3$”
- $\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3)$ “Scalar Multiplication in $\mathbb{R}^3$”

Now, to be precise, we should also define vector addition and scalar multiplication for vectors. Since the definition is identical (except for placing arrows above the $x$ and $y$) we omit this. As you can see, always making a distinction between a point and a vector can be cumbersome.

**Problem 1.** Let $\vec{x} = (1,2)$ and $\vec{y} = (5,2)$ and sketch $\vec{x}$, $\vec{y}$, $-\vec{x}$, $2\vec{y}$.

Consider the two vectors, $\vec{x}$ and $\vec{y} - \vec{x}$. Both vectors have the same direction and the same magnitude. They are different because the vector $\vec{y} - \vec{x}$ has its base at the origin and its tip at the point $y - x$ while $\vec{x}$ has its base at $x$ and its tip at $y$.

**Problem 2.** Sketch $\vec{x} + \vec{y}$, $\vec{x} - \vec{y}$ and $\vec{x}y$ where $\vec{x} = (2,3)$ and $\vec{y} = (4,2)$.

**Definition 6.** Let $n \in \mathcal{N}$. A function from $\mathbb{R}$ to $\mathbb{R}^n$ is called a parametric curve.
**Problem 3.** Let \( \vec{v}(t) = (2t, 3t) \). Sketch \( \vec{v} \) for all \( t \in \mathbb{R} \). If \( t \) represents time and \( \vec{v}(t) \) represents the position of a llama at time \( t \) then how fast is the llama traveling?

**Problem 4.** Let \( \vec{v}(t) = (1,2) + (4,5)t \). Sketch \( \vec{v} \) for all \( t \in \mathbb{R} \) and give the speed of a lemur whose position in the plane at time \( t \) seconds is given by \( \vec{v}(t) \).

**Problem 5.** What is the distance between the point \((1,2,3)\) and the origin? What is the distance between \((1,2,3)\) and \((4,5,6)\)?

**Problem 6.** Sketch \( \vec{v}\(t\) = \(t(1,2,3)\) + \((1-t)(3,4,-5)\) \) for all \( t \in \mathbb{R} \). Compute \( \vec{v}(0) \) and \( \vec{v}(1) \).

**Problem 7.** What is the distance between \( x = (x_1,x_2,x_3) \) and \( y = (y_1,y_2,y_3) \)?

**Problem 8.** Let \( x = (2,5,-1) \) and \( y = (1,2,4) \).

1. Write an equation, \( \vec{L} \), for the line in \( \mathbb{R}^3 \) passing through \( x \) and \( y \) with \( \vec{L}(0) = x \) and \( \vec{L}(1) = y \).

2. Write an equation, \( \vec{M} \), for the line in \( \mathbb{R}^3 \) passing through \( x \) and \( y \) so that \( \vec{M}(0) = x \) and the speed of an object with position determined by the line is twice the speed of an object with position determined by the line in part 1 of this problem.

**Problem 9.** Find infinitely many parametric equations for the line passing through \((a,b,c)\) in the direction \((x,y,z)\).

**Problem 10.** Plot some points in order to graph \( \vec{f}(t) = (\sin(t), \cos(t), t) \) for \( t \in [0, 6\pi] \). How would you write the set representing the range of \( \vec{f} \)?

**Note to Instructor (R)** Here we discuss the domains and ranges of various “types” of functions including examples of all the types of function we have seen such as functions from \( \mathbb{R} \to \mathbb{R} \), from \( \mathbb{R}^2 \to \mathbb{R} \), from \( \mathbb{R} \to \mathbb{R}^2 \). We discuss the coordinate planes, \( x = 0, y = 0, \) and \( z = 0 \) along with translations of these planes such as \( x = 1, z = 3, \) and \( y = -4 \). Then we graph \( f(x,y) = 2x + y^2 \) by graphing the slices \( x = -2, x = 0, x = 2, \) and \( y = 0 \). We’ll note that this might be written as \( z = 2x - y^2 = 0 \) which really means the set, \( \{(x,y,z) \in \mathbb{R}^3 : z = 2x - y^2 = 0 \} \). I try to be very precise when I lecture, demonstrating the independent variables and writing things out completely. We compose this function with \( l(t) = (0,0) + (1,1)t \) and discuss that at any time \( t \), a bug standing at position \( l(t) \) might see her friend directly above (or below) her at height, \( f(l(t)) \) and position \( (t, f(l(t))) \). Such discussions are my attempt at giving a conceptual intuition to the many types of functions and surfaces that will arise throughout the course. Because most of my students are engineers who will see ample applications, we don’t spend lots of time assigning “real-world” applications. We do spend time discussing the types of real-world problems to which the mathematics we do can be applied.

In Calculus I and II you studied functions from \( \mathbb{R} \) to \( \mathbb{R} \), for example \( f(x) = x^3 \), and functions from \( \mathbb{R} \) to \( \mathbb{R}^2 \), for example \( \vec{f}(t) = (\cos(t), \sin(t)) \). Now we have added functions from \( \mathbb{R} \) to \( \mathbb{R}^3 \) such as the examples in the previous few problems. We wish to add functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \) to our list next. Such functions are called real-valued functions of several variables.

**Definition 7.** A real valued function of several variables is a function \( f \) from \( \mathbb{R}^n \to \mathbb{R} \). We will primarily consider \( n = 2 \) and \( n = 3 \) in this course.

**Here are examples of some functions we will encounter and their names.**

1. real valued functions defined on \( \mathbb{R} \)
   \[ f : \mathbb{R} \to \mathbb{R}, \ f(t) = t^2 + e^t \]
2. parametric curves (also called vector valued functions) defined on $\mathbb{R}$
   (a) planar curves
   $$\mathbf{f} : \mathbb{R} \to \mathbb{R}^2, \quad \mathbf{f}(t) = (e^{2t+1}, 3t + 1)$$
   (b) space curves
   $$\mathbf{f} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{f}(t) = (2t + 1, (3t + 1)^3, 4t + 1)$$

3. functions of several variables, or multivariate functions
   (a) real valued functions of two variables
   $$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^2 + \sqrt{y}$$
   (b) vector valued functions of three variables
   $$f : \mathbb{R}^3 \to \mathbb{R}^2, \quad f(x, y, z) = (2xy, 3x + 4y^2)$$
   (c) vector valued functions of several variables
   $$f : \mathbb{R}^n \to \mathbb{R}^m$$

Problem 11. Graph the set of all points $(x, y, z)$ in $\mathbb{R}^3$ that satisfy $x + y + z = 1$.

Problem 12. Graph the set of all points $(x, y, z)$ in $\mathbb{R}^3$ that satisfy $z = 4$.

Definition 8. If $\mathbf{x}$ is a vector in $\mathbb{R}^3$ then $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This is called the magnitude, length, or norm of $\mathbf{x}$.

We defined the norm on vectors, but the same definition is valid for points in $\mathbb{R}^3$.

Note to Instructor (I) We define the dot product of two vectors and discuss the difference between the words that we will inevitably use interchangeably, orthogonal and perpendicular. We give the simple example that if $x = (-1, 1)$ and $y = (1, 1)$ then they have zero dot product, hence are orthogonal. Of course, a graph of $x$ and $y$ as vectors along with simple trigonometry shows that they are also perpendicular. More discussion on the relationship between dot products and projections will be forthcoming. In the problems, students are asked to prove that two orthogonal vectors are in fact perpendicular.

Theorem 1. Law of Cosines Given any triangle with sides of lengths $a, b,$ and $c$, and having an angle of measure $\alpha$ opposite the side of length $a$, the following equation holds: $a^2 = b^2 + c^2 - 2bc \cos(\alpha)$.

Problem 13. Sketch in $\mathbb{R}^2$ the vectors $(1, 2)$ and $(3, 5)$ and find the angle between these vectors by using the law of cosines.

Problem 14. Sketch in $\mathbb{R}^3$ the vectors $(1, 2, 3)$ and $(-2, 1, 0)$ and find the angle between these vectors by using the law of cosines.

Definition 9. If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ then the dot product of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$

Again, we defined the dot product on vectors, but the same definition is valid for points in $\mathbb{R}^3$.

Definition 10. We say the vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$.

Problem 15. Show that if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^3$ then
$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2.$$
Problem 16. Find two vectors orthogonal to \( (1,2) \). How many are there?

Problem 17. Find three vectors orthogonal to \((1,2,3)\). How many are there?

Problem 18. Use Theorem 1 + Problem 15 to show that if \( \vec{x}, \vec{y} \in \mathbb{R}^3 \) then \( \vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \theta \) where \( \theta \) is the angle between \( \vec{x} \) and \( \vec{y} \).

Problem 19. Find two vectors orthogonal to both \((1,4,3)\) and \((2,-3,4)\). Sketch all four vectors.

Note to Instructor (R) Because we allow students to look at other materials, it is common for a student to find a vector orthogonal to two vectors by using the cross product which we have not developed yet. Typically, they compute the determinant of the “matrix” with rows \((\hat{i}, \hat{j}, \hat{k})\), \(\vec{u}\), and \(\vec{v}\). When this happens, we use the problem as an opportunity to show the class how to compute determinants of 2x2 and 3x3 matrices. Some of my students have had linear algebra, but many have not. Next we set up the equations one would need to solve in order to demonstrate how the previous problem could be worked by brute force. And I say that while we can use this trick, we won’t know (as mathematicians) that it is valid until we resolve Problem 22 that shows how to find a vector perpendicular to two vectors. The key point is that in defining the cross product we won’t know (as mathematicians) that it is valid until we resolve Problem 22 that shows how to find a vector orthogonal to two vectors. The point is that the dot product has an important meaning associated with projections. We will use this very important fact in demonstrating how a vector line integral is used to compute the work in moving a particle through a vector field as well as with Green’s Theorem and Gauss’ Divergence Theorem. I attempt to make every fact that we study early in the course a recurring theme throughout the course and we attempt to relate it to other subjects. Of course projections, cross products, and determinants in this setting are powerful motivations for a deep understanding of linear algebra. If they have taken linear algebra, then it is an argument for taking the second semester; if not, it is a basis for understanding the first semester.

Problem 20. Show that if two non-zero vectors \( \vec{x} \) and \( \vec{y} \) are orthogonal then the angle between them is \( 90^\circ \). Hence any two orthogonal vectors are perpendicular vectors.

Problem 21. Given the vectors \( \vec{u}, \vec{v} \in \mathbb{R}^3 \), find the area of the parallelogram with sides \( \vec{u} \) and \( \vec{v} \) and diagonals \( u+v \) and \( u-v \). The vertices of this parallelogram are the points: the origin, \( u \), \( v \), \( u+v \), and \( u-v \).

Problem 22. Assume \( \vec{u}, \vec{v} \in \mathbb{R}^3 \). Find a vector \( \vec{x} = (x,y,z) \) so that \( \vec{x} \perp \vec{u} \) and \( \vec{x} \perp \vec{v} \) and \( x+y+z = 1 \).

Problem 23. Prove or give a counter example to each of the following where \( \vec{u}, \vec{v} \in \mathbb{R}^3 \) and \( c \in \mathbb{R} \):

1. \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \)
2. \( \vec{u}(\vec{w} \cdot \vec{v}) = (\vec{u} \cdot \vec{w})(\vec{u} \cdot \vec{v}) \)
3. \( c(\mathbf{u} \cdot \mathbf{v}) = (c \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c \mathbf{v}) \)

4. \( \mathbf{u} + (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{w}) \)

5. \( \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \)

**Problem 24.** Let \( f(x, y) = x^2 + y^2 \). Sketch the intersection of the graph of \( f \) with the planes: \( z = 0; \ z = 4; \ z = 9; \ y = 0; \ y = -1; \ y = 1; \ x = 0; \ x = -1; \ x = 1 \). Now sketch all of these together in one 3-D graph.

In the previous problem, the intersection of the graph with \( z = 0, z = 4 \), and \( z = 9 \) are called level curves because each represents the path you would take if you walked around the graph always remaining at a certain height or level.

Recall composition of functions and the chain rule from Calculus I. We’ll extend these to the functions we study this semester.

**Definition 11.** If \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) are differentiable functions then \((f \circ g)(t) = f(g(t))\) and \((f \circ g)'(t) = f'(g(t))g'(t)\)

**Note to Instructor (R)** Here a discussion arose about the difference between a surface and a function. I told them that a surface is a two dimensional topological manifold; unfortunately, three out of four of those words are beyond the scope of the course – take topology! I told them to think of a surface as a set of points in three-space where at every point one could place a tangent plane; think smooth and differentiable. We are most interested in being sure we understand when a given set of points in three space is a function. Backing up to 2-space, we discussed the example of a circle in the plane as an equation that is not a function, re-emphasizing that \( x^2 + y^2 = 4 \) means \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\} \). We observe that if we have an equation in three variables, \( x, y, \) and \( z \) such as \( x^2 + y^2 + z^2 = 9 \) then solving for \( z \) may or may not result in a function, just as solving the circle \( x^2 + y^2 = 9 \) for \( y \) does not yield a function. Of course, we know of (and sketch) other examples of surfaces by considering \( x^2 + y^2 = 9 \) and \( 2x - 3y = 5 \) in three space as cylinders and planes that are not functions.

**Problem 25.** Let \( g(x, y) = x^2 + y^3 \) and \( \mathbf{T}(t) = (0, 1)t \). Compute \( g \circ \mathbf{T} \). Graph \( g \), \( \mathbf{T} \), and \( g \circ \mathbf{T} \).

**Problem 26.** Compute \((g \circ \mathbf{T})'(t)\) and \((g \circ \mathbf{T})'(2)\). What is the significance of this number with respect to your graphs from the previous problem?

Here is a reminder from Calculus II of the definition of arc length.

**Definition 12.** If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a function which is differentiable on \([a, b]\), then the arc length of \( f \) on \([a, b]\) is \( \int_a^b \sqrt{1 + (f'(x))^2} \, dx \). If \( \mathbf{c} : \mathbb{R} \rightarrow \mathbb{R}^2 \) is a vector valued function which is differentiable on \([a, b]\) so that \( \mathbf{c}'(t) = (x(t), y(t)) \) then the arc length of \( \mathbf{c} \) on \([a, b]\) is \( \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \).

**Problem 27.** A man walks along a path on the surface \( f(x, y) = 4 - 2x^2 - 3y^2 \) from one point on the \( x \)-axis to a second point on the \( x \)-axis, always remaining directly above the \( x \)-axis. Graph the path and write an integral expression for the distance he walked and compute the distance he walked.

**Problem 28.** A lady walks along the surface from the previous problem staying exactly 3 units above the \( xy \)-plane. Write an integral expression for the distance she walks if she starts and stops at \((0, \frac{1}{\sqrt{3}}, 3)\) and never retraces her steps.
Problem 29. Redo Problem 25 and Problem 26 with \( \vec{l}(t) = (-1, 1)t \).

Problem 30. Find the slope of the line tangent to \( f(x, y) = x^3 + 3y^2 \) at \( (1, 2, 13) \) that lies above the line \( \vec{l}(t) = (1, 2) + (1, 1)t \).

Problem 31. Given \( \vec{a} = \langle 4, 3 \rangle, \vec{b} = \langle 1, -1 \rangle, \) and \( \vec{c} = \langle 6, -4 \rangle \), determine the angle between \( \vec{b} \) and \( \vec{c} \).

In the next problem the notation, \( | \cdot | \), is used for both the absolute value (on the left side of the equation) and the norm (on the right side of the equation). Is this bad notation? Consider the definition for the norm, that
\[
| (x_1, x_2) | = \sqrt{x_1^2 + x_2^2}.
\]
Suppose we take the norm of a vector in \( \mathbb{R}^1 \), such as \( (x_1) \). Then,
\[
| (x_1) | = \sqrt{x_1^2} = \text{the absolute value of the number } x_1.
\]
Thus, the absolute value of \( x \) is the norm of \( x \) so you have been studying norms since high school (elementary school?) without knowing it!

Problem 32. Show that if \( \vec{u} \) and \( \vec{v} \) are vectors in \( \mathbb{R}^2 \) then \( | \vec{u} \cdot \vec{v} | \leq | \vec{u} | | \vec{v} | \). Can you show this for vectors in \( \mathbb{R}^3 \)?

The next result is known as the Triangle Inequality and it states essentially that the shortest distance between two points is the straight line. Look at a graph of \( \vec{u}, \vec{v}, \vec{u} + \vec{v}, \) and \( \vec{u}(u+v) \). If you travel from the origin, along the vector \( \vec{u} \) and then along the vector \( \vec{u}(u+v) \) then you have traveled further than if you traveled along the vector \( \vec{u} + \vec{v} \).

Problem 33. Triangle Inequality Show that if each of \( \vec{u} \) and \( \vec{v} \) are vectors in \( \mathbb{R}^2 \) then \( | \vec{u} + \vec{v} | \leq | \vec{u} | + | \vec{v} | \).
Chapter 2

Cross Product and Planes

In Problem 22, you were given two vectors and asked to find a vector that was perpendicular to both. Because there are infinitely many vectors perpendicular to any two given vectors, we added another condition ($x_1 + x_2 + x_3 = 1$) so that the answer would be unique. This problem was a warm-up for the definition of the cross product.

**Definition 13.** The **cross product** of two vectors is the vector that is perpendicular to both of them and has length that is the area of the parallelogram defined by the two vectors.

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ then the cross product of $\vec{u}$ and $\vec{v}$ may be computed by one of two methods:

**Method One**

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

For the second method, you need a tool from linear algebra, determinants. If I have not done so already, ask me during class to show you how to compute the determinant of 2 by 2 and 3 by 3 matrices.

**Method Two** If we define $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, and $\vec{k} = (0,0,1)$ then we may compute the cross product as:

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$ 

**Problem 34.** Prove or give a counter example for each statement, assuming $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$ and $k \in \mathbb{R}$.

1. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$
2. $\vec{u} \times \vec{v} = - (\vec{v} \times \vec{u})$
3. $\vec{u} \times \vec{u} = \vec{u} \times \vec{0} = \vec{0} \times \vec{u} = 0$
4. $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$
5. $k + (\vec{u} \times \vec{v}) = (k + \vec{u}) \times (k + \vec{v})$
6. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
7. $\vec{u} \times (\vec{v} \cdot \vec{w}) = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{w})$
8. \( \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w} \)

**Note to Instructor** (I) We give both algebraic and geometric developments for planes.

Algebraic: We launch from lines, noting that the slope-intercept form, \( y = mx + b \) doesn’t give us all lines, while \( ax + by = c \) does. We graph one simple line \( 2x - 3y = 9 \) and graph the \( x- \) and \( y- \) intercepts. Moving to three dimensions, we note that \( z = ax + by + c \) won’t give us all planes, but \( ax + by + cz = d \) will. And we need only plot three simple points, the \( x- \), \( y- \), and \( z- \) intercepts to sketch most planes.

Geometric: The slope of a line gives us the orientation of the line and a point on the line uniquely determines the line among all lines with that slope. With planes, the perpendicular to a plane gives us its orientation in three-space but we still need a point to nail down which plane we are discussing, just as we did with lines. We typically work one example, finding a plane orthogonal to \((-2,-2,2)\) and containing \((2,3,5)\) by sketching the orthogonal, the point, and the plane. We observe that any point \((x,y,z)\) on the plane must satisfy \((x,y,z) - (2,3,5)) \cdot (-2,-2,2) = 0\) and we simplify this expression to place the equation in standard form, observing that this yields the algebraic form we discussed earlier. I’ll query them as to what they observe about the coefficients and see if they notice that the coefficients are indeed a scalar multiple of the orthogonal to the plane. If not, we’ll discover it soon enough. From this example, we see what it means for a vector to be perpendicular to a plane. We say a vector is perpendicular to a plane when it is perpendicular to every vector that lays in the plane. •

One might think of planes in \( \mathbb{R}^3 \) as analogous to lines in \( \mathbb{R}^2 \). This is because a line is a one-dimensional object in \( \mathbb{R}^2 \) – that is it has dimension one less than the dimension of the space. In \( \mathbb{R}^3 \) a plane is two-dimensional – it has dimension one less than the dimension of the space.

**Definition 14.** Given \( a,b,c \in \mathbb{R} \) where \( a \) and \( b \) are not both zero, the **line** determined by \( a,b, \) and \( c \) is the collection of all points \((x,y) \in \mathbb{R}^2 \) satisfying \( ax+by=c \).

Given this definition we can define a **plane** in the same manner.

**Definition 15.** Given \( a,b,c,d \in \mathbb{R} \) where \( a,b \) and \( c \) are not all zero, the **plane** determined by \( a,b,c,d \) is the collection of all points \((x,y,z) \in \mathbb{R}^3 \) satisfying \( ax+by+cz=d \).

If algebraically we think of a plane as all \((x,y,z) \in \mathbb{R}^3 \) satisfying \( ax+by+cz=d \) where not all of \( a,b \) and \( c \) are zero, then geometrically we can think of a plane as uniquely determined by a vector and a point where the plane is perpendicular to the vector and contains the point. We need both because there are infinitely many planes perpendicular to a given vector, but knowing one point in the plane uniquely determines the plane.

**Problem 35.** Show that \((3,-2,5)\) is perpendicular to the plane \(3x-2y+5z=7\) by choosing two points, \( x = (x_1,x_2,x_3) \) and \( y = (y_1,y_2,y_3) \), in the plane and showing that \((3,-2,5) \cdot \dot{x} - \dot{y} = 0\).

**Problem 36.** Show that \((a,b,c) \in \mathbb{R}^3\) is orthogonal to the plane \( ax+by+cz=d \).

**Problem 37.** Determine whether these two planes are parallel.

1. \(2x-3y+\frac{5}{2}z=9\)
2. \(x-\frac{3}{2}y+\frac{5}{4}z=12\)

**Problem 38.** Write in standard form \((ax+by+cz=d)\) the equation of a plane \( \perp \) to the first plane from previous problem and containing the point \((9,2,3)\).
Note to Instructor (R) Just as occurs with the formula for computing the cross product of two vectors, often a student will use a formula from a book for the distance between two planes. If this occurs, I praise the student for getting this formula for us to use and agree that we can use it from now on. Again, I emphasize that it might be wrong (perhaps Wikipedia has a typo?) and that until we prove it, we won’t know if it is valid. Then we’ll address how we might find the distance without a formula and that by doing it this way, we can take a step toward deriving the formula. There are a few nice ways to do this. I like to create a parametric line that is orthogonal to both planes and passes through a point in one plane at time $t = 0$. Then we can determine the time at which the line intersects the other plane and use the formula for the distance between two points to find the distance. This method easily generalizes to derive the formula. Of course, one can use projections and trigonometry as well.

Problem 39. Find all the planes parallel to the plane $x + y - z = 4$ and at a distance of one unit away from the plane. When does the “distance between two planes” make sense?

Problem 40. Find both angles between these two planes:

1. $2x - 3y + 4z = 10$
2. $4x + 3y - 6z = -4$

Problem 41. Find the equation of the plane containing $(2, 3, 4)$, $(1, 2, 3)$ and $(6, -2, 5)$.

For next two problems, there is a formula on the web or in some book, but it’s probably wrong because of a typographical error. Find a vector $\vec{v}$ to both planes, determine the equation of a parametric line, $\vec{l}$, passing through both planes. Find the points where $\vec{l}$ intersects each plane. Find the distance between these points.

Problem 42. Find the distance between the two planes, $x + y + z = 1$ and $x + y + z = 2$.

Problem 43. Find the distance between the two planes, $3x - 4y + 5z = 9$ and $3x - 4y + 5z = 4$.

Problem 44. Find an equation for the distance between two planes, $ax + by + cz = e$ and $ax + by + cz = f$.

Problem 45. Find the equation of the plane containing the line $\vec{l}(t) = (1 + 2t, -1 + 3t, 4 + t)$ and the point $(1, -1, 5)$.

The next problem asks for the intersection of two planes. What are all the possibilities for the intersection of any two planes? One possibility is a line. To find the equation of the line, there are a couple of ways we could think about this. First, we could find two points in the intersection and then find the equation of that line. Or we could observe that the intersection of two planes must be contained in each plane and thus is parallel to both planes. How can we find a vector that is parallel to both planes?

Problem 46. Find the intersection of the two planes $3x - 2y + 6z = 1$ and $3x - 4y + 5z = 1$. 
Chapter 3

Limits and Derivatives

For the next two definitions, suppose that $x, y, z : \mathbb{R} \to \mathbb{R}$ are differentiable functions.

**Definition 16.** The limit of the vector valued function of one variable $\vec{f}(t) = (x(t), y(t), z(t))$, as $t$ approaches $a$ is defined by

$$\lim_{t \to a} \vec{f}(t) = \left( \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)$$

as long as each of these limits exists. If any one of the limits does not exist, then the limit of $\vec{f}$ does not exist at $a$.

**Definition 17.** The derivative of the vector valued function of one variable $\vec{f}(t) = (x(t), y(t), z(t))$, is defined by

$$\vec{f}'(t) = (x'(t), y'(t), z'(t))$$

as long as each of the derivatives exists. If any one of the derivatives does not exist, then the derivative of $\vec{f}$ does not exist.

**Problem 47.** Let $\vec{f}(t) = (t^2 - 4, \sin(t), \sqrt{t+1}^3)$. Compute $\lim_{t \to 0} \vec{f}(t)$.

**Problem 48.** Compute $\vec{f}'$ and $\vec{f}'(0)$ for $\vec{f}$ from the previous problem.

**Note to Instructor (R)** Once this problem has been presented, we give the geometrical interpretation of $\vec{f}'(0)$ as the tangent to the parametric curve. Parametric curves and polar coordinates were introduced in the second semester of calculus, so here we might remind them that if $c(t) = (x(t), y(t))$ is a planar curve, then $c'(t) = (x'(t), y'(t))$ is tangent to $c$ and each of $(-y'(t), x'(t))$ and $(y'(t), -x'(t))$ are normals. I attempt Socratic, question-and-answer lectures. “What do you suppose $f'(0)$ means?” (After listening and waiting for a long time, if there is no response...) “What did $f'(0)$ tell us about $f(x) = x^2 + 1$?” If a series of questions won’t generate success, then sketching the elliptical example $c(t) = (3 \sin(t), 4 \cos(t))$ and reminding them of the definition of the derivative $c'(t) = \lim_{h \to 0} \frac{c(t+h)-c(t)}{h}$ can help motivate just why $c'$ is tangent to $c$.

We now wish to develop the rules for limits and derivatives that parallel the rules from calculus in one dimension. Of course, we know you have not forgotten any of these rules, so the ones we develop should look familiar! The good news is that the really hard work in proving these was done in the first semester of calculus and thus the work here is more notational than mathematical!

**Problem 49.** Let $\vec{f}(t) = (t^2, t^3 - 1, \sqrt{t-1})$ and $\vec{g}(t) = (2-t^2, t^3, \sqrt{t+1})$. 
1. Compute \( \lim_{t \to 2} \overrightarrow{f}(t) \).

2. Compute \( \lim_{t \to 2} \overrightarrow{g}(t) \).

3. Compute \( \lim_{t \to 2} \overrightarrow{f}(t) + \lim_{t \to 2} \overrightarrow{g}(t) \).

4. Compute \( \lim_{t \to 2} [\overrightarrow{f}(t) + \overrightarrow{g}(t)] \).

5. What can you conjecture about \( \lim_{t \to a} [\overrightarrow{f}(t) + \overrightarrow{g}(t)] \) for arbitrary choices of \( a \), \( \overrightarrow{f} \), and \( \overrightarrow{g} \)?

**Problem 50.** State 5 rules for limits of the vector valued functions, \( \overrightarrow{f}, \overrightarrow{g} : \mathbb{R} \to \mathbb{R}^3 \) that parallel the limit rules from Calculus I and prove one of these conjectures. You may grab a book or look on the web to remind you of the rules from Calculus I.

**Problem 51.** Compute \( \overrightarrow{f}'(2) \) and \( \overrightarrow{g}'(2) \) where \( \overrightarrow{f} \) and \( \overrightarrow{g} \) are from Problem 49. Compute \( (\overrightarrow{f} + \overrightarrow{g})'(2) \). What can you conjecture about \( (\overrightarrow{f} + \overrightarrow{g})'(t) \) for arbitrary choices of \( \overrightarrow{f} \) and \( \overrightarrow{g} \)?

**Problem 52.** State 5 rules for derivatives of vector valued functions that parallel the derivative rules from Calculus I. Prove one of these conjectures. You may grab a book or look on the net to remind you of the rules from Calculus I.

You may assume that which ever one you do not prove will end up on the next test. Yes, it is a well known fact that all calculus teachers can read students’ minds. How else would we always be able to schedule our tests on the same days as your physics tests?

**Limits of Functions of Several Variables**

Recall from Calculus I the various ways in which you computed limits. If possible, you substituted a value into the function. If not, perhaps you simplified the function via some algebra or computed a limit table or graphed the function. Or you might have applied L’Hopital’s Rule. Your instructor probably used the Squeeze Theorem to obtain the result that \( \lim_{t \to 0} \frac{\sin(t)}{t} = 1 \). Recall that if the left hand limit equaled the right hand limit then the limit existed.

In three dimensions, the difficulty is that there are more paths to consider than merely left and right. For the limit to exist at a point \((a, b)\), we need that the limit as \((x, y)\) approaches \((a, b)\) exists regardless of the path we take as we approach \((a, b)\). We could approach \((a, b)\) along the x-axis for example, setting \( y = 0 \) and taking the limit as \( x \to 0 \). Or we could take the limit along the line, \( y = x \). The limit exists if the limit as \((x, y) \to (a, b)\) along every possible path exists. We will see an example where the limit toward \((a, b)\) exists along every straight line, but does not exist along certain non-linear paths!

**Note to Instructor (I)** To introduce limits, we start with \( f(x) = \frac{|x|}{x} \) from Calculus I and consider the left- and right-hand limits. Before talking about three dimensions, I’ll ask them if anyone has seen a house where if you approach it from one direction you enter on one floor, but if you enter from another direction, you enter on a different floor. Usually someone knows someone who lives in a house on a hill where such construction is standard. If so, they can visualize different paths that lead to different heights at the origin. Then we consider the problem in three-space

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]
by computing the limits along the lines $x = 0$, $y = 0$, and $y = x$. We address how, just as in Calculus I, the limit exists only if the limit exists along every possible path and we foreshadow upcoming problems by warning them that there are problems where the limits exist and agree along every straight line path, but there is a non-linear path along which we get a limit that does not agree. Because they don’t have the tools to check every possible path, on tests I’ll only ask them to compute limits along specific paths and make their best guess as to whether a limit exists or does not exist.

**Definition 18.** If $(a, b) \in \mathbb{R}^2$ and $L \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then we say that

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if $f(x,y)$ approaches $L$ as $(x,y)$ approaches $(a,b)$ along every possible path.

**Problem 53.** Sketch $f(x,y) = x^2 + y^2$, indicate the point $(2, 3, f(2, 3))$ and compute $\lim_{(x,y) \to (2,3)} x^2 + y^2$.

**Problem 54.** Use any software to graph the function from the previous example near $(0,0)$. Print and use a highlighter to mark the paths $x = 0$, $y = 0$, and $y = x$. If you use Maple, available in our lab, then the command to plot $f(x,y) = x^2 + y^2$ would be “plot3d($x^2 + y^2, x = -1..1, y = -1..2$);”.

**Problem 55.** Convert the previous problem to polar coordinates via the substitution $x = r \cos(\theta)$ and $y = r \sin(\theta)$ and then compute the limit as $r \to 0$.

**Problem 56.** Let $f(x,y) = \frac{x + y^2 + 2}{x - y + 2}$.

1. **Graph** $f$ using any software and state the domain.

2. **Compute** $\lim_{(x,y) \to (-1,2)} f(x,y)$.

Recall your Calculus I definition of continuity for functions of one variable.

**Definition 19.** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$.

This definition says that for $f$ to be continuous at $a$ three things must happen. First, the function must be defined at $a$. This means that $a$ must be in the domain of the function so that $f(a)$ is a number. Second, the limit of the function as we approach $a$ must exist. And third, $f(a)$ must equal the limit of $f$ at $a$. The same statement defines continuity for all functions.

**Definition 20.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ if $\lim_{x \to a} f(x) = f(a)$.

**Problem 57.** Consider $\lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

1. **Compute** this limit along the lines: $x = 0$, $y = 0$, $y = x$, and $y = -x$.

2. **Convert** to polar coordinates and check the limit.

3. **Graph** using any software.

4. **Why isn’t** this function continuous at $(0, 0)$?

5. **How can you modify** $f$ in such a way as to make it continuous at $(0, 0)$?

**Problem 58.** Compute $\lim_{(x,y) \to (0,0)} \frac{xy + y^3}{x^2 + y^2}$ if it exists.
Problem 59. Determine whether the function $f$ continuous at $(x,y) = (0,0)$ by considering the paths $y = kx^2$ for several choices of $k$. 

$$f(x,y) = \begin{cases} 
\frac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\
0 & (x,y) = (0,0)
\end{cases}$$

Note to Instructor (I) This lecture ties together many concepts that have been developed (limits, the use of vectors as directions, composition of functions, and parametric curves) while refreshing concepts from Calculus I such as the limit definition of the derivative.

Letting $g(x) = x^3$ we use the limit definition of the derivative to compute the slope of the tangent line to $g$ at $(2,8)$. Letting $f(x,y) = x^3 + y^2$ we use the limit

$$\lim_{h \to 0} \frac{f(2,0+h) - f(2,0)}{h}$$

to compute the slope of the line tangent to $f$ at the point $(2,0,8)$ and in the plane $y = 0$. We illustrate this with a careful graph, observing that it is the same as our previous result for $g$ because we are computing the slope of a line that is tangent to a slice of $f$ that is precisely the graph of $g$. We then use limits to compute the directional derivative of $f$ in the direction $(0,-1)$ (the negative $y$-direction) observing that the result has the same absolute value, but the opposite sign of our previous results. This is a departure from Calculus I where we did not talk about the derivative at $x$ in the left or right directions. There was only one derivative at any point $x$ at which it was differentiable. We then compute the slope of the tangent line in the direction $(1,0)$ (the positive $x$-direction) at the same point. We conclude with the general definition for the directional derivative of $f$ at vector $u$ in direction $v$, defining the various notations for functions of two variables:

1. the directional derivative of $f$ in the direction $v$ at $u$ is $D_v f(u) = \lim_{h \to 0} \frac{f(u + hv) - f(u)}{h}$
2. directional derivative in the x-direction is $D_{(1,0)} f(u) = df/dx = f_x = f_1$
3. directional derivative in the y-direction is $D_{(0,1)} f(u) = df/dy = f_y = f_2$
4. gradient of $f$ at $u$ is $\nabla f(u) = (f_1, f_2, f_3, ..., f_n)$

To toss in a tad of history, we mention that this idea is generalizable to spaces other than $\mathbb{R}^n$ and is called the Gateaux Derivative and that there are numerous notions of differentiability, including the Frechet derivative. A function that is Frechet differentiable is also Gateaux differentiable since Frechet equates to having, in some sense, a “total” derivative, while Gateaux only indicates that we have directional derivatives in every direction. If I have honors students, these can make good projects for study.

Directional Derivatives

Suppose we have the function $f(x,y) = x^2 + y^2$ and we are sitting on that function at some point $(a,b, f(a,b))$ (other than $(0,0,0)$). Then there are many directions we can walk while remaining on the surface. Depending on the direction of our path, the rate of increase of our height, or slope of our path, may vary. Some paths will move us uphill and others downhill. Suppose while we sit at the point, $(a,b, f(a,b))$, we decide to walk in a direction that will not change the $y$ coordinate, but only changes the $x$ coordinate. Thus, we are walking on the surface and staying within the
plane, \( y = b \). Walking in this way, we could go in one of two directions. Either we could go in the direction that increases \( x \) or decreases \( x \). Let’s go in the direction that increases \( x \). Let’s go in the direction that increases \( x \).

Now, consider the tangent line to the curve at this point that lies in the plane, \( y = b \). As we take our first step along the curve our rate of increase in height, \( z \), will be the same as the slope of that tangent line. This slope is the \textit{directional derivative of the function at the point \((a,b)\) in the \( x \) direction}. If we had decided to fix \( x = a \) and walk in the direction that increases \( y \), then the slope of the line tangent to the function and in the plane \( x = a \) is the \textit{directional derivative of \( f \) at \((a,b)\) in the \( y \) direction}.

The next definition formalizes this discussion and the problem immediately following it is an example that will make these notions of directional derivative precise!

\textbf{Definition 21.} \textit{If \( f: \mathbb{R}^2 \to \mathbb{R} \) is a function and \((a,b)\) is in the domain of \( f \) then the \textit{derivative of \( f \) in the \((1,0)\) direction at \((a,b)\) is the slope of the line tangent to \( f \) at the point \((a,b, f(a,b))\) and in the plane, \( y = b \).}

\textbf{Notation} Suppose that \( f: \mathbb{R}^2 \to \mathbb{R} \) is a function as in the previous definition. There are many phrases and notations used to denote the \textit{derivative of \( f \) in the \((1,0)\) direction at \((a,b)\)}. For example,

- \( f_1 \) – the \textit{derivative of \( f \) with respect to the first variable}
- \( f_x \) – the \textit{the derivative of \( f \) with respect to \( x \)}
- \( \frac{df}{dx} \) – the \textit{partial derivative of \( f \) with respect to \( x \)}

Similarly, \( f_2 \), \( f_y \), and \( \frac{df}{dy} \) would denote the same concepts where the derivative was taken in the \((0,1)\) direction.

The \textit{derivative of \( f \) with respect to \( x \) at \((a,b)\)} can be computed via the limit:

\[ f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \]

or, since \( y \) is being held constant and \( x \) is changing, one may just compute the derivative of \( f \) as if \( x \) is the variable and \( y \) is a constant.

\textbf{Problem 60.} Let \( f(x,y) = x^2 + y^2 \) and \((a,b) = (2, -3)\).

1. Sketch \( f \) and sketch the line tangent to \( f \) at the point \((2, -3, f(2, -3))\) that is in the plane, \( y = -3 \).
2. Let \( g(x) = f(x, -3) \) and compute \( g'(2) \).
3. What part of the graph of \( f \) is the graph of \( g \)?

\textbf{Problem 61.} Let \( f(x,y) = \frac{y}{x} \) and compute \( f_x(1,2) \) using the limit described following Definition 21.

\textbf{Problem 62.} Compute \( f_x \) and \( f_y \) for each function.

1. \( f(x,y) = x^3 - 4x^2 \)
2. \( f(x,y) = e^{xy^2} \)
3. \( f(x,y) = \frac{x^2}{\sin(xy)} \)
4. \( f(x,y) = e^{x^2 + y^2} \)

**Definition 22.** Just as in Calculus I, “second derivatives” are merely derivatives of the first derivatives. Thus \( f_{xx} = (f_x)_x \). I.e. \( f_{xx} \) is the derivative of \( f_x \) with respect to \( x \). Similarly, \( f_{xy} = (f_x)_y \) and \( f_{yx} = (f_y)_x \).

Other standard notations are:

\[
\begin{align*}
  f_x &= \frac{\partial f}{\partial x}, & f_y &= \frac{\partial f}{\partial y}, & f_{xx} &= \frac{\partial^2 f}{\partial x^2}, & f_{yy} &= \frac{\partial^2 f}{\partial y^2}, & f_{xy} &= \frac{\partial^2 f}{\partial x \partial y}
\end{align*}
\]

**Theorem 2. Clairaut's Theorem** For any function, \( f \), whose second derivatives exist, we have \( f_{xy} = f_{yx} \).

**Note to Instructor (R)** While the next problem seems trivial, it is a spring board that we use to discuss where all these derivatives show up in physical applications, Maxwell’s equation, Laplace’s equation, etc.

**Problem 63.** Find \( f_{xx}, f_{xy}, f_{yx}, \) and \( f_{yy} \) for each function.

1. \( f(x,y) = e^{x^2 + y^2} \)
2. \( f(x,y) = \sin(xy + y^3) \)
3. \( f(x,y) = \sqrt{3x^2 - 2y^3} \)
4. \( f(x,y) = \cot \left( \frac{x}{y} \right) \)

The study of partial differential equations is the process of finding functions that satisfy some equation that has derivatives with respect to multiple variables. For example, *Laplace’s Equation* is the equation \( u_{xx} + u_{yy} + u_{zz} = 0 \) and solutions give us information about the steady state of heat flow in a three dimensional object.

**Problem 64.** Which of these functions satisfy \( u_{xx}(x,y) + u_{yy}(x,y) = 0 \) for all \((x,y) \in \mathbb{R}^2?\)

1. \( u(x,y) = x^2 + y^2 \)
2. \( u(x,y) = x^2 - y^2 \)
3. \( u(x,y) = x^3 + 3xy^2 \)
4. \( u(x,y) = \ln(\sqrt{x^2 + y^2}) \)
5. \( u(x,y) = \sin(x)\cosh(y) + \cos(x)\sinh(y) \)
6. \( u(x,y) = e^{-x}\cos(y) - e^{-y}\cos(x) \)
7. Find a solution to this equation other than the one’s above.

**Note to Instructor (R)** While it may seem repetitive, we find that an occasional review of the various types of functions we have studied along with their derivatives is needed. We know and emphasize that there is really only one definition of a function so we are really only talking about functions with different domains and ranges. If this leads to a discussion, we’ll define the following. Given two sets, \( A \) and \( B \), a relation on \( A \times B \) is a subset of \( A \times B \). A function on \( A \times B \) is a relation on \( A \times B \) in which no two elements of the relation have the same first coordinates. While accurate, this curt, systematic explanation does not make the language (real-valued, parametric, vector valued, space curve, planar curve, etc.) any easier so we consider these examples,
1. \( f(x) = \sin(x^3) \)
2. \( g(t) = (4\cos(t), 3\sin(t)) \)
3. \( f(x, y) = \sin^2(xy) - \sqrt{\cos(x^2)} \)
4. \( f(x, y) = (\cos(xy), \sin(xy)) \)
5. \( 3x^2 + 5y^2 + 6z^2 = 1 \) (not a a function!)

In the last two problems, we studied functions of two variables and we defined derivatives in each direction, the \( x \) direction and the \( y \) direction. If \( f \) were a function of three variables, then there would be partial derivatives with respect to each of \( x, y, \) and \( z \). Let’s extend the notion of the partial derivatives with respect to \( x \) and \( y \) to functions with domain \( \mathbb{R}^n \) where \( n > 2 \). If \( f : \mathbb{R}^n \to \mathbb{R} \) then the domain of \( f \) is \( \mathbb{R}^n \) so there is a partial derivative of \( f \) with respect to the first variable, the second variable, and so on, up to the partial derivative of \( f \) with respect to the \( n^{th} \) variable. We use the notation, \( f_1, f_2, f_3, \ldots, f_n \) or \( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n} \) to denote the derivative of \( f \) with respect to each variable.

**Problem 65.** Let \( f(x_1, x_2, x_3, x_4, x_5) = x_3 \sqrt{(x_1)^2 + (x_2)^2 + x_4 e^{x_5}} \). Compute the five partial derivatives, \( f_1, f_2, \ldots, f_5 \).

This allows us to define the derivative for functions of \( n \) variables.

**Definition 23.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is a function and each partial of \( f \) exists then the **gradient** of \( f \) is the function
\[
\nabla f : \mathbb{R}^n \to \mathbb{R}^n
\]
and is defined by
\[
\nabla f = (f_1, f_2, f_3, \ldots, f_n) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n} \right).
\]

**Problem 66.** For each function below, state the domain of the function, compute the gradient and state the domain of the gradient.

1. \( f(x, y) = x^2y^2 - x^2y^3 \)
2. \( g(x, y, z) = xz^3 - 3xyz + \ln(x^2yz^3) \)

For any function of two variables, \( f \), let’s assume once again that we are at a point on the function, \((a, b, f(a, b))\). We know that the slope of the line tangent to \( f \) at \((a, b, f(a, b))\) and above the line \( y = b \) is \( f_x(a, b) \). And we know that the slope of the line tangent to \( f \) at \((a, b, f(a, b))\) and above the line \( x = a \) is \( f_y(a, b) \). Now consider a a line in the \( xy \)-plane passing through \((a, b)\) in some direction \((c, d)\) that is not parallel to either the \( x \) axis or the \( y \) axis. What would the slope of the line tangent to \( f \) at \((a, b, f(a, b))\) and above this line be? From the point \((a, b)\) there are infinitely many directions that we might travel, not just the directions parallel to the \( x \) and \( y \) axes. We can define such a direction from \((a, b)\) by a vector, \((c, d)\). The slopes of the lines tangent to \( f \) at \((a, b, f(a, b))\) in the direction \((c, d)\) are called the **directional derivatives** of \( f \) at \((a, b)\) in the direction \((c, d)\). The partial derivatives of \( f \) with respect to \( x \) and \( y \) are your first examples of directional derivatives where the directions were \( i = (1, 0) \) and \( j = (0, 1) \). Here is the formal definition of the directional derivative.

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**Definition 24.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The **directional derivative** of $f$ at $\vec{u}$ in the direction $\vec{v}$ is given by:

$$D_{\vec{v}} f(\vec{u}) = \lim_{h \to 0} \frac{f(\vec{u} + h\vec{v}) - f(\vec{u})}{h},$$

where $\vec{v}$ must be a unit vector.

Of course, not every limit exists, so directional derivatives may exist in some directions but not others.

**Problem 67.** Using the definition just stated, compute the directional derivative of $f(x, y) = 4x^2 + y$ at the point $\vec{u} = (1, 2)$ in the direction $\vec{v}$ for each $\vec{v}$ defined below.

1. $\vec{v} = (0, 1)$
2. $\vec{v} = (1, 0)$
3. $\vec{v} = (1, 1)$
4. $\vec{v} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

**Note to Instructor (R)** As soon as this problem goes on the board, we state and give an example of Theorem 5 below. The point of the previous problem is to demonstrate that if we compute the directional derivatives without using unit vectors for the direction then we don’t actually get the slope of the tangent line. Even though $(1, 1)$ and $(\sqrt{2}/2, \sqrt{2}/2)$ represent the same direction, the directional derivative would depend on the magnitude of the direction vector, which is why the definition requires that we use a unit vector.

Important! In the previous problem, you got different answers for the directional derivatives in parts 3 and 4 even though those represent the same direction. For this reason, it is a convention to always use a unit vector for the direction when computing directional derivatives.

**Problem 68.** Use Definition 24, Definition 21, and the discussion immediately following Definition 21 to show that if $f : \mathbb{R}^2 \to \mathbb{R}$ and $\vec{i} = (1, 0)$, then $D_{\vec{i}} f(\vec{v}) = f_x(\vec{v})$.

**Non-Definition:** An analytical definition of $f$ is **differentiable** at $u$ is beyond the scope of this course, but a geometrical definition is not. In two dimensions (Calculus I), $f$ was differentiable at $a$ if there was a tangent line to $f$ at $(a, f(a))$. In three dimensions (Calculus III), $f$ is differentiable at $\vec{u}$ if there is a tangent plane to $f$ at $(u, f(u))$.

**Definition 25.** An $\varepsilon$-**neighborhood of** $\vec{u}$ is the set of all points with a distance from $\vec{u}$ of less than $\varepsilon$. I.e. $N_\varepsilon(\vec{u}) = \{\vec{v} : |\vec{u} - \vec{v}| < \varepsilon\}$.

How will we be able to tell when a function is “nice,” that is, when a function has a derivative?

**Theorem 3.** If $\nabla f$ exists at $\vec{u}$ and at all points in some $\varepsilon$-neighborhood of $\vec{u}$ then $f$ is differentiable at $\vec{u}$.

**Problem 69.** Is $f(x, y) = y^3(x - \frac{1}{2})^2$ differentiable at $(1, 2)$?

**Problem 70.** For each of the following problems, either prove it or give a counterexample by finding functions and variables for which it does not hold. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable. Assume $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Assume $\vec{x}, \vec{y}, \vec{x} + \vec{y}$ are in the domain of $f$ and $g$.

1. $\nabla f(\vec{x} + \vec{y}) = \nabla f(\vec{x}) + \nabla f(\vec{y})$
2. $\nabla(f + g)(\vec{x}) = \nabla f(\vec{x}) + \nabla g(\vec{x})$
3. $\nabla(cf)(\vec{x}) = c\nabla f(\vec{x})$
4. $\nabla(f(x,y))(\vec{x}) = \partial_x f(x,y)\hat{i} + \partial_y f(x,y)\hat{j}$
5. $\nabla(f(x,y))(\vec{x}) = \partial_x f(x,y)\hat{i} + \partial_y f(x,y)\hat{j}$
6. $\nabla(f \cdot g)(\vec{x}) = \nabla f(\vec{x})g(\vec{x}) + f(\vec{x})\nabla g(\vec{x})$
7. $\nabla\left(\frac{f}{g}\right)(\vec{x}) = \frac{\nabla f(\vec{x})g(\vec{x}) - f(\vec{x})\nabla g(\vec{x})}{g^2}$

As in Calculus I, it is very nice to know when and where a function is continuous. The following theorem answers that question in both cases.

**Theorem 4.** From Calculus I, if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x$ then $f$ is continuous at $x$. In Calculus III, if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}$, then $f$ is continuous at $\vec{x}$.

The next theorem is a very important one. In Calculus I, we first computed derivatives using the definition and then proved rules to help us differentiate more complex functions; we do the same here. We won’t prove this theorem, but it gives a very easy way to compute directional derivatives by using the dot product and the gradient of the function.

**Theorem 5.** $D_{\vec{v}} f(\vec{u}) = \nabla f(\vec{u}) \cdot \vec{v}$ for any $\vec{u}, \vec{v} \in \mathbb{R}^3$ where $|\vec{v}| = 1$.

**Problem 71.** Redo Problem 67 using Theorem 5.

To date we have studied the derivatives of functions from $\mathbb{R}$ to $\mathbb{R}$, from $\mathbb{R}$ to $\mathbb{R}^2$, and from $\mathbb{R}^2$ to $\mathbb{R}$. Of course there is nothing special about $\mathbb{R}^2$ here. We might as well have studied $\mathbb{R}^n$ as all the derivatives would follow the same rules. Now let’s consider the derivative of a function, $f : \mathbb{R}^2 \to \mathbb{R}^2$.

**Definition 26.** If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is any vector valued function of two variables defined by $f(x,y) = (u(x,y), v(x,y))$ then the derivative of $f$ is given by

$$Df = \begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix}$$

**Problem 72.** Compute the derivative of $f(x,y) = (x^2 \sin(xy), \frac{e^y}{\tan(x)})$.

Because of the number of different domains and ranges of functions we are studying, we have several variations of the chain rule. Before we begin, I would like to take this opportunity to apologize for the number of notations used by mathematicians, physicists, and engineers for derivatives, partial derivatives, total derivatives, gradients, Laplacians, etc. There are a number of notations and all are convenient at one time or another. I attempt to adhere for the most part to the functional notation for derivatives ($f_1, f_x, f_y$, etc.), but Leibnitz notation, $\frac{df}{dx}$, is a convenient notation as well. Table 1 illustrates my preferred notations, where $L(\mathbb{R}^2, \mathbb{R}^2)$ denotes the set of all 2x2 matrices.

**Note to Instructor (I)** We spend a fair amount of time demonstrating the chain rule in various forms via examples, emphasizing that while it may be written differently due to the different notations that we use for the derivative of functions with differing domains, there is really only one chain rule: $(f \circ g)' = (f' \circ g) \cdot g'$ where the ' and · must be interpreted correctly depending on the domains of $f$ and $g$. And each time the chain rule is used in class, we reiterate this theme.
Table 3.1: Derivative Notation

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f : \mathbb{R} \to \mathbb{R} )</td>
<td>( f' : \mathbb{R} \to \mathbb{R} )</td>
</tr>
<tr>
<td>( f : \mathbb{R}^2 \to \mathbb{R} )</td>
<td>( \nabla f : \mathbb{R}^2 \to \mathbb{R}^2 )</td>
</tr>
<tr>
<td>( f : \mathbb{R}^2 \to \mathbb{R} )</td>
<td>( Df : \mathbb{R}^2 \to \mathbb{L}(\mathbb{R}^2, \mathbb{R}^2) )</td>
</tr>
</tbody>
</table>

First we do a Calculus I example, letting \( h(x) = \cos(\sqrt{x^3 - 3}) \) and breaking \( h \) into \( f \circ g \) to clearly demonstrate the chain rule. Second, we do a more complicated composition, letting \( f(x,y) = e^{xy} \) and \( \overrightarrow{g}(t) = (t^2 + 1, 3t^3) \) and computing \( (f \circ \overrightarrow{g})' \) in two ways. We compose the two functions and compute the derivative and then we use Theorem 7. In one final example, we demonstrate both ways, letting \( f(x,y) = \sin(x^2 + y) \) and \( \overrightarrow{g}(s,t) = (s^2t, s + t) \) and computing \( \nabla(f \circ \overrightarrow{g}) \) by first composing and then taking the derivative and by applying Theorem 8. 

The chain rule you learned in Calculus I, which is stated next, applies to each of the types of functions we just discussed; that is, given any two functions with domains so that their composition actually makes sense and so that they are differentiable at the appropriate places, we can compute their derivative using the same chain rule that you learned in Calculus I with one warning. When the domains and ranges of the functions change, the derivatives change. Thus, in the following statement, depending on the domain of \( f \), sometimes \( f' \) means the derivative of a parametric curve, but sometimes it means the gradient of \( f \), \( \nabla f \), and sometimes it means the matrix of derivatives of \( f \). The same holds for the derivative of \( g \). And finally, the symbol \( \cdot \) might mean multiplication or the dot product or matrix multiplication. You’ll know from context which one. The point is to realize that no matter how many different notational ways we have of writing the chain rule, it always boils down to this one.

**Theorem 6. Chain Rule** If \( f, g : \mathbb{R} \to \mathbb{R} \) are differentiable functions then

\[
(f \circ g)' = (f' \circ g) \cdot g'
\]

or with the independent variable displayed,

\[
(f \circ g)'(x) = f'(g(x)) \cdot g'(x).
\]

Next we state the chain rule for differentiating functions that are the composition of a function of two variables with a planar curve. Notice that this theorem is exactly the same as the original theorem, but restated for functions with different domains. Because \( f' : \mathbb{R}^2 \to \mathbb{R} \) we replace the \( f' \) from the previous theorem with \( \nabla f \) and because \( \overrightarrow{g} : \mathbb{R} \to \mathbb{R}^2 \) we replace the \( g' \) in the previous theorem with \( \overrightarrow{g}' \). Thus the theorem still says (in English) that “the derivative of \( f \) composed with \( g \)” is the (derivative of \( f \)) evaluated at \( g \) times the derivative of \( g \).”

**Theorem 7. Chain Rule** If \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( \overrightarrow{g} : \mathbb{R} \to \mathbb{R}^2 \) are both differentiable, then

\[
(f \circ \overrightarrow{g})' = ((\nabla f) \circ \overrightarrow{g}) \cdot \overrightarrow{g}'.
\]

Writing this with the independent variable \( t \) in place we could write:

\[
(f \circ \overrightarrow{g})'(t) = (\nabla f) \left( \overrightarrow{g}'(t) \right) \cdot \overrightarrow{g}'(t).
\]

On the right hand side of the last line of the Chain Rule, we have the composition of \( \nabla f \) with \( g(t) \). Because we are multiplying vectors, “\( \cdot \)” represents dot product and not multiplication.

**Problem 73.** Let \( f(x,y) = x^2 - 3y^2 \) and \( \overrightarrow{g}(t) = (2, 3) + (4, 5)t \). Compute \( (f \circ \overrightarrow{g})' \) both by direct composition and by using Theorem 7.
Problem 74. Compute \((w \circ \overrightarrow{g})'\) where \(w(x,y) = e^x \sin(y) - e^y \sin(x)\) and \(\overrightarrow{g}(t) = (3, 2)t\). Write a complete sentence that says what line \((w \circ \overrightarrow{g})'(-1)\) is the slope of.

Problem 75. Redo the following problems using this theorem and paying special attention to the use of unit vectors in both cases.

2. Problem 29.

Next we state the chain rule for differentiating functions that are the composition of a function of several variables with a function from the plane into the plane. Notice that this theorem is exactly the same as the original theorem, but restated for functions with different domains. Because \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) we replace the \(f'\) from the original theorem with \(\nabla f\) and because \(\overrightarrow{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) we replace the \(g'\) in the previous theorem with \(D \overrightarrow{g}\). Thus the theorem still says that “the derivative of \((f \text{ composed with } g)\) is the (derivative of \(f\)) evaluated at \(g\) times the derivative of \(g\)” Because \(D \overrightarrow{g}\) is a matrix, the right hand side of this is now a vector times a matrix.

Note to Instructor (I) After we have seen several chain rule problems at the board, we attempt to take some of the mystery out of the notation. Suppose \(L : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) and \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\). Then the derivatives of \(f\) with respect to the first and second variables are:

\[
(f(L(s,t)))_s = \nabla f(L(s,t)) \cdot L_s(s,t) \quad \text{and} \quad (f(L(s,t)))_t = \nabla f(L(s,t)) \cdot L_t(s,t).
\]

If you don’t like subscripts representing the derivatives, we can eliminate these.

\[
\frac{\partial}{\partial s} (f(L(s,t))) = (\nabla f(L(s,t))) \cdot \frac{\partial}{\partial s} L(s,t) \quad \text{and} \quad \frac{\partial}{\partial t} (f(L(s,t))) = (\nabla f(L(s,t))) \cdot \frac{\partial}{\partial t} L(s,t)
\]

Eliminating the independent variables, and using the Leibnitz notation, we have:

\[
\frac{\partial f}{\partial s} = (\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y}) \cdot (\frac{\partial x}{\partial s} \cdot \frac{\partial y}{\partial s}) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}
\]

and

\[
\frac{\partial f}{\partial t} = (\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y}) \cdot (\frac{\partial x}{\partial t} \cdot \frac{\partial y}{\partial t}) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

Theorem 8. Chain Rule If \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) and \(\overrightarrow{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) are both differentiable, then

\[
\nabla (f \circ \overrightarrow{g}) = \left(\nabla f \circ \overrightarrow{g}\right) \cdot D \overrightarrow{g}.
\]

Writing this with the independent variables displayed,

\[
\nabla \left(f(g(s,t))\right) = \left(\nabla f\right)(g(s,t)) \cdot (D \overrightarrow{g})(s,t).
\]

Problem 76. Let \(f(x,y) = 2x^2 - y^2\) and \(\overrightarrow{g}(s,t) = (2s + 5t, 3st)\). Compute \(\nabla (f \circ \overrightarrow{g})\) in two ways. First, compute by composing and then taking the derivative. Second, apply Theorem 8.

Problem 77. Let \(w(x,y) = \ln(x+y) - \ln(x-y)\) and \(g(s,t) = (te^s, -e^{2t})\). Compute \(\nabla (w \circ \overrightarrow{g})\) in two ways. First, compute by composing and then taking the derivative. Second, apply Theorem 8.
Problem 78. Prove or give a counterexample to the statement that \( f'(\overrightarrow{t}(t)) = (f(\overrightarrow{t}(t)))' \) for all differentiable functions \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( \overrightarrow{t} : \mathbb{R} \to \mathbb{R}^2 \).

The next problem tells us something important. If you are sitting on some function in three space and you are trying to decide what direction you should travel to go uphill at the steepest possible rate, then the gradient tells us this direction.

Problem 79. Use Problem 18 and Theorem 5 to show that \( D\overrightarrow{v}f(\overrightarrow{u}) \) is largest when \( \overrightarrow{v} = \nabla f(\overrightarrow{u}) \) and smallest when \( \overrightarrow{v} = -\nabla f(\overrightarrow{u}) \).

Note to Instructor (I): By now we have been foreshadowing relations and surfaces for some time. This is a central theme behind the way the course is run. Rather than introducing a new concept, such as surfaces, just in time to introduce planes, we have foreshadowed the idea for some time. When the concept becomes necessary, it is already understood at a conceptual, intuitive level. We first graph \( y = x^2 \) and \( x = y^2 \). One is a function, the other is a relation (which we define as a set of points in the plane). We recall that every function is a relation, but not every relation is a function. Then we move to three-space. We rewrite \( f(x,y) = x^2 + y^2 \) as \( z = x^2 + y^2 \) and remind them that this short hand really means \( \{(x,y,z) : x^2 + y^2 = z\} \). We ask, “What equation would result in the same shape, but rotated so that it is symmetric about the x-axis?” The class (sometimes prompted by a game of hangman if they are not forthcoming) suggests \( x = y^2 + z^2 \). Then we graph a few simple surfaces in three-space and ask which are functions: \( z = 2x, y = x^2, z = \sin(x), y = \sin(z) \).

Surfaces Earlier we gave a list of the types of functions we have studied so far. Of course, there is only one definition for a function, so we are really talking about functions with different domains and ranges as was illustrated by the need for different chain rules for functions with different domains. In linear algebra, we see functions with domain the set of matrices (the determinant function) and \( T(f) = \int_0^1 f(x) \, dx \) is a function from the set of all continuous functions into the real numbers.

In earlier courses, you studied not only functions, but relations such as \( x^2 + y^2 = 1 \). What was the difference? Well, a function has a unique \( y \) for each \( x \) while a relation may have several \( y \) values for a given \( x \) value. In three dimensions a function will have a unique \( z \) for a given coordinate pair, \( (x,y) \). When one has multiple \( z \) values for a given \( (x,y) \) value, we call it a surface. We have thus far studied mostly functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \) such as \( f(x,y) = x^2 + y^2 \) or \( f(x,y) = ye^x \), but surfaces are equally important. Surfaces are to functions in three-space as relations were to functions in two-space.

Problem 80. Sketch these functions in \( \mathbb{R}^3 \).

1. \( f(x,y) = |x| - |y| \)
2. \( h(x,y) = \sqrt{xy} \)
3. \( g(x,y) = \sin(x) \)
4. \( i(x,y) = 2x^2 - y^2 \)

Problem 81. Sketch these surfaces in \( \mathbb{R}^3 \).

1. \( x^2 + y^2 + z^2 = 1 \)
2. \( y^2 + z^2 = 4 \)
3. \( x^2 - y + z^2 = 0 \)
4. \(|y| = 1 \)

Surfaces may be expressed as \( F(x,y,z) = k \); For example, in the previous problem, we could rewrite these as

1. \( F(x,y,z) = 1 \) where \( F(x,y,z) = x^2 + y^2 + z^2 \).
2. \( F(x,y,z) = 4 \) where \( F(x,y,z) = y^2 + z^2 \).
3. \( F(x,y,z) = 0 \) where \( F(x,y,z) = x^2 - y + z^2 \).
4. \( F(x,y,z) = 1 \) where \( F(x,y,z) = |y| \).

**Note to Instructor (I)** This mini-lecture gives away too much in my opinion, but there is simply too much material in the syllabus for them to discover everything so we simply show them how to compute tangent planes to functions and surfaces.

Suppose we want to find the tangent plane to a function such as \( f(x,y) = x^2 + y^2 \) at the point, \( p = (2,3,13) \). We know that to find the equation of a plane it is enough to have a vector that is orthogonal to the plane and a point on the plane. We already have the point \( p \), so all we need is an orthogonal vector. Our strategy will be to find two vectors that are tangent to the surface at the point \( p \) and thus are in the plane. Their cross product will yield the orthogonal vector. We know that to find the equation of a plane it is enough to have a vector that is orthogonal to the plane and pointing in the x-direction. Similarly, \( (0,1,6) \) is parallel to our plane and points in the y-direction. Taking the cross product will yield our vector that is perpendicular to the function and the plane at \( p \). This vector is \((-4,-6,1)\) so our plane is \((x - 2, y - 3, z - 13) \cdot (-4, -6, 1) = 0 \).

To demonstrate how to find tangent planes to surfaces, we’ll first note that every function may be written as a surface, where surfaces will always be written as \( F(x,y,z) = k \) where \( F : \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( k \in \mathbb{R} \). Thus our function, \( f(x,y) = x^2 + y^2 \) may be written as \( F(x,y,z) = 0 \) where \( F(x,y,z) = x^2 + y^2 - z \). Note that our point \( p = (2,3,13) \) is on this surface since \( F(2,3,13) = 0 \). Now suppose that \( c(t) = (x(t), y(t), z(t)) \) is a curve on this surface. Then \( F(c(t)) = 0 \) so \( (x(t))^2 + (y(t))^2 - z(t) = 0 \). Taking the derivative of both sides we have \( 2x(t)x'(t) + 2y(t)y'(t) - z'(t) = 0 \). Rewriting this as a dot product we have \((2x(t), 2y(t), -1) \cdot (x'(t), y'(t), z'(t)) = 0 \) or \( \nabla F(c(t)) \cdot c'(t) = 0 \). Thus for any curve \( c \) on the surface, the tangent to the curve is in the tangent plane and is orthogonal to the gradient of \( F \) evaluated at \( c(t) \). The tangent plane may now be computed by simply using the fact that \( \nabla F(2,3,13) \) is an orthogonal vector to the plane and \((2,3,13) \) is a point on the plane. Of course it is the same plane we found when we worked it the other way, treating \( f \) as a function. Once this is complete, we point out that there is nothing special about our surface or curve. For any surface, \( F(x,y,z) = k \) and any curve \( c \) on the surface, we have that \( F(c(t)) = k \) so \( \nabla F(c(t)) \cdot c'(t) = 0 \) so \( \nabla F(c(t)) \) must always be a vector orthogonal to the tangent plane to \( F \).

**Tangent Planes to Functions**

**Problem 82.** Find the equation of the plane tangent to the function \( f(x,y) = 25 - x^2 - y^2 \) at the point \((3,1,15)\). Sketch the graph of the function and the plane.

**Problem 83.** Find the equation of the plane tangent to the function \( f(x,y) = \sqrt{x^2 + y^2} \) at the point \((3,4,5)\). Sketch the graph of the function and the plane.
Note to Instructor (I) Here we spend the time to prove that the gradient of a surface is orthogonal
to the surface. Let’s look at an example of a tangent plane to a surface in three-space. Suppose
\( F(x, y, z) = x^2 + y^2 + z^2 \) and consider the surface (a sphere), \( F(x, y, z) = 1 \). Consider also the curve
on the surface given by \( \vec{r}(t) = (\frac{1}{2}\cos(t), \frac{1}{2}\sin(t), \frac{\sqrt{3}}{2}) \). First, let’s verify that \( \vec{r} \) is actually a curve
on the surface by computing the composition of \( F \) and \( \vec{r} \). We obtain,
\[
F(\vec{r}(t)) = \left(\frac{1}{2}\cos(t)\right)^2 + \left(\frac{1}{2}\sin(t)\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1,
\]
so \( \vec{r} \) is a curve on the surface. Now, consider the point on the surface and on the curve at time
\( t = \frac{\pi}{4} \), or \( \vec{r}(\frac{\pi}{4}) \). Can we find a vector that is orthogonal to this surface and this curve at this
point? For the sphere, the vector originating at the origin and passing through the point \( \vec{r}(\frac{\pi}{4}) \)
is orthogonal to the surface, so \( (\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}) \) is both a point on the surface and a vector that is
orthogonal to the surface.

Next, let’s do something that does not immediately appear related. Let’s compose the gradient
of the surface, \( \nabla F \), with the curve \( \vec{r} \).
\[
\nabla F(x, y, z) = (2x, 2y, 2z)
\]
thus
\[
\nabla F(\vec{r}(t)) = (\cos(t), \sin(t), \sqrt{3}).
\]
Now let’s compute \( \vec{r}' \).
\[
\vec{r}'(t) = (-\frac{1}{2}\sin(t), \frac{1}{2}\cos(t), 0).
\]
Imagine these two graphically. Since \( \vec{r} \) is a curve on the surface, \( \vec{r}' \) represents a direction tangen-
tial to the curve, \( r \). And \( \nabla F \) represents the direction in which \( F \) increases the most rapidly. Does it
seem natural that these two directions would be orthogonal? Let’s check.
\[
\nabla F(\vec{r}(t)) \cdot \vec{r}'(t) = (\cos(t), \sin(t), \sqrt{3}) \cdot (-\frac{1}{2}\sin(t), \frac{1}{2}\cos(t), 0) = 0.
\]
As you might expect, this is not a unique phenomena. Any time you have a surface \( F(x, y, z) = k \)
and a curve \( \vec{r} \) on that surface, you will have that \( \nabla F(\vec{r}(t)) \cdot \vec{r}'(t) = 0 \). This says that \( \nabla F(\vec{r}(t)) \)
is orthogonal to \( \vec{r}'(t) \) for every curve \( \vec{r} \) on the surface, \( F \). In other words, \( \nabla F \) is an orthogonal
vector to the surface. This is how we will define the tangent plane. But first, let’s show that it works
in general. Suppose \( \vec{r}(t) = (x(t), y(t), z(t)) \) is a parametric equation. Suppose \( F(x, y, z) = k \) is a
surface.
Composing:
\[
F(\vec{r}(t)) = k
\]
Differentiating:
\[
\frac{d}{dt} F(\vec{r}(t)) = \frac{d}{dt} k
\]
\[
\frac{d}{dt} F(\vec{r}(t)) = 0
\]
\[
\nabla F(\vec{r}(t)) \cdot \vec{r}'(t) = 0
\]
We conclude that: \( F(\vec{r}(t)) \perp \vec{r}'(t) \) which is the tangent to \( \vec{r} \).
Summarizing, if we are sitting on the point $p$ on the surface $F(x,y,z) = k$, then every curve $R$ that passes through $p$ is $\perp$ to $\nabla F(p)$. The next definition uses this observation to define the tangent plane.

As an example, we compute the tangent plane to $f(x,y) = x^2 + y^2$ at the point $(1,2,13)$. To use our definition of the tangent plane to a surface, we first rewrite $f$ as a surface. Rewriting $f$ as $z = x^2 + y^2$ and yields $x^2 + y^2 - z = 0$ so if we define $F(x,y,z) = x^2 + y^2 - z$ then we have rewritten $f$ as the surface, $F(x,y,z) = 0$. The gradient of the surface is orthogonal to the surface, so $(2x, 2y, -1)$ is orthogonal to the surface. Evaluating at $(1,2,5)$ gives us our orthogonal vector, $(1,4, -1)$. Now we know a vector orthogonal to the plane and a point on the plane and we are done! •

**Tangent Planes to Surfaces**

**Definition 27.** If $F(x,y,z) = k$ is a surface, then the **tangent plane** to $F$ at $u = (x,y,z)$ is the plane passing through $u$ with normal vector, $\nabla F(u)$.

**Problem 84.** Find the equation of the tangent plane to the surface $x^2 - 2y^2 - 3z^2 + xyz = 4$ at the point $(3,-2,-1)$.

**Problem 85.** Find the equations of two lines perpendicular to the surface in the previous problem at the point $(3,-2,-1)$ on the surface.

**Problem 86.** Find the equations of two lines perpendicular to the surface $z + 1 = ye^y \cos(z)$ at the point $\vec{p} = (1,0,0)$. 
Chapter 4

Optimization and Lagrange Multipliers

Probably the most applied concept in all of calculus is finding the maxima and minima of functions. In industry, these can be problems from engineering, such as trying to design an airplane wing that yields the maximum lift and stability while at the same time minimizing the drag coefficient. This way, we build a plane that flies easily while using less fuel. Since planes measure fuel consumption in gallons per second, a small change in wing design can result in considerable profit for the company (and a big raise for you). Have you noticed the addition of the upward turned tips at the ends of the airplane wings in recent years?

In the financial markets, mutual funds are sets of stocks. People may buy shares of the fund instead of buying shares of individual stocks. Mutual fund managers (and their clients) want to choose groups of stocks that will increase in value and make them (and their clients) rich. Thus, a fund manager wants to design a mutual fund that will maximize profits for the investors, but because investors fear volatility (large fluctuations in the value of their portfolios), the manager also wants to minimize the volatility of the mutual fund. This is an example of an optimization problem with what is called a constraint because you want to maximize profit but are constrained by the customers’ concerns about volatility.

Mathematicians have spent a considerable amount of time in industry working on both of these interesting problems that are representative of “real-world” applications. In mathematics, the difference between being able to understand or apply a formula to such a problem and the ability to derive or create your own formulas for the problems is the difference between working as an engineer, mutual fund manager, or biologist on a team (a wonderful job in it’s own right) and working in a think tank such as Bell Labs or Los Alamos or MSRI (the Mathematical Sciences Research Institute) where you are tackling the problems that no one can solve and creating the mathematics that will be implemented by the teams in industry.

In Calculus I you solved optimization or max-min problems by setting the derivative of a function to zero to tell you where the rate of change (or slope) of the function was zero. In Calculus III we do exactly the same thing. Except that instead of solving \( f' = 0 \) we are solving \( \nabla f = 0 \) and we are seeking a horizontal tangent plane instead of a horizontal tangent line. The procedures are very similar and both find the point in the domain of the functions where potential maxima and minima are attained.

**Definition 28.** A **critical point** of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is any point \( \vec{x} \in \mathbb{R}^2 \) where \( \nabla f(\vec{x}) \) is zero or undefined.

Since \( \nabla f = 0 \) translates to \( \nabla f(x,y) = (f_x(x,y), f_y(x,y)) = (0,0) \) we are seeking points \( (x,y) \) in the plane where both \( f_x(x,y) = 0 \) and \( f_y(x,y) = 0 \), or where at least one is undefined. Review the definitions for “local minimum, local maximum, and inflection point” for functions from \( \mathbb{R} \) to \( \mathbb{R} \) (i.e. from Calculus I). You may look these up in a book or on the web.
Definition 29. An \( \varepsilon \)-neighborhood of the point \((s, t)\) in \(\mathbb{R}^2\) is the set of all points in \(\mathbb{R}^2\) that are a distance of less than \(\varepsilon\) away from \((s, t)\).

Definition 30. If \(f : \mathbb{R}^2 \to \mathbb{R}\) and \(x\) is in the domain of \(f\), then we say \((\vec{x}, f(\vec{x}))\) is a local minimum if \(f(\vec{x}) < f(\vec{y})\) for every \(\vec{y}\) in some \(\varepsilon\)-neighborhood of \(\vec{x}\).

Local maximum is defined similarly. A function may have infinitely many local minima and maxima.

Problem 87. Find the minimum of \(f(x, y) = x^2 - 2x + y^2 - 4y + 5\) in two ways.

1. Complete the square to write \(f(x, y) = (x - a)^2 + (y - b)^2\).
2. Set \(\nabla f(x, y) = 0\) and solve for \((x, y)\).

Problem 88. Let \(f(x, y) = 8y^3 + 12x^2 - 24xy\).

1. Find all critical points of \(f\).
2. Sketch \(f\) using any software to verify your answers.
3. Use any software to solve the whole problem. In other words, use software to compute your partial derivatives and solve for the roots of these partial derivatives.

Recall in Calculus I that if \(x\) were a critical point for \(f\) then \((x, f(x))\) could have been a minimum, maximum, or an inflection point for \(f\). Inflection points were critical points where the function \(f\) switched concavity (i.e. where the first derivative is zero and the second derivative switches signs). In Calculus III the analogous points are critical points that are neither maxima nor minima and we call these saddle points.

Definition 31. If \(f : \mathbb{R} \to \mathbb{R}^2\) is differentiable at \(x\) then we say \((\vec{x}, f(\vec{x}))\) is a saddle point if \(\nabla f(\vec{x}) = 0\) and no matter how small an \(\varepsilon > 0\) we choose, there are points \(\vec{y}\) and \(\vec{z}\) in the \(\varepsilon\)-neighborhood of \(\vec{x}\) so that \(f(\vec{y}) < f(\vec{z})\) and \(f(\vec{y}) > f(\vec{z})\).

This definition of a saddle point says that if we are at a saddle point and we decide to walk away from it, then there are paths away from the critical point along which \(f\) increases and paths away from the point along which \(f\) decreases. Given a critical point, how do we determine if it was a maximum, minimum, or saddle point? That is, how do we classify the critical points of \(f\)? How did we do it in Calculus I? We restate the Second Derivative Test which is a slick way to classify the the critical points of the single-variable, real-valued functions from Calculus I. Notice how nicely it parallels the next theorem for classifying the multi-variable, real-valued functions of Calculus III.

Theorem 9. Second Derivative Test I If \(f : \mathbb{R} \to \mathbb{R}\) is differentiable and \(f'(x) = 0\) then

1. If \(f''(x) > 0\) then \((x, f(x))\) is a minimum.
2. If \(f''(x) < 0\) then \((x, f(x))\) is a maximum.
3. If \(f''\) switches signs at \(x\) then \((x, f(x))\) is an inflection point.

Here is a sweet theorem for classifying critical points of functions of two variables.

Theorem 10. Second Derivative Test II If \(f : \mathbb{R}^2 \to \mathbb{R}\) and \(\nabla f(\vec{u}) = 0\) and \(D = f_{xx}(\vec{u})f_{yy}(\vec{u}) - (f_{xy}(\vec{u}))^2\), then
1. \( (\nabla f, \vec{u}') \) is a local min if \( D > 0 \) and \( f_{xx}(\vec{u}') > 0 \).
2. \( (\nabla f, \vec{u}') \) is a local max if \( D > 0 \) and \( f_{xx}(\vec{u}') < 0 \).
3. \( (\nabla f, \vec{u}') \) is a saddle point if \( D < 0 \).
4. No information if \( D = 0 \).

Note to Instructor (I) As a class exercise, we attempt to compute and classify all critical points of a function \( f \) with gradient \( \nabla f(x, y) = (xy - 2x - 3y + 6, yx - y + 4x - 4) \). The problem is intentionally misleading. We quickly solve for the critical points, but when we attempt to classify them by computing the determinant of the Hessian, \( \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \) we find that \( f_{xy} \neq f_{yx} \). We ask, “How could this happen? Don’t we have a theorem that says that \( f_{xy} = f_{yx} \)” After some thought, someone will suggest that perhaps there is not a function with that gradient and we integrate the first component of our gradient with respect to \( x \) and the second component of the gradient with respect to \( y \) to demonstrate that in fact, there is no function having this gradient. This provides an example for solving two equations in two variables and foreshadows the proof technique needed later to show that a vector field \( f = (P, Q) \) is conservative if \( P_y = Q_x \).

Problem 89. Compute the critical points of \( f(x, y) = xy^2 - 6x^2 - 3y^2 \) and classify these critical points as local maxima, local minima, or saddle points.

Problem 90. Compute and classify the critical points of \( f(x, y) = xy + \frac{2}{x} + \frac{4}{y} \).

Problem 91. Compute and classify the critical points of \( f(x, y) = e^{-(x^2+y^2-4y)} \).

The next problem reminds you of an important aspect of max/min problems from Calculus I. If you wanted the local maxima and minima of a function like \( f(x) = x^2 \) on \([-1, 3]\) then you checked not only the places where \( f' = 0 \) but also the values of \( f \) at the endpoints (boundary) of the interval. The same must be done for functions of several variables. When you applied this technique, you were applying the Extreme Value Theorem.

Theorem 11. Extreme Value Theorem for functions of one variable If \( f : [a, b] \rightarrow \mathbb{R} \) is a differentiable function then \( f \) has a maximum and a minimum on this interval.

The corresponding theorem for real-valued functions of two variables requires at least an intuitive idea of the notions of what it means for a set to be closed and bounded. A set \( S \) is bounded if there is a number \( M \) so that \( |x| \leq M \) for all \( x \in S \). A set \( S \) is closed if it contains its boundary points. Think of the open and closed intervals. An open interval does not contain its end points, the points on the boundary of the set. A closed interval does contain its boundary points. The set of all points \((x, y)\) satisfying \( x^2 + y^2 \leq 9 \) is closed, while the set of all points \((x, y)\) satisfying \( x^2 + y^2 < 9 \) is not closed.

Theorem 12. Extreme Value Theorem for functions of two variables If \( M \) is a closed and bounded set and \( f : M \rightarrow \mathbb{R} \) is a differentiable function then \( f \) has a maximum and a minimum on \( M \).

Definition 32. A set \( S \) in \( \mathbb{R}^2 \) is closed if it contains all its boundary points.

Note to Instructor (I) This seminal lecture reinforces composition of functions, graphing and differentiation while introducing Lagrange multipliers. Our goal is to optimize \( f(x, y) = 2x^2 + y^2 + 1 \) subject to \( g = 0 \) where \( g(x, y) = x^2 + y^2 - 1 \) and we solve the problem in four distinct ways:
1. We solve \( g = 0 \) for \( y \), substitute the result into \( f \) and optimize the resulting function of one variable.

2. We parameterize \( g = 0 \) as \( c(t) = (\cos(t), \sin(t)) \) and optimize the composition \( f \circ c \).

3. We graph \( f \) and locate the extrema visually.

4. We apply Lagrange Multipliers, saving the theory of why it works for later.

Because we solve for \( y \), Method 1 finds only two of the critical points unless we go back and solve \( g = 0 \) for \( x \) and repeat the process. I don’t tell them this. We simply move on to solve it another way, confident that our two extrema are correct. Method 2 then surprises us when we find four critical points. At this point, we may revisit Method 1, asking how we can find the other two. Whether we go back or not depends on the students’ questions. Method 3, graphing, then verifies that there are exactly four solutions since the graph of \( f \) constrained by \( g \) looks like a Pringle potato chip with two maxima and two minima. At this point, someone usually realizes that we would have found all critical points using Method 1 if we had solved for \( x \) as well but all these methods seem onerous even for such a simple problem.

Method 2 foreshadows the proof that the method of Lagrange Multipliers works. Consider functions \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( g : \mathbb{R}^2 \to \mathbb{R} \), both differentiable everywhere. If we parameterize \( g = 0 \) as \( r(t) = (x(t), y(t)) \) then our critical points will occur at any point \( t \) in the domain of \( r \) where \((f \circ r)' = 0 \) or where \( \nabla f(r(t)) \cdot r'(t) = 0 \). Since \( r \) is a parametrization of \( g = 0 \), we know that \( g(r(t)) = 0 \) and thus \( \nabla g(r(t)) \cdot r'(t) = 0 \). Thus, both \( \nabla f(r(t)) \) and \( \nabla g(r(t)) \) are orthogonal to the same vector \( r'(t) \) and therefore they must be scalar multiples of one another. Hence we have Lagrange’s Theorem, that solving the system, \( \nabla f = \lambda \nabla g \) and \( g = 0 \) is sufficient to find all critical points.

**Problem 92.** Let \( T(x, y) = 2x^2 + y^2 - y \) be the temperature at the point \( (x, y) \) on the circular disk of radius 1 centered at \((0,0)\).

1. Find the critical points of \( T \).

2. Find the minimum and maximum of \( T \) over the circle (perimeter), \( x^2 + y^2 = 1 \) by parameterizing the circle.

3. Find the minimum and maximum of \( T \) over the disk, \( x^2 + y^2 \leq 1 \).

For the next problem, you will need to parameterize each of the four line segments that form the line and check the maximum and minimum of the function over not only the interior of the square, but also over each of the four lines.

**Problem 93.** Find the maximum and minimum of \( f(x, y) = 2x^2 - 3y^2 + 10 \) over the square disk, \( S = \{(x, y)|0 \leq x \leq 3, 2 \leq y \leq 4\} \).

**Problem 94.** Find all maxima and minima of \( f(x, y) = x^2 - y^2 + 4y \) over the rectangle \( R = \{(x, y)|-1 \leq x \leq 1, -3 \leq y \leq 3\} \).

In Calculus I an \( n^{th} \) degree polynomial will have at most \( n - 1 \) critical points. What about in Calculus III? Here is a \( 6^{th} \) degree polynomial which has far more than 5 critical points. All of these may be found by hand. Grab some free graphing software from the web to help verify these.

**Problem 95.** Find all thirteen critical points of \( f(x, y) = x^3y^3 - x^3y - 3xy^3 + 3xy + 1 \).

**Problem 96.** Let \( f(x, y) = x^3 - y^3 \) and \( p = (2,4) \). Find the direction in which \( f \) increases the most rapidly.
Problem 97. Ted is riding his mountain bike and is at altitude (in feet) of $A(x,y) = 5000e^{-(3x^2+y^2)/100}$. What is my slope of descent or ascent if I am riding in the direction $(-1,1)$ starting at the point $(10,10,5000e^{-4})$? In what direction should I travel to ascend the most rapidly? To descend the most rapidly?

Problem 98. Let $T(x,y,z) = 10x^2 + y^2 + z^2$ and $\vec{r}(t) = (t\cos(\pi t), t\sin(\pi t), t)$. If $T(x,y,z)$ represents the temperature in space at the point $(x,y,z)$ and $\vec{r}(t)$ represents Ted’s position at time $t$, then compute $(T \circ \vec{r})'(3)$ and explain what this number represents in a complete sentence. To check your answer, compute it in two ways. First compose the functions and take the derivative. Second, use the chain rule.

We have tackled optimization problems before when we found maxima and minima of functions like $f(x,y) = x^2 + x^2y + y^2 + 4$ from Problem 87. We have also sought maxima and minima of curves (or paths) on surfaces when we found the maxima and minima of $T(x,y) = 2x^2 + y^2 - y$ over the circle $x^2 + y^2 + 1$ as in Problem 92. Problem 92 is called a constrained optimization problem because we want a maxima or a minima of $T$ subject to the constraint that it is above the unit circle. The method of Lagrange multipliers is a slick way to tackle constrained optimization problems.

Theorem 13. Lagrange Suppose that $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$. To maximize (or minimize) the function $f$ subject to the constraint $g = 0$ we solve the two equations,

1. $\nabla f(\vec{x}) = \lambda \nabla g(\vec{x})$
2. $g(\vec{x}) = 0$

for $\vec{x}$ and for $\lambda$. The variable $\lambda$ is called the Lagrange multiplier, and $\vec{x}$ is the point at which $f$ is maximized (or minimized).

Note to Instructor (R) At some point another example in class for Lagrange Multiplies is nice. We minimize $f(x,y) = y^2 - x^2$ over the region $g(x,y) \leq 0$ where $g(x,y) = \frac{x^2}{4} + y^2 - 1$. •

Problem 99. Find the max and min of $f(x,y) = x^2 + y^2$ subject to $xy = 3$ in three ways. First use Lagrange multipliers by putting $g(x,y) = xy - 3$ so that the constraint is $g(x,y) = 0$. Second, substitute $y = 3/x$ into the equation and solve. Third, sketch $f$ and identify the portion of $f$ that is above the equation, $xy = 3$.

Problem 100. Find any minima and maxima of $f(x,y) = 4x^2 + y^2 - 4xy$ subject to $x^2 + y^2 = 1$. It may be helpful to eliminate $\lambda$ first.

Problem 101. Find the minimum of $f(x,y,z) = 3x + 2y + z$ subject to $9x^2 + 4y^2 - z = 0$ via Lagrange multipliers.
Chapter 5

Integration

Calculus III continues to parallel Calculus I and here we are at integration. Let's review integration in one variable before we tackle integration in several variables. By now you have been using antiderivatives (and hence the Fundamental Theorem of Calculus) to compute integrals for so long that it is worth remembering the original definition for the definite integral. If \( f: \mathbb{R} \rightarrow \mathbb{R} \) is a function, then we define \( \int_{a}^{b} f(x) \, dx \) as the limit of Riemann sums. If \( f \) is positive, then this is the limit of sums of areas of rectangles. Now we will define our integrals once again as limits of sums, but this time we will take limits of sums of volumes.

First, let's formally restate the definition of the definite integral from Calculus I.

**Note to Instructor (I)** To introduce multi-variable integration, we launch from the roots of Calculus I by reviewing the definition of a the definite integral as the limit of sums of rectangles, and computing \( \int_{0}^{2} x^2 \, dx \) using summation notation and proving that the limit of the sums of the areas of the rectangles is \( \frac{8}{3} \). Now that we truly know what the area is, we recompute it in several ways using the Fundamental Theorem by showing that

\[
\int_{0}^{2} x^2 \, dx = \int_{0}^{2} \int_{0}^{x^2} 1 \, dy \, dx = \int_{0}^{4} \int_{\sqrt{y}}^{4} 1 \, dx \, dy.
\]

To explain the integrand of “1”, we sketch in three-space the solid of height one above the region bounded by \( y = x^2 \), \( y = 0 \), and \( x = 4 \). The volume of this solid equals the area we first computed. We intentionally leave out integrating with respect to \( y \) because this is a forthcoming problem. Finally, we define the double integral as it appears in the notes below as a limit of sums of volumes of right cylinders with square bases.

**Definition 33.** If \([a, b]\) is a closed interval then a partition \( P \) of \([a, b]\) is an ordered sequence \( a = x_0 < x_1 < x_2 < \ldots < x_{n−1} < x_n = b \). The norm or mesh of \( P \) is \( \max\{x_i − x_{i−1} : i = 1, 2, \ldots, n\} \) and is denoted \( \|P\| \).

**Definition 34.** The integrand of \( f \) over \([a, b]\), denoted by \( \int_{a}^{b} f \) is defined (assuming the limit exists) as

\[
\int_{a}^{b} f = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\hat{x}_i) \cdot (x_i − x_{i−1})
\]

where \( \hat{x}_i \in [x_{i−1}, x_i] \) and \( n \) is the number of divisions of the partition \( P \). If the limit does not exist, we say that \( f \) is not integrable.

Of course, this is the limit of the sum of the areas of a collection of rectangles where the width of the rectangles tends toward zero as the mesh of the partition tends toward zero. In Calculus III,
we do the same except that we must partition both the x and y-axes and we are summing volumes over rectangles rather than sum areas over intervals.

**Definition 35.** Given any two sets, A and B,

\[ A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}. \]

In the case where A and B are closed intervals these are simply rectangles in the plane,

\[ [a, b] \times [c, d] = \{ (x, y) \mid x \in [a, b], y \in [c, d] \}. \]

**Definition 36.** If R is a rectangle, say R = [a, b] × [c, d] then a partition, U, of R is a partition of \([a, b], P = \{a = x_0 < x_1 < \ldots < x_n = b\}\) along with a partition of \([c, d], Q = \{c = y_0 < y_1 < y_2 < \ldots < y_m = d\}\). The norm or mesh of U is the largest width of any of the rectangles, \([x_{i-1}, x_i] \times [y_{j-1}, y_j]\) where \(i = 1, 2, \ldots n\) and \(j = 1, 2, \ldots m\).

**Definition 37.** If \(f \in C_R\) (i.e. \(f\) is continuous on the rectangle R) then

\[
\int_R f \, dA = \lim_{\|U\| \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\hat{x}_i, \hat{y}_j)(x_i - x_{i-1})(y_j - y_{j-1})
\]

where

\((\hat{x}_i, \hat{y}_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j].\)

Thankfully, in Calculus III we won’t have to compute any such limits because there is a theorem that states:

\[
\int_R f \, dA = \int_a^d \int_c^b f(x,y) \, dx \, dy
\]

which gives us a straightforward way to compute integrals using anti-derivatives by using the Fundamental Theorem of Calculus from Calculus I that stated:

**Theorem 14.** **Fundamental Theorem of Calculus** If \(f\) is continuous on \([a, b]\) and \(F\) is any antiderivative of \(f\), then \(\int_a^b f = F(b) - F(a)\).

The next problem reminds us that in Calculus I we can integrate with respect to either \(x\) or \(y\) and obtain the same result. These techniques are especially handy in Calculus III.

**Problem 102.** Compute the area bounded by \(y = x^2\), the x-axis, and \(x = 2\) two ways. First, use an integral with respect to \(x\), \(\int_{-\sqrt{2}}^{\sqrt{2}} \, dx\), then use an integral with respect to \(y\), \(\int_{-2}^{2} \, dy\). Sketch a picture to help explain your endpoints of integration.

Most of the theorems that you proved in Calculus I have an analog in Calculus III. The following theorem provides a list of properties for double integrals that won’t surprise you. We won’t state them again, but they also hold for any integrable function \(f : \mathbb{R}^n \to \mathbb{R}\) where \(n > 2\).

When we are integrating over a two dimensional region \(R\), we will write \(\int_R f \, dA\), where the \(A\) reminds us that we are integrating over a region with area. This will be a double integral. When we are integrating over a three dimensional region, we will write \(\int_R f \, dV\), where the \(V\) reminds us that we are integrating over a region with volume. This will be a triple integral.

**Theorem 15.** Suppose \(f\) is integrable on a closed and bounded rectangular region \(R\). Then

1. \(\int_R [f(x,y) + g(x,y)] \, dA = \int_R f(x,y) \, dA + \int_R g(x,y) \, dA\)
2. \( \int_R c[f(x,y)] \, dA = c \int_R f(x,y) \, dA \) where \( c \) is a real number

3. \( \int_R f(x,y) \, dA \geq \int_R g(x,y) \, dA \) if \( f(x,y) \geq g(x,y) \) for all \( (x,y) \in R \)

4. \( \int_R f(x,y) \, dA = \int_{R_1} f(x,y) \, dA + \int_{R_2} f(x,y) \, dA \), where \( R_1 \) and \( R_2 \) are rectangular regions such that \( R_1 \) and \( R_2 \) have no points in common except for points on parts of their boundaries and \( R = R_1 \cup R_2 \)

**Note to Instructor (I)** In order to expedite multiple integrals, we discuss the notations \( dA \) and \( dV \). We also work through several examples of integrals although they are unlikely to all occur in one lecture.

1. We use the fundamental theorem to demonstrate Fubini’s theorem by integrating \( \int_0^3 \int_1^2 x^2 + y^2 \, dx \, dy \) and \( \int_0^3 \int_1^2 x^2 + y^2 \, dy \, dx \) and illustrate graphically the volume that these integrals represent.

2. We graph and find the volume under \( 4x + 3y + 6z = 12 \) and above \( xy \)-plane.

3. We compute \( \int_R x^2 + y^2 \, dA \) over \( R = [0,1] \times [1,2] \)

4. We compute \( \int_R x^2 y \, dA \) over the region \( R \) bounded by \( y = x^2, x = 0 \) and \( y = 4 \).

**Problem 103.** Compute \( \int_1^3 \int_0^2 2x + 3y \, dx \, dy \) and \( \int_0^2 \int_1^3 2x + 3y \, dy \, dx \). This is called “reversing the order of integration.” Sketch the solid that you found the volume of.

**Problem 104.** Compute and compare these three integrals:

1. \( \int_0^{x-2} \int_0^1 2x + 3y \, dy \, dx \)

2. \( \int_0^1 \int_0^{x-2} 2x + 3y \, dx \, dy \)

3. \( \int_0^{y+2} \int_1^0 2x + 3y \, dx \, dy \)

As you can see from the previous two problems, when the limits of integration contain variables care must be taken in reversing the order of integration. When the limits of integration are numbers, we can reverse the order of the integrals and obtain the same result. This is known as Fubini’s Theorem.

**Theorem 16. Fubini’s Theorem** Suppose \( f \) is a continuous function of two variables, \( x \) and \( y \), defined on the rectangle \( R = [a,b] \times [c,d] \). Then

\[
\int_R f(x,y) \, dA = \int_a^b \left[ \int_c^d f(x,y) \, dy \right] \, dx = \int_c^d \left[ \int_a^b f(x,y) \, dx \right] \, dy.
\]

**Problem 105.** Evaluate \( \int_B 4x \ln(y)z \, dV \) over the box \( B = [0,2] \times [1,4] \times [-2,5] \). Now change the order of integration and verify the result. How many choices are there for orders of integration?
Problem 106. Let \( f(x, y) = 4 - x^2 - y^2 \) over the region \( R = [0, 1] \times [0, 1] \). Compute \( \int_0^1 \int_0^1 f(x, y) \, dy \, dx \). Sketch the solid that this integral represents the volume of.

Problem 107. Let \( f(x, y) = 4 - x^2 - y^2 \). Write an integral for the volume of the solid that that is bounded by \( f \) and by the planes \( z = 0, x = 0, x = 2, y = 0, \) and \( y = 2 \). Compute this integral.

Problem 108. Compute \( \int_0^1 \int_0^1 \frac{y}{(xy + 1)^2} \, dx \, dy \).

Problem 109. Compute \( \int_0^{\ln(3)} \int_0^1 xye^{xy^2} \, dy \, dx \).

Problem 110. Compute \( \int_R \sin(x+y) \, dA \) where \( R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \).

Problem 111. Compute \( \int_R xy \sqrt{1 + x^2} \, dA \) where \( R = [0, \sqrt{3}] \times [1, 2] \).

Thus far we have integrated primarily over domains that were rectangles, but we can also integrate over more general domains.

Problem 112. Let \( f(x, y) = 4x + 2y \).

1. Sketch the region in the xy-plane bounded by by \( x = 2, x = 4, y = -x, y = x^2 \).

2. Compute the volume of the solid below \( f \) and above this region.

Problem 113. Evaluate \( \int_S xy \, dA \) where \( S \) is the region bounded by \( y = x^2 \) and \( y = 1 \).

Problem 114. Evaluate \( \int_S \frac{2}{1 + x^2} \, dA \) over the region \( S \) determined by the triangle with vertices \( (0,0), (2,2), \) and \( (0,2) \) in the x-y plane. Sketch the solid that you found the volume of.

Note to Instructor (I) A simple example to help with the different ways to write volumes is to find an integral expression for the volume of one-eighth of the sphere of radius 2 in three ways:

\[
\int_0^\pi \int_0^\pi \int_0^1 1 \, dx \, dy \, dz
\]

Problem 115. Fill in the blanks in order to change the order of integration.

\[
\int_0^1 \int_0^{1-x} \int_0^{1-x-2y} f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \int_0^{f(x,y,z)} f(x, y, z) \, dz \, dy
\]

Problem 116. Sketch and compute the volume of the solid bounded by \( x^2 = 4y, z = 0, \) and \( 5y + 9z - 45 = 0 \). Write the integral both as \( \int_0^\pi \int_0^\pi \int_0^\pi \) and \( \int_0^\pi \int_0^\pi \int_0^\pi \) dy dx.

Problem 117. Fill in the blanks:

\[
\int_0^1 \int_{-y}^y f(x, y) \, dx \, dy = \int_{-y}^y \int_0^\pi f(x, y) \, dy \, dx
\]

\[
\int_0^2 \int_{-y^2}^{2y} f(x, y) \, dx \, dy = \int_{-y^2}^{2y} \int_0^\pi f(x, y) \, dy \, dx
\]
Problem 118. Compute $\int_0^1 \int_{-x}^{x} e^{x+y} \, dy \, dx$ directly and by reversing the order of integration.

Coordinate Transformations

Loosely speaking, a **coordinate transformation** is a transformation from one coordinate system to another coordinate system and is also called **change of variables**. A cleverly chosen coordinate transformation can make a difficult integral easy. There are infinitely many, but the three most common are: the conversion from rectangular coordinates to polar coordinates (used in double integrals), from rectangular coordinates to spherical coordinates (used in triple integrals), and from rectangular coordinates to cylindrical coordinates (used in triple integrals).

In Calculus I when you did a trigonometric substitution, you did a change of variable. Written as a theorem, it would look like this.

**Theorem 17.** If $f$ is an integrable function over $[a,b]$ and $u : [a,b] \rightarrow [c,d]$ is a differentiable function then

$$
\int_a^b f(x) \, dx = \int_c^d f(u(t))u'(t) \, dt
$$

where we are making the substitution $x = u(t)$ and $u(c) = a$ and $u(d) = b$.

Work the following problem, using the given substitution, but keeping all the independent variables in tact, i.e. don’t toss out the “$t$” when you replace $x$ by $u(t)$.

**Problem 119.** Compute:

1. $\int_3^4 \frac{1}{x^2 + 9} \, dx$ using the substitution $u(t) = 3 \tan(t)$

2. $\int_3^4 \frac{1}{\sqrt{x^2 + 9}} \, dx$ using the substitution $u(t) = 3 \tan(t)$

For transformation of two variables the theorem is similar.

**Theorem 18.** If $f$ is an integrable function over the domain $B$ and $x$ and $y$ are differentiable functions transforming the region $B$ to the region $B'$, then

$$
\int_B \int f(x,y) \, dx \, dy = \int_{B'} \int f(x(u,v), y(u,v))J(u,v) \, du \, dv
$$

where we are making the substitution $x = x(u,v)$ and $y = y(u,v)$ and

$$
J(u,v) = \det \begin{pmatrix}
x_u(u,v) & x_v(u,v) \\
y_u(u,v) & y_v(u,v)
\end{pmatrix} = x_u(u,v)y_v(u,v) - y_u(u,v)x_v(u,v)
$$

**Note to Instructor (I)** To demonstrate change of variables in two dimensions, we rewrite (but don’t compute) $\int_0^1 \int_{x-3}^{x+3} \sqrt{x+2y(y-x)^2} \, dy \, dx$ using $u = x+2y$ and $v = y-x$. We first solve the two equations for $x$ and $y$ before computing $J(u,v)$. Then we note that $J(u,v) = \frac{1}{J(x,y)}$.

**Problem 120.** Integrate $\int_0^4 \int_{\frac{y}{2}}^{\frac{y+1}{2}} \frac{2x-y}{2} \, dx \, dy$. Now make the change of variable, $u = \frac{2x-y}{2}$ and $v = \frac{y}{2}$ and integrate the result.
Polar Coordinates Refresher

In polar coordinates, the point in the plane \( P = (x, y) \) is denoted by \((r, \theta)\) where \( r \) is the signed distance from \((0, 0)\) to \( P \) and \( \theta \) is the angle between the vector \( \overrightarrow{P} \) and the positive x-axis. We restrict \( \theta \) to the interval \([0, 2\pi]\). We say \( r \) is the signed distance to allow equations such as \( r = 4 \sin(\theta) \) where if \( \theta = \frac{11\pi}{6} \) then \( r = -2 \). The corresponding point would be \((-2, \frac{5\pi}{6})\) which is the same point as \((2, \frac{11\pi}{6})\). It follows that \( x, y, r, \) and \( \theta \) are related by the equations:

1. \( x = r\cos \theta \),
2. \( y = r\sin \theta \),
3. \( r = \sqrt{x^2 + y^2} \), and
4. \( \theta = \arctan \left( \frac{y}{x} \right) \).

Note to Instructor (I) We spent little time on polar coordinates during the last semester, so we filled in gaps here. First we discussed a few graphs, \( r = a \) (circles); \( \theta = b \) (lines); \( r = \pm \cos(n\theta) \) (roses); \( r = a \pm b \sin(n\theta) \) and \( r = a \pm b \cos(n\theta) \) (limacons and cardiods). We graph one, perhaps \( r = 4 \sin(2\theta) \), carefully labeling every point. Then we move on to calculus, finding four integral expressions for the area below \( x^2 + y^2 = 9 \). We know that the answer is \( 9\pi/2 \) so writing the integrals is an exercise to reinforce their understanding of the integral expressions.

1. Single Rectangular: \( \int_{-3}^{3} \sqrt{9 - x^2} \, dx \)

2. Double Rectangular: \( \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} 1 \, dx \, dy \)

3. Double Polar: \( \int_{0}^{\pi} \int_{0}^{3} J(r, \theta) \, dr \, d\theta = \int_{0}^{\pi} \int_{0}^{3} r \, dr \, d\theta \)

What about single variable polar integration for this area? Since \( r = 3 \) is the equation of the circle, we might try \( \int_{0}^{\pi} 3 \, d\theta \) but this yields \( 3\pi \) which is incorrect. What’s wrong? We return to Riemann Sums and illustrate the problem by dividing \( r = f(\theta) \) into sectors of circles that

\[
\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 (\theta_{i+1} - \theta_i) = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta
\]

Thus,

4. Single Polar: \( \frac{1}{2} \int_{0}^{\pi} (3)^2 \, d\theta \)

Problem 121. Sketch each of the following pairs of polar functions on the same graphs.

1. \( r = 5 \cos(\theta) \) and \( \theta = 2\pi/3 \)
2. \( r = 2 \sin(\theta) \) and \( r = 2 \cos(\theta) \)
3. \( r = 2 + 2 \cos(\theta) \) and \( r = 1 \)
Problem 122. Compute \( \int_{0}^{\frac{\pi}{2}} \int_{0}^{\cos(\theta)} r^2 \sin(\theta) \, dr \, d\theta \).

Problem 123. Convert the previous integral to rectangular coordinates and recompute to verify your answer.

Problem 124. To make a change of variables from rectangular to polar coordinates, we let \( x \) and \( y \) be the functions, \( x(r, \theta) = r \cos(\theta) \) and \( y(r, \theta) = r \sin(\theta) \). Show that \( J(r, \theta) = r \) (from Theorem 18) by filling in the missing computation:

\[
J(r, \theta) = \det \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \cdots = r.
\]

Using the previous problem and Theorem 18, we now see that when making a change of variable from rectangular to polar coordinates we have,

\[
\int_{B} f(x, y) \, dx \, dy = \int_{B'} f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta
\]

where \( B' \) is the region \( B \) represented in polar coordinates. The next problem demonstrates an intuitive geometric argument supporting this result.

Problem 125. Recall that the area of the sector of a circle with radius \( r \) spanning \( \theta \) radians is \( A = \frac{1}{2} r^2 \theta \). Let \( 0 < r_1 < r_2 \) and \( 0 < \theta_1 < \theta_2 < \frac{\pi}{2} \). Sketch the region in the first quadrant bounded by the two circles \( r = r_1 \), \( r = r_2 \), and the two lines \( \theta = \theta_1 \), and \( \theta = \theta_2 \). Show that the area of the bounded region is \( \left( \frac{r_1 + r_2}{2} \right) (r_2 - r_1)(\theta_2 - \theta_1) \).

Definition 38. Two circles are \textit{concentric} if they share a common center and distinct radii.

Definition 39. An \textit{annulus} is a region in the plane trapped between two concentric circles.

Problem 126. Let \( r_1 \) and \( r_2 \) be real numbers with \( 0 < r_1 < r_2 \). Let \( R \) be the portion of the annulus centered at the origin, between the two circles of radii \( r_1 \) and \( r_2 \), and above the x-axis. Convert \( \int_{R} e^{x^2+y^2} \, dA \) over the region \( R \) to polar coordinates and compute. Write down the endpoints of integration in rectangular coordinates.

Problem 127. Convert to polar and compute \( \int_{0}^{\frac{\pi}{2}} \int_{0}^{\sqrt{1-y^2}} \sin(x^2 + y^2) \, dx \, dy \).

Problem 128. Sketch the region bounded by \( r = 2 \) and \( r = 2(1 + \cos(\theta)) \) from \( \theta = 0 \) to \( \theta = \pi \). Compute \( \int_{R} y \, dA \) via polar coordinates.

Problem 129. Sketch \( \theta = \frac{\pi}{6} \) and \( r = 4 \sin(\theta) \). Compute the area of the smaller of the two regions bounded by the curves in two ways:

1. First compute \( \int_{R} r \, dr \, d\theta \).

2. Check your answer by using the formula for the area inside a polar graph, \( \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta \).

Problem 130. Convert to polar and compute \( \int_{1}^{2} \int_{0}^{\sqrt{2x-x^2}} (x^2 + y^2)^{-\frac{1}{2}} \, dy \, dx \).
Problem 131. Find the volume of the solid under \( z = 3xy \), above \( z = 0 \), and within \( x^2 + y^2 = 2x \).

Problem 132. Find the volume of the solid in the 1st octant (i.e. where \( x > 0, y > 0, z > 0 \)) under \( z = x^2 + y^2 \) and inside the surface \( x^2 + y^2 = 9 \).

Cylindrical and Spherical Coordinates

You already know how to represent points in the plane in two different ways, rectangular and polar coordinate systems. We wish to represent points in three-space in two new ways referred to as cylindrical and spherical coordinates.

Definition 40. The Cylindrical Coordinate Representation for a point \( P = (x, y, z) \) is denoted by \( (r, \theta, z) \) where \( (r, \theta) \) are the polar coordinates of the point \( (x, y) \) in the \( x-y \)-plane and \( z \) remains unchanged (i.e. \( z \) is the height of the point above or below the \( x-y \)-plane).

Definition 41. The Spherical Coordinate Representation for a point \( P = (x, y, z) \) is denoted by \( (\rho, \phi, \theta) \) where \( \rho \) is the distance from the point to the origin, \( \phi \) is the angle between \( \overrightarrow{P} \) and positive \( z \)-axis, and \( \theta \) is the angle between the \( x \)-axis and the projection of \( \overrightarrow{P} \) onto the \( x-y \)-plane. To avoid multiple representations of a single point in space we restrict \( \rho \geq 0 \), \( 0 \leq \phi \leq \pi \), and \( 0 \leq \rho \leq 2\pi \).

Problem 133. Sketch (or simply describe in words) each of the following regions in spherical coordinates.

1. the region between \( \rho = 2 \) and \( \rho = 3 \)
2. the region between \( \phi = \pi/2 \) and \( \pi/4 \)
3. the region below \( \rho = 10 \) and above \( \phi = \pi/3 \)

Note to Instructor (I) We draw the pictures and demonstrate the derive relationships between the coordinates in rectangular, cylindrical and spherical coordinates. We convert

\[
\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{4-x^2-y^2} \, dz \, dy \, dx
\]

to both cylindrical and spherical coordinates. First we sketch the region defined by \( x \in [0, 2], y \in [0, \sqrt{4-x^2}], \) and \( z \in [0, \sqrt{4-x^2-y^2}] \). The obvious choice seems to be a conversion to spherical coordinates, so we do that first to obtain

\[
\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\rho^2} \rho^3 \cos \phi \sin^2 \phi \sqrt{4-\rho^2 \sin^2 \phi} \, d\rho \, d\phi \, d\theta
\]

which we can’t easily integrate! In cylindrical it becomes

\[
\int_0^{\pi/2} \int_0^{\sqrt{4-x^2}} \sqrt{4-r^2} \, r \, dz \, d\theta
\]

which is easily integrated. •

From Problem 124, we know that the Jacobian for polar coordinates is

\[
J(r, \theta) = \det \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix} = r.
\]
The next two problems generate the Jacobian for cylindrical and spherical coordinate systems. Warning: If you haven’t had linear algebra yet, the next two problems require computing the determinant of $3 \times 3$ matrices so you might skip them or look up Cramer’s Rule for computing determinants of matrices.

**Problem 134.** To make a change of variables from rectangular to cylindrical coordinates, we let $x = x(r, \theta) = r \cos \theta$, $y = y(r, \theta) = r \sin \theta$, and $z = z$. Show that $J(r, \theta, z) = r$ (from Theorem 18) by filling in the missing computation:

$$J(r, \theta, z) = \det \begin{pmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{pmatrix} = \cdots = r.$$

**Problem 135.** Consider the solid trapped inside the cylinder $x^2 + y^2 = 9$, above the function $f(x, y) = 4 - x^2 - y^2$ and below the plane $z = 10$.

1. Write a triple integral in rectangular coordinates for the volume, but don’t compute.
2. Write a triple integral in cylindrical coordinates for the volume and compute it.
3. Write the volume as the volume of the cylinder minus the volume under the upside down paraboloid.

**Problem 136.** To make a change of variables from rectangular to spherical coordinates, we let $x(\rho, \phi, \theta) = \rho \sin \phi \cos \theta$, $y(\rho, \phi, \theta) = \rho \sin \phi \sin \theta$, and $z(\rho, \phi, \theta) = \rho \cos \phi$. Show that $J(\rho, \phi, \theta) = \rho^2 \sin(\phi)$ (from Theorem 18) by filling in the missing computation:

$$J(\rho, \phi, \theta) = \det \begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix} = \cdots = \rho^2 \sin(\phi).$$

Summarizing, when integrating with respect to polar, cylindrical, or spherical coordinates, we will always use the appropriate Jacobian as illustrated below.

1. Polar:

$$\int \int \cdots r \, dr \, d\theta, \text{ where } x = r \cos(\theta), y = r \sin(\theta)$$

2. Cylindrical:

$$\int \int \int \cdots r \, dr \, d\theta \, dz, \text{ where } x = r \cos(\theta), y = r \sin(\theta), \& z = z$$

3. Spherical:

$$\int \int \int \cdots \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta, \text{ where } x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, \& z = \rho \cos \phi$$

Note that the order of integration could change and you will find times when the choice of the order of integration transforms an apparently hard problem into an easy one.

**Problem 137.** Compute the volume of the region bounded by the two spheres $\rho = 4$ and $\rho = 6$ using spherical coordinates. Verify using the formula for the volume of a sphere, $V = \frac{4}{3} \pi r^3$. 

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Problem 138. Sketch and find the volume that is bounded below by the cone $\phi = \frac{\pi}{3}$ and bounded above by the sphere $\rho = 6$.

Problem 139. Convert to spherical coordinates then evaluate:

$$
\int_{-3}^{3} \int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2}} (x^2 + y^2 + z^2)^{\frac{3}{2}} \, dy \, dz \, dx
$$

Problem 140. Convert to spherical coordinates and compute: $\int e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dV$ where $V$ is the region $x^2 + y^2 + z^2 \leq 1$

Problem 141. Find the volume of the ellipsoids:

1. $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} \leq 1$ via the change of variables $x = 2a$, $y = 3b$, $z = 6c$

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ via the change of variables $x = ua$, $y = vb$, $z = wc$

Note to Instructor (I) We imagine the x-axis as an infinite, massless beam and suppose that we have two masses: $m_1$ at $(x_1,0)$ and $m_2$ at $(x_2,0)$. At what point $(x,0)$ would we place a fulcrum in order to balance the beam (axis)? To determine this we need to solve $(x-x_1) \cdot m_1 = (x_2-x) \cdot m_2$ for $x$ which yields $x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$. We call $M_y = \sum m_i x_i$ the moment with respect to $y$. If we do the same thing with $n$ weights in the plane at $(x_1,y_1),...,(x_n,y_n)$ then we have:

1. the moment with respect to $y$, $M_y = \sum_{i=1}^n m_i x_i$,

2. the moment with respect to $x$, $M_x = \sum_{i=1}^n m_i y_i$, and

3. the center of mass is $(M_y/m, M_x/m)$ where $m = \sum m_i$.

We often skip the rather interesting case where we wish to find the center of mass (centroid) of an area bounded by a function $f$ on $[a,b]$ which turns out to be

$$
M_y = \sum \text{area} \ast \text{density} \ast (\text{distance from y-axis})
= \sum (x_{i+1} - x_i) f(x_i) \rho(x_i) x_i \rightarrow \int_a^b x \rho(x) f(x) \, dx
$$

$$
M_x = \sum \text{area} \ast \text{density} \ast (\text{distance from x-axis})
= \sum (x_{i+1} - x_i) f(x_i) \rho(x_i) \frac{1}{2} f(x_i) \rightarrow \frac{1}{2} \int_a^b \rho(x) (f(x))^2 \, dx = \int \rho y \, dy.
$$

Instead of doing this, we move on to the formula’s for center of mass in three dimensions, figuring that the simple finite case is enough of a transition. The goal is, once again, to emphasize the transition from finite sums to integrals. •

An Integration Application

The mass of an object can be found by integrating the density function, $\delta$, over the entire object. If $\delta$ is the density function, then the center of mass is given by $(\hat{x}, \hat{y}, \hat{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$ where

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1. \( m = \text{mass} = \int \delta(x,y,z) \, dx \, dy \, dz \),

2. \( M_{xy} = \int \int z \, \delta(x,y,z) \, dx \, dy \, dz \),

3. \( M_{xz} = \int \int y \, \delta(x,y,z) \, dx \, dy \, dz \), and

4. \( M_{yz} = \int \int x \, \delta(x,y,z) \, dx \, dy \, dz \).

The numbers, \( M_{xy} \), \( M_{xz} \), and \( M_{yz} \) are called the moments of the object with respect to \( z \), \( y \), and \( x \) respectively.

**Problem 142.** Find the center of the mass of the solid inside \( x^2 + y^2 = 4 \), outside \( x^2 + y^2 = 1 \), below \( z = 12 - x^2 - y^2 \), and above \( z = 0 \), assuming the constant density function \( \rho(x,y,z) = k \). Graph the solid and mark the center of mass to see if your answer makes sense!

**Problem 143.** Find the mass of the solid bounded by \( z = 2 - \frac{1}{2} x^2 \), \( z = 0 \), \( y = x \), and \( y = 0 \), assuming the density is \( \delta(x,y,z) = k z \) where \( k > 0 \). Challenge: Write this integral with the various orders of integration, \( dz \, dy \, dx \), \( dy \, dx \, dz \), \( dz \, dx \, dy \), and \( dx \, dz \, dy \).

**Problem 144.** Find the center of the mass of the solid from the previous problem.
Chapter 6

Line Integrals, Flux, Divergence, Gauss’ and Green’s Theorem

The phrases scalar field and vector field are new to us, but the concept is not. A scalar field is simply a function whose range consists of real numbers (a real-valued function) and a vector field is a function whose range consists of vectors (a vector-valued function).

Definition 42. Let \( n \) and \( m \) be integers greater than or equal to 2. A scalar field on \( \mathbb{R}^n \) is a function \( f : \mathbb{R}^n \to \mathbb{R} \). A vector field on \( \mathbb{R}^n \) is a function \( f : \mathbb{R}^n \to \mathbb{R}^m \).

Note to Instructor (I) We introduce the new notation with a few simple examples. \( f(x, y) = x^2 - 2xy \) is a scalar field. \( f(x, y, z) = (2x - y, y\cos(z^2), x - yz) \) is a vector field on \( \mathbb{R}^3 \). For each particle in the ocean at location \((x, y, z)\), the vector field \( \vec{f}(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)) \) might represent the current at this point.

We now modify our definition for \( \nabla \). If \( f : \mathbb{R}^2 \to \mathbb{R} \) then we previously defined \( \nabla f = (f_x, f_y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \). We now redefine \( \nabla \) as an operator that acts on the set of differentiable functions and write

\[
\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\]

This encompasses our previous notation as we will write

\[
\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (f_x, f_y).
\]

We previously defined the dot product only between vectors (and points) in \( \mathbb{R}^n \) and now redefine that notation as well. If \( \vec{f} : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( \vec{f}(x, y) = (p(x, y), q(x, y)) \) then we will write

\[
\nabla \cdot \vec{f} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (p(x, y), q(x, y)) = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = p_x(x, y) + q_y(x, y).
\]

The next definition formalizes what we just wrote, stating it for \( \mathbb{R}^3 \).

Definition 43. If \( \vec{f}(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)) : \mathbb{R}^3 \to \mathbb{R}^3 \) is a differentiable vector field then the divergence of \( \vec{f} \) is the scalar field from \( \mathbb{R}^3 \to \mathbb{R} \) defined by:

\[
\nabla \cdot \vec{f} = p_x + q_y + r_z.
\]

And the next definition extends our definition of the cross product, \( \times \).

Definition 44. If \( \vec{f}(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)) : \mathbb{R}^3 \to \mathbb{R}^3 \) then the curl of \( \vec{f} \) denoted by \( \nabla \times \vec{f} \) is defined by the vector valued function

\[
\nabla \times \vec{f} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (p, q, r) = \left(\frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}, \frac{\partial r}{\partial x} - \frac{\partial p}{\partial z}, \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right).
\]
Notice that the divergence of $\mathbf{f}$, denoted by $\nabla \cdot \mathbf{f}$ is a scalar field while the curl of $\mathbf{f}$, denoted by $\nabla \times \mathbf{f}$, is a vector field.

**Problem 145.** Compute the divergence and curl of $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

1. $\mathbf{f}(x, y, z) = (x^2yz, x^2 + y + \sqrt{z}, \pi x^3/yz)$
2. $\mathbf{f}(x, y, z) = (e^x, \ln(xyz), \sin(x^2y^2))$

**Definition 45.** Let $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^3$. Assume $f$, $\mathbf{g}$, $\mathbf{h}$ are differentiable. Then

1. $(f \mathbf{g})(x, y, z) = f(x, y, z)\mathbf{g}(x, y, z)$
2. $(\mathbf{g} \cdot \mathbf{h})(x, y, z) = \mathbf{g}(x, y, z) \cdot \mathbf{h}(x, y, z)$
3. $(\mathbf{g} \times \mathbf{h})(x, y, z) = \mathbf{g}(x, y, z) \times \mathbf{h}(x, y, z)$

**Problem 146.** Let $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^3$. Prove or give a counter example:

1. $\nabla \cdot (f \mathbf{g} + \mathbf{h}) = \nabla \cdot f \mathbf{g} + \mathbf{g} \cdot \mathbf{h}$
2. $\nabla \cdot (f \mathbf{g}) = f(\nabla \cdot \mathbf{g})$
3. $\nabla \times (f \mathbf{g} + \mathbf{h}) = \nabla \times f \mathbf{g} + \nabla \times \mathbf{h}$
4. $\nabla \cdot (f \mathbf{g}) = f(\nabla \cdot \mathbf{g}) + \mathbf{g} \cdot \nabla f$
5. $\nabla \times (f \mathbf{g}) = f\nabla \mathbf{g} + \nabla f \times \mathbf{g}$
6. $\nabla \times (\mathbf{g} \cdot \mathbf{h}) = (\nabla \times \mathbf{g}) \cdot \mathbf{h} + \mathbf{g} \times (\nabla \cdot \mathbf{h})$

**Arc Length and Line Integrals over Scalar Fields**

Definition 12 gave the definitions for the arc length of functions and parametric curves.

**Problem 147.** Verify that the arc length of $f(x) = x^2$ from $(-1, 1)$ to $(1, 1)$ and the arc length of $\mathbf{f}(t) = (t, t^2)$ for $-1 \leq t \leq 1$ are equal by computing both.

**Note to Instructor (I)** This is our introduction to the concept behind the line integral. Our goal is to compute the area of a wall that has as its base the curve $c(t) = (t, t^2)$, $0 \leq t \leq 1$ and has a height of $t^3$ at the point $(t, t^2)$. For entertainment’s sake, we imagine the scenario where a mathematician lives in a house and likes the neighbor across the street, but does not care for the side neighbor. Thus she hires our class to build a fence with a parabolic base that has its vertex in her front lawn and curves around her side lawn. The fence is to have height zero in front of her house at the vertex of the parabola and to get cubically taller as it wraps around the side of her house. We’ll need to estimate materials, so we’ll need to know the area of the fence. By using a simple, physically possible example, we will see the importance of the arc-length in the process. When we ask what the area of this wall will be, students often say the area trapped between $f(x) = x^3$, the x-axis, and $x = 1$. This, however, does not compensate for the curvature of the base of the wall. Thus, we return to Riemann Sums and compute from first principles. We partition the base of the wall into $N$ pieces, $\{c(t_0), c(t_1), \ldots, c(t_N)\}$. Then the area of the wall will simply be the sum of the areas of the sections of the wall, where the $i^{th}$ section of the wall has base the portion of
the parabola between \(c(t_{i-1})\) and \(c(t_i)\) and has approximate height, \(f(t_i) = t_i^3\). If \(d\) represents the formula for the distance between two points in the plane and \(g(x) = x^2\), then an approximation of the area based on this partition would be:

\[
\text{area} \approx \sum_{i=1}^{N} d(c(t_{i-1}), c(t_i)) f(t_i) = \sum_{i=1}^{N} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2} f(t_i) = \sum_{i=1}^{N} \sqrt{1 + \left(\frac{t_i - t_{i-1}}{t_i - t_{i-1}}\right)^2} f(t_i)(t_i - t_{i-1})
\]

\[
\rightarrow \int_{0}^{1} \sqrt{1 + g'(t)^2} f(t) \, dt \quad \text{as } n \to \infty
\]

Of course, after this derivation, we see that we have simply ended up integrating \(f(x) = x^3\) with respect to the arc-length of \(g\). While we have previously shown in this semester and in Calculus II that the arc-length of an function \(g\) from \(a\) to \(b\) is given by \(\int_{a}^{b} \sqrt{1 + g'(x)^2} \, dx\), I think the repetition in the different scenario is valuable. After the derivation, or in a future class, we discuss other physically admissible examples of scalar integrals – integrating the electrical charge of a piece of bent wire in three-space or integrating the density of a piece of wire to obtain the mass.

**Problem 148.** For the fencing example worked in class, suppose we replace the base \(c(t) = (t, g(t))\) with the curve \(c(t) = (u(t), v(t))\) where \(u\) and \(v\) are some real-valued functions and we replace the height \(f(t) = t^3\) with some real valued function \(h\). What would the area of the wall be now? Your answer will be an integral in terms of \(u\), \(v\), and \(h\).

**Problem 149.** Two scalar line integrals:

1. A wall over \(\vec{c}(t) = (3 \sin(t), 3 \cos(t))\) from \(t = 0\) to \(t = \frac{\pi}{2}\) has height \(h(x, y) = x^2 y\). Graph the wall and determine its area.

2. A wall over \(\vec{c}(t) = (\sqrt{9 - t^2}, t)\) from \(t = 0\) to \(t = 3\) has height \(h(x, y) = x^2 y\). Determine its area.

What you have just been computing are called line integrals over scalar fields.

**Definition 46.** The line integral of a scalar field \(g : \mathbb{R}^2 \to \mathbb{R}\) over the curve \(\vec{c}(t) = (x(t), y(t))\) is defined by \(L = \int_{c} g(\vec{c}(t)) \cdot |\vec{c}'(t)| \, dt\).

The line integral over a scalar field is computed with respect to arc-length and we use the notation,

\[
\int_{c} g \, ds \quad \text{to mean} \quad \int_{c} g(\vec{c}(t)) \cdot |\vec{c}'(t)| \, dt.
\]

We may use line integrals to compute the mass of a curved piece of wire in a similar manner. Suppose we have a piece of wire in the plane that is bent into the shape of a curve, \(\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^2\). If \(\delta : \mathbb{R}^2 \to \mathbb{R}\) and \(\delta(x,y)\) represents the density of the wire at \((x,y)\), then the line integral of the density with respect to the arc length is the mass of the wire.
Problem 150. Recall from the last section of the last chapter that mass is the integral of density. Find the mass of a wire in the plane with density at \( \delta(x, y) = 2xy \) and shape \( \vec{c}(t) = (3 \cos(t), 4 \sin(t)) \) from \( t = 0 \) to \( t = 2\pi \). Could such a wire exist?

Problem 151. Find the mass of the wire in three-space with density at \( \delta(x, y, z) = 3z \) and in the shape \( \vec{c}(t) = (2 \cos(t), 2 \sin(t), 5t), t \in [0, 4\pi] \). Find its center of mass as well.

We now have many different types of integrals to compute. Here is a table of notations to help you determine which integral we are computing. Some of these notations we have already seen, some are coming soon.

- \( dx \) or \( dt \) means the usual, for example:
  \[
  \int_1^2 x^2 \, dx = \int_1^2 t^2 \, dt
  \]

- \( ds \) means we integrate with respect to arc length, for example:
  \[
  \int_{\vec{c}} f \, ds = \int_{\vec{c}} f(\vec{c}(t)) |\vec{c}'(t)| \, dt
  \]

- \( d\vec{c} \) means a line integral over a vector field along some curve, for example:
  \[
  \int_{\vec{c}} \vec{f} \cdot d\vec{c} = \int_{\vec{c}} \vec{f}(\vec{c}(t)) \cdot \frac{\vec{c}'(t)}{|\vec{c}'(t)|} |\vec{c}'(t)| \, dt = \int_{\vec{c}} \vec{f}(\vec{c}(t)) \cdot \vec{c}'(t) \, dt
  \]

- \( dA \) means we integrate over a two dimensional domain, for example the double integral:
  \[
  \int_D f \, dA = \int \int f(x,y) \, dx \, dy
  \]

- \( dV \) means we integrate over a three dimensional domain, for example the triple integral:
  \[
  \int_D f \, dV = \int \int \int f(x,y,z) \, dx \, dy \, dz
  \]

Definition 47. A vector field is said to be conservative if it is the gradient of some function. Conservative fields are also called gradient fields.

Problem 152. \( \vec{f}(x, y) = (-2xy + 2e^y, -x^2 + 2xe^y) \) is conservative since \( \vec{f} = \nabla g \) where \( g \) is given by \( g(x, y) = -x^2y + 2xe^y \).

Problem 153. Is \( \vec{f}(x, y) = (xy, x - y) \) conservative? If \( \vec{f} \) is conservative then there must be a function \( g \) so that \( \nabla g = \vec{f} \). Is there a function \( g \) so that \( g_x(x, y) = xy \) and \( g_y(x, y) = x - y \)? Why or why not?

Problem 154. Is \( \vec{f}(x, y) = (x^2 + y^2, 2xy) \) conservative?

Problem 155. Show that if \( \vec{f}(x, y) = (p(x, y), q(x, y)) \) and \( p_y(x, y) = q_x(x, y) \) then \( \vec{f} \) is a gradient field.

Problem 156. Is \( \vec{f}(x, y) = \left( \frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right) \) a gradient field?
Note to Instructor (I) We use the physics of work and force as the motivation for line integrals over vector fields. To give an intuition for work, we work the two examples following this note: the work done in stretching a spring and the work done in pumping water out of a cone. The recurring theme is first principles, the idea of adding up the force required to move one small slice over some distance. To introduce line integrals over vector fields, we sketch a vector field and a curve \( c \) passing through the field, considering \( c(t) \) as a boat’s position at time \( t \) as it moves through a turbulant bay with current \( f(x, y) = (p(x, y), q(x, y)) \) at \((x, y)\). We point out that if the direction of the boat \( c'(t) \) is perpendicular to the motion of the water \( f(c(t)) \), then no work is being done as we are not traveling against a current. We neglect the wetted surface area, the coefficient of drag and other hydrodynamic factors affecting how much force is required to move the boat through the water. To find the total work associated with the current, we recall the fact that if \( x \) and \( y \) are vectors and \( y \) is a unit vector then the length of the projection of \( x \) onto \( \{ \lambda y: \lambda \in \mathbb{R} \} \) is given by \( x \cdot y \). Therefore, if we make \( c'(t) \) a unit vector by dividing by it’s magnitude and integrate \( f(c(t)) \cdot \frac{c'(t)}{|c'(t)|} \) with respect to arc length (to accommodate for the length of path), then we have \( \int f(c(t)) \cdot \frac{c'(t)}{|c'(t)|} |c'(t)| \, dt \) and \( |c'(t)| \) cancels leaving us with the total work as \( \int f(c(t)) \cdot c'(t) \, dt \). We have thus illustrated that sometimes computing work requires writing out the sums and sometimes we may compute the work without resorting to Riemann sums. 

The next two examples remind us that work may be computed by integrating force using two examples that you may well have seen in a previous calculus course.

Example 1. From physics we have Work = Force \times Distance. So lifting a 5 lb book 2 feet requires 10 foot-lbs of work. Suppose we have a spring that has force \( f \) proportional to the square of the distance it is stretched from equilibrium so that \( f(x) = kx^2 \) (where \( k > 0 \) is the spring’s coefficient). If \( 0 = x_0 \leq x_1 \leq \cdots \leq x_N = 4 \) then the work done to stretch it 4 units would be

\[
W = \lim_{N \to \infty} \sum_{i=1}^{N} kx_i^2 \left( x_{i+1} - x_i \right) \text{ where } \hat{x}_i \in [x_i, x_{i+1}]
\]

\[
= \int_0^4 kx^2 \, dx \text{ by the definition of the integral.}
\]

Thus in one dimension, work is the integral of force: \( W = \int f(x) \, dx \).

Example 2. Suppose we have a cone full of water with radius 10 ft, height 15 ft, its point on the ground, and standing so that it holds the water. How much work is done in pumping all the water out of the cone? Hints: Force = Mass \times Acceleration (due to gravity) and Mass of one slice = area of slice \times Density of H_2O. Add (integrate) the amount or work done in moving each horizontal slice of water out of the tank. The density of water is 62.5 lbs/ft^3 and acceleration due to gravity is 32 ft/sec^2.

We now move to the study of vector (force) fields because we would like to be able to compute the total work done by an object as it moves through some field that acts on the object by either aiding or hindering its motion. As examples, think of a metal ball passing through a magnetic field, an electron passing through an electric field, a submarine passing through a fluid field, or a man passing through a gravitational field.

Suppose we have an object passing along a curve through a vector field. Let \( \vec{f} : \mathbb{R}^2 \to \mathbb{R}^2 \) be our (differentiable) vector field and \( c : \mathbb{R} \to \mathbb{R}^2 \) be our (differentiable) planar curve or path of the object. Therefore \( \vec{f} \) is of the form, \( \vec{f}(x, y) = (p(x, y), q(x, y)) \) for some \( p, q : \mathbb{R}^2 \to \mathbb{R} \) and \( c \) is...
of the form, \( c(t) = (a(t), b(t)) \) for some \( a, b : \mathbb{R} \to \mathbb{R} \). Thus we may compute the work done as the particle moves through the vector field by integrating over the field evaluated at the particle, dotted with the speed of the particle. This gives the component of the force that is acting in the direction of travel of the particle.

\[
W = \int c'(t) \cdot \mathbf{F}(t) \, dt
\]

Second, if we \( f(x, y) = (p(x, y), q(x, y)) \) and \( c(t) = (x(t), y(t)) \) then

\[
W = \int \mathbf{F} \cdot \mathbf{v} \, dt = \int \mathbf{F}(t) \cdot c'(t) \, dt = \int \left( p(x(t), y(t)) \cdot x'(t) + q(x(t), y(t)) \cdot y'(t) \right) \, dt
\]

**Problem 157.** Suppose we have a particle traveling through the force field, \( \mathbf{F}(x, y) = (xy, 2x - y) \).

1. Compute the work done as the particle travels through the force field \( \mathbf{F} \) along the curve \( c(t) = (t, t^2) \) from \( t = 0 \) to \( t = 1 \).

2. Compute the work done as the particle travels through the force field \( \mathbf{F} \) along the curve \( c(t) = (2t, 4t^2) \) from \( t = 0 \) to \( t = \frac{1}{2} \).

What you have just computed is called a line integral over a vector field.

**Definition 48.** The line integral of the vector field \( \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( \mathbf{F}(x, y) = (p(x, y), q(x, y)) \) over the curve \( c : \mathbb{R} \to \mathbb{R}^2 \) given by \( c(t) = (x(t), y(t)) \) is defined by

\[
L = \int \mathbf{F}(c(t)) \cdot c'(t) \, dt = \int p(x(t), y(t)) \cdot x'(t) + q(x(t), y(t)) \cdot y'(t) \, dt
\]

**Problem 158.** Compute the line integral where \( c \) is the rectangle with corners at \((0, 0), (1, 0), (1, 2), (0, 2)\) and force field \( \mathbf{F}(x, y) = (4x + y, x + 2y) \). Start your path at the origin and work counter-clockwise around the rectangle. You will need to compute four integrals along four parameterized lines.

You have been using the Fundamental Theorem of Calculus for three semesters now. This powerful result in one dimension is all that is needed to prove the version we need for three dimensions.

**Theorem 19.** The Fundamental Theorem of Calculus If \( F \) is any anti-derivative of \( f \) then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]
Theorem 20. The Fundamental Theorem of Calculus for Line Integrals If \( \vec{f} \) is a gradient field and \( g \) is any anti-derivative of \( \vec{f} \) (i.e. \( \nabla g = \vec{f} \)) then
\[
\int_{c} \vec{f} (\vec{c}) \cdot d\vec{c} = g(\vec{c}(b)) - g(\vec{c}(a)).
\]

Problem 159. Prove Theorem 20.

Problem 160. Compute the line integral from Problem 158 using Theorem 20.

The next two theorems are important results associated with line integrals. We don’t assert all the necessary hypothesis – curves are smooth, functions are assumed to be integrable, etc.

Theorem 21 states that if we reverse the path along which we compute a line integral, then we change the sign of the result. Thinking of the line integral as work, this makes sense, because the work done by traveling one direction along a path should be the opposite of the work done traveling the other way. When we have a curve \( \vec{c} \) and we write \( -\vec{c} \) we mean the same set of points in the plane, but we simply reverse the direction. If we have the curve \( \vec{c}(t) = (x(t), y(t)) \) from \( t = a \) to \( t = b \) then \( -\vec{c} \) is simply the same curve where \( t \) goes from \( t = b \) to \( t = a \).

Theorem 21. If \( \vec{c} : \mathbb{R} \rightarrow \mathbb{R}^2 \) is a parametric curve and \( \vec{f} \) is a vector field then
\[
\int_{c} \vec{f} (\vec{c}) \cdot d\vec{c} = -\int_{-c} \vec{f} (\vec{c}) \cdot d\vec{c}.
\]

Problem 161. Show that \( \int_{c} \vec{f} (\vec{c}) \cdot d\vec{c} = -\int_{-c} \vec{f} (\vec{c}) \cdot d\vec{c} \) where \( \vec{f}(x, y) = (xy, y-x) \), \( \vec{c}(t) = (t^2, t) \), \( t \in [0, 2] \). Remember that since \( c \) is the path from \((0, 0)\) to \((4, 2)\), then \(-c\) is the same path but from \((4, 2)\) to \((0, 0)\).

Theorem 22 states that line integrals over conservative fields are independent of path. Suppose you and I start at the same point at the bottom of a mountain and we walk to the top following different paths. Did we do the same amount of work? Yes. Because gravity is a conservative (gradient) field it does not matter what path we follow as long as we both start at the same place and end at the same place. By the same logic, if we start at a point on the mountain, walk around, and return to the same spot, then the total work is zero.

Theorem 22. If \( \vec{c}_1 \) and \( \vec{c}_2 \) : \( \mathbb{R} \rightarrow \mathbb{R}^2 \) are two (distinct) paths beginning at the point \((x_1, y_1)\) and ending at the point \((x_2, y_2)\) in the plane and \( \vec{f} \) is a gradient field then
\[
\int_{c_1} \vec{f} (\vec{c}) \cdot d\vec{c} = \int_{c_2} \vec{f} (\vec{c}) \cdot d\vec{c}.
\]

Problem 162. Let \( f \) be the vector field \( \vec{f}(x, y) = (3x^2y + x, x^3) \) and \( c \) be the planar curve \( \vec{c}(t) = (\cos(t), \sin(t)) \) from \( t = 0 \) to \( t = \pi \).

1. Write out and simplify, but don’t compute, the line integral of \( f \) over \( \vec{c} \).
2. Compute this line integral by choosing the simpler path \( p(t) = (-t, 0) \) from \( t = -1 \) to \( t = 1 \) and applying Theorem 22.
3. Recompute this line integral by finding a function \( g \) so that \( \nabla g = (3x^2y + x, x^3) \) and applying Theorem 20.

Problem 163. Let \( \vec{f}(x, y) = (x - y, x + y) \).
1. Compute \( \int_{c_1} f(\vec{c})\,d\vec{c} \) where \( \vec{c}_1(t) = (t, t^2),\ t \in [0, 1]. \)

2. Compute \( \int_{c_2} f(\vec{c})\,d\vec{c} \) where \( \vec{c}_2(t) = (\sin(t), \sin^2(t))\ t \in [2\pi, \frac{5\pi}{2}] \).

3. Explain your answer.

When we compute line integrals, we often integrate along a curve that is the boundary of some region. Here are a few of the buzz words about curves that we will use.

**Definition 49.** Let \( \vec{c} : [a, b] \to \mathbb{R}^2 \). We say that \( \vec{c} \) is a **simple closed curve** if \( \vec{c} \) starts and ends at the same point (i.e. \( \vec{c}(a) = \vec{c}(b) \)) and never crosses itself.

When we are integrating around a simple closed curve \( \vec{c} \) with respect to the arc-length of the curve, we will use the notation \( \oint_{\vec{c}} \cdots \,ds \)

**Definition 50.** Let \( \vec{c} : [a, b] \to \mathbb{R}^2 \) be a simple closed curve on \( [a, b] \). We say that \( \vec{c} \) is **positively oriented** if as \( t \) increases from \( a \) to \( b \) and we traverse the curve \( \vec{c} \) then the enclosed region is on our left.

**Note to Instructor (I)** Rather than proving Gauss’ Divergence Theorem, we spend considerable time trying to motivate the physical interpretation, explaining that if we are discussing the movement of particles inside a region via the function \( f \) then Gauss’ Theorem says that the integral of the divergence of the flow within the region equals the integral of the flow across the boundary of the region. To put this in context, we give brief lectures on computing the flux (flow across a boundary), the divergence of elementary vector fields, and computing normals to curves. Then we illustrate with a simple example.

1. **FLUX:** Suppose we cover the plane with a one-foot deep pool of water that is flowing to the right at 3 ft/sec. How much water flows across the line segment from \((2, 1)\) to \((2, 3)\)? We can compute this via common sense as

\[
\text{flow} \cdot \text{width} \cdot \text{height} = 3 \text{ft/sec} \cdot 2 \cdot 1 = 6 \text{ft}^3/\text{sec}.
\]

We can also compute this by considering the flow as a vector,

\[
\text{flow} \cdot (\text{unit normal to flow}) \cdot \text{width} = (3, 0) \cdot (1, 0) \cdot 2.
\]

What would be the flow across the line segment from \((0, 0)\) to \((2, 0)\). By logic, it is zero, which is also what we get when we consider the vector computation

\[
(3, 0) \cdot (0, -1) \cdot 2.
\]

What if we consider the flow across the line segment from \((1, 1)\) to \((2, 0)\)? Using our vector computation,

\[
(3, 0) \cdot (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \cdot \sqrt{2} = 3.
\]

This result makes sense because the flow across the line segment from \((0, 0)\) to \((0, 1)\) is also \(3 \text{ft}^3/\text{sec}\) and whatever flows across this line segment must also flow across the line segment from \((1, 1)\) to \((2, 0)\).
2. DIVERGENCE: Suppose that we have the vector field, \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( f(x,y) = (p(x,y), q(x,y)) \). Recall that the divergence, \( \nabla \cdot f = p_x + q_y \), represents the sum of the velocity in the \( x \) direction of the first component of the flow and the velocity in the \( y \) direction of the second component of the flow. Thus, divergence is a measure of the velocity of the flow. We sketch vector fields for:

(a) \( f(x,y) = (1,0) \) – all flow to the right and \( \nabla \cdot f = 0 \) since at every point, the flow in = the flow out

(b) \( f(x,y) = (x,0) \) – all flow outward and \( \nabla \cdot f = 1 > 0 \) since at every point not on the \( y \)-axis, there is more flow out than in

(c) \( f(x,y) = (x,y) \) – all flow outward and \( \nabla \cdot f = 2 \)

(d) \( f(x,y) = (-x,-y) \) – all flow inward and \( \nabla \cdot f = -2 \)

3. NORMALS TO CURVES: Suppose \( c(t) = (x(t), y(t)) \) is a parametric curve. We already know that the tangent to the curve is given by \( c'(t) = (x'(t), y'(t)) \). For any time \( t \), the two vectors, \( (-y'(t), x'(t)) \) and \( (y'(t), -x'(t)) \) both perpendicular to \( c'(t) \). By working a simple example, such as \( c(t) = (5\cos(t), 2\sin(t)) \) we can determine that the second one is the outward normal.

4. EXAMPLE: Verify Gauss’ Divergence theorem, \( \int_C \vec{F} \cdot n\, ds = \int_D \nabla \cdot \vec{F} \, dA \), for \( f(x,y) = (-2x, -2y) \) over the circle of radius 1 centered at (0,0).

As we will show in a forthcoming lecture, Gauss’ Divergence Theorem is equivalent to Green’s Theorem. They are simply the same theorem stated in two different ways. Gauss’ Divergence Theorem says that if we have a certain simple closed curve representing the boundary of a region over which we have a vector field (a flow), then the flow across boundary (the flux) must equal the integral of the divergence of the fluid (or the electricity or whatever) over the region enclosed by the boundary.

For both Green’s and Gauss’ Theorems, when integrating along the boundary of the region the simple closed curve must be positively oriented. That is, you must integrate in a counter-clockwise direction so that the region is on your left as you travel along the parametric curve.

**Theorem 23. Gauss’ Divergence Theorem for the Plane** If \( F \) is a vector field and \( c \) is a simple closed curve and \( n \) is the unit normal to \( c \) then \( \int_C \vec{F} \cdot n\, ds = \int_D \nabla \cdot \vec{F} \, dA \)

**Problem 164.** Verify Gauss’ Divergence Theorem by computing both integrals (the flux integral and the divergence integral) with \( \vec{F}(x,y) = -4(x,y) \) and \( c(t) = (\cos(t), \sin(t)) \) assuming \( t \in [0, 2\pi] \).

**Problem 165.** Verify Gauss’ Divergence Theorem for \( f(x,y) = (xy, 2x-y) \) over the region \( D \) in the first quadrant bounded by \( y = 0 \), \( x = 0 \) and the line \( y = 1 - x \).

**Problem 166.** Verify Gauss’ Divergence Theorem for the flow \( f(x,y) = (0,y) \) over the rectangular region, \( \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 3\} \).

**Note to Instructor (I)** We sketch pictures for both Green’s Theorem and Gauss’ theorem to illustrate that they are close cousins. The line integral associated with Gauss’ Theorem has us integrating the component of \( f \) that is perpendicular to the boundary, while the line integral associated with Green’s Theorem has us integrating the component of \( f \) that is parallel to the boundary. Then we verify Green’s theorem for two problems.
1. \( f(x,y) = (x - y, x + y) \) over the circle \( c(t) = (2\sin(t), 2\cos(t)) \)

2. \( f(x,y) = (4y, 3x + y) \) over the region bounded by the x-axis and \( y = \cos(x) \)

For the first example we get \( 8\pi \) one way and \(-8\pi \) the other way. Why? Because our circle is not positively oriented. The second example reminds us how to integrate by parts!

While we don’t prove either Gauss’ or Green’s Theorem, we do show that they are equivalent. We may use the last few minutes of a class period to state the two theorems, reemphasize that they are very similar by reminding them that the line integral associated with Gauss’ Theorem has us integrating the component of \( f \) that is perpendicular to the boundary, while the line integral associated with Green’s Theorem has us integrating the component of \( f \) that is parallel to the boundary. This and starting the theorem by writing the first two lines below may be enough that a student can present it and if not, then after we present problems the next day, I will present it. Even if I present it, then I assign as homework for the class to present the other direction, that Green’s implies Gauss’. The proof we give follows.

Suppose we have proven Gauss’ Theorem and wish to prove Green’s where \( f(x,y) = (u(x,y), v(x,y)) \) and \( c \) is the simple, closed, positively oriented curve \( c(t) = (x(t), y(t)) \).

\[
\int_C f(\vec{c}) \cdot d\vec{c} = \int_C f(x(t), y(t)) \cdot (\vec{x}'(t), \vec{y}'(t)) \, dt
\]

\[
= \int_C (px' + qy') \, dt
\]

\[
= \int_D \nabla \cdot g \, dA \quad \text{by Gauss’ Theorem}
\]

\[
= \int_D \left( \frac{\partial}{\partial x} (px' + qy') \right) \, dA
\]

\[
= \int_D q_x - p_y \, dA \quad \bullet
\]

\textbf{Theorem 24. Green’s Theorem} If \( \vec{f} : \mathbb{R}^2 \to \mathbb{R}^2 \), \( \vec{f}(x, y) = (p(x, y), q(x, y)) \), is a vector field and \( \vec{c} : \mathbb{R} \to \mathbb{R}^2 \) is a positively oriented simple closed curve so that \( \vec{c}(t) = (x(t), y(t)) \) and \( D \) is the region enclosed by \( \vec{c} \) then

\[
\int_C \vec{f}(\vec{c}) \cdot d\vec{c} = \int_D q_x(x,y) - p_y(x,y) \, dA
\]

\textbf{Problem 167.} Verify Green’s Theorem for the flow \( f(x,y) = (-y^2, xy) \) over the region bounded by \( x = 0, y = 0, x = 3, \) and \( y = 3 \).

\textbf{Problem 168.} Verify Green’s Theorem for the vector field \( f(x,y) = (-y, x) \) over the region bounded by the curve \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \).
Problem 169. Verify Green’s Theorem for the vector field \( f(x, y) = (xy, 3x) \) over the region formed by the three points, \((-1, 0), (1, 0), \text{ and } (0, 4)\).

Note to Instructor (I) If we have a simple, closed, positively oriented parametric curve, \( c(t) = (x(t), y(t)) \), then the area enclosed by the curve may be computed by: \( A = \frac{1}{2} \int_a^b x \, dy - y \, dx \).

Proof: We have intentionally written this formula as an engineer or physicist might. To prove it, the first step we will take is to rewrite it as a mathematician.

\[
\frac{1}{2} \int x \, dy - y \, dx = \frac{1}{2} \int x(t)y'(t) - y(t)x'(t) \, dt \\
= \frac{1}{2} \int (-y(t), x(t)) \cdot (x'(t), y'(t)) \, dt \\
= \frac{1}{2} \int f(x(t), y(t)) \cdot (x'(t), y'(t)) \, dt \text{ where } f(x, y) = (-y, x) \\
= \frac{1}{2} \int f(c(t)) \cdot c'(t) \, dt \text{ where } c(t) = (x(t), y(t)) \\
= \frac{1}{2} \int f(c) \, dc \\
= \frac{1}{2} \int v_x - u_y \, dA \text{ by Green’s Theorem} \\
= \frac{1}{2} \int 1 - (-1) \, dA \\
= \int 1 \, dA = \text{ area!}
\]

Problem 170. Find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) via the formula \( A = \frac{1}{2} \int x \, dy - y \, dx \) and the parametrization \( \vec{c}(t) = (a \cos(t), b \sin(t)), t \in [0, 2\pi] \).

Note to Instructor (I) Before introducing Gauss’ Theorem in higher dimensions, there should probably be some problems on surface integrals. At the very least, one needs to define and explain that if we have a simple parameterized surface, \( h(s, t) = (s, t, g(s, t)) \) then \( dS = \sqrt{1 + g_s^2 + g_t^2} \). We typically mention this while working through two examples.

Example 1. Verify Gauss’ Theorem for \( F(x, y, z) = (x, y, z) \) over the sphere of radius 1.

\[
\int_{\partial S} F \cdot n \, dS = \int_{\partial S} (x, y, z) \cdot \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \, dS \\
= \int_{\partial S} \frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} \, dS \\
= \int_{\partial S} 1 \, dS \\
= \text{surface area of the sphere} \\
= 4\pi 1^2 = 4\pi
\]

and

\[
\int_S \nabla \cdot F \, dV = \int_S 3 \, dV \\
= \frac{3}{4} \pi 1^3 \\
= 3 \text{ volume of a sphere}
\]
Example 2. Let $S$ be the solid bounded by $x^2 + y^2 = 4$, $z = 0$, and $x + z = 6$. Let $F(x,y,z) = (x^2, xy, e^z)$. We can compute one side easily,
\[
\int_S \nabla F \cdot n \, dV = \int_S 3x \, dV = \int_0^{2\pi} \int_0^2 \int_0^{6-r\cos(\theta)} 3r\cos(\theta) r \, dz \, dr \, d\theta = \cdots = -12\pi.
\]

We can set up the other side, although the integrals are sufficiently difficult that both Maple and a TI Voyage 200 numerically approximate the solutions. 

Gauss’ Divergence Theorem is valid in higher dimensions as well, although it is often the case that integrating over certain parts of the boundary is challenging.

**Theorem 25. Gauss’ Divergence Theorem for Three Dimensions** If $S$ is a solid in three dimensional space, we write
\[
\int_{\delta S} f \cdot n \, dS = \int_S \nabla \cdot f \, dV
\]
where $\delta S$ denotes the boundary of $S$, $n$ denotes the unit outward normal and $dV$ indicates that we are integrating over the entire volume of the solid.

**Problem 171.** Verify Gauss’ Divergence Theorem for the vector field $f(x,y,z) = (x^2y, 2xz, y^3)$ over the three dimensional box, $0 \leq x \leq 1, 0 \leq y \leq 2$, and $0 \leq z \leq 3$.

**Problem 172.** Let $S$ be the cylinder of radius 2 with base centered at the origin and height 3. Let $f(x,y,z) = (x^3 + \tan(yz), y^3 - e^{xz}, 3z + x^3)$. Use the divergence theorem to compute the flux across the side of the cylinder.

**Problem 173.** Let $S$ be the solid bounded by $2x + 2y + z = 6$ in the first octant. Let $f(x,y,z) = (x, y^2, z)$. Sketch the solid and verify Gauss’ Divergence Theorem.

Congratulations, for some of you this note constitutes having independently worked through three semesters of Calculus, which is quite an accomplishment.
Chapter 7

Practice Problems

This chapter consists of practice problems for students to work after we have covered a topic. Section 7.X has practice problems with solutions for Chapter X.

7.1 Vectors, Lines, and Functions Drill

Vectors

1. Let \( \vec{x} = (-1, 3) \) and \( \vec{y} = (4, 1) \) and graph \( \vec{x}, \vec{y}, \vec{x} - \vec{y} \).
2. Find the norm of each vector in the previous problem.
3. Graph \( (1, -2, 1) \) and \( (-2, 4, 3) \) and find the angle between them.

Lines

1. Sketch the vectors \( x = (1, 3, 5) \) and \( y = (2, 4, -3) \) and the line through these points.
2. Find a parametric equation, \( \vec{t} \), for the line in the previous problem so that \( \vec{t}(0) = x \) and \( \vec{t}(1) = y \).
3. Plot \( \vec{t}(t) = (2, -1, 4) + (1, 0, 0)t \). Assuming this represents the position of an object, compute the speed of the object.
4. Find an equation for the position of the object that has: the same speed as the object in the previous problem, the same position at time 0, and is travelling in the opposite direction.

Functions

1. Sketch a graph of \( f(x, y) = 2x^2 + 3y^2 \).
2. Sketch the intersection of \( f \) from the previous problem with the plane, \( z = 4 \).
3. Sketch the function \( g(x, y) = x^3 + y^2 \). Be sure and label a few points.
4. Compute the composition \( g \circ \vec{t} \) where \( g \) is from the previous problem and \( \vec{t}(t) = (2, -1) + (1, 0)t \).
5. Compute \( (g \circ l)'(3) \) and indicate its meaning.
6. Sketch \( l(t) = (0, 1) + (1, 0)t \) for \( t \geq 0 \).
7. Sketch \( c(t) = (3 \cos(t), 3 \sin(t)) \) for \( 0 \leq t \leq 2\pi \).

8. Sketch \( e(t) = (4 \cos(t), 2 \sin(t)) \) for \( 0 \leq t \leq 2\pi \).

9. Graph \( \vec{r}(t) = (\cos(t), 2t, \sin(t)) \) for \( t \in [0, 6\pi] \); if \( \vec{r}(t) \) is the position of an object at time \( t \) then show that the speed of the object is constant.
Vectors, Lines, and Functions Solutions

Vectors

1. a parallelogram with diagonals
2. $\|x\| = \sqrt{10}$ and $\|y\| = \sqrt{17}$
3. approximately 122 degrees

Lines

1. 
2. $\vec{l}(t) = (1, 3, 5) + (1, 1, -8)t$ ($\infty$ possible solns)
3. $\|\vec{l}’(t)\| = 1$
4. $\vec{p}(t) = (2, -1, 4) - (1, 0, 0)t$ ($\infty$ possible solns)

Functions

1. a squished paraboloid
2. an ellipse
3. a water slide
4. $(t + 2)^3 + 1$
5. 75, the slope of the line (in three-space) that lies above the line $\vec{l}$ and is tangent to the graph of $g$ at the point $(5, -1, 126)$
6. $l =$ line
7. $c =$ circle, radius = 3
8. $e =$ ellipse
9. $s =$ spiral; a spring wrapped around the y-axis in three-space; speed is $\sqrt{5}$
7.2 Cross Products, Planes, and Limits Drill

Cross Products

1. Compute the cross product of $\vec{u} = (3, 2, -1)$ and $\vec{v} = (2, \pi, -3)$ and state the significance of the resulting vector.

2. Find two vectors of unit length orthogonal to both $\vec{a} = (4, -3, 2)$ and $\vec{b} = (2, 5, -3)$.

Planes

1. Find the equation of a plane parallel to $3x + 2z = 4 - y$ and containing $(2, 4, -2)$.

2. Find the equation of the line that represents the intersection of $3z - 4x + 2y = 4$ and $2z - x = 6 - y$.

3. Find an equation for the plane containing $(1, 2, 3), (2, 3, 4)$, and $(-1, 1, 1)$. Find another equation for the same line.

4. Find intersection of $2x - 3y + 5z = 4$ and $2x + 5y - 10z = 5$.

Limits

1. Consider the function defined by $f(x, y) = \begin{cases} \frac{x^2}{x^2 + y} & x^2 + y \neq 0 \\ 0 & (x, y) = (0, 0) \end{cases}$

Does the limit exist as $(x, y) \rightarrow (0, 0)$ along $y = kx$ exist for every $k \in \mathbb{R}$? Does the limit exist as $(x, y) \rightarrow (0, 0)$ along $y = kx^2$ exist for every $k \in \mathbb{R}$? Is the function $f$ continuous at $(x, y) = (0, 0)$?
Cross Products, Planes, and Limits Solutions

Cross Products

1. \((\pi - 6, 7, 3\pi - 4)\)
2. \(\pm \frac{1}{\sqrt{933}}(-1, 16, 26)\)

Planes

1. \(3x + y + 2z = 6\)
2. \((4, 10, 0) + (-2, -10, 4)t\) (infinitely many other ways to write this one line; check yours by substituting it into each plane)
3. \(z - x = 2\)

Limits

1. No, for a chosen value of \(k\), the limit along the path \(y = kx^2\) would be \(\frac{1}{1+k}\). Therefore the limit along different parabolic paths (different values of \(k\)) would yield different results.
7.3 Domains, Graphing, and Derivatives Drill

Domains of Functions

1. Find the domain of \( f(x, y) = \ln(x^2 + y^2 - 1) \).
2. Find the domain of \( g(x, y) = \tan^{-1}\left(\frac{x}{y}\right) \).
3. Find the domain of \( h(x, y) = \frac{4}{|x| - |y|} \).
4. Find the domain of \( k(x, y, z) = \frac{2xy}{z^2 + z - 1} \).
5. Find the domain of \( \ell(x, y, z) = \frac{xz}{\sqrt{1 - y^2}} \).
6. Determine the domain over which \( f(x, y) = \ln(x^3y^4) \) is continuous.
7. Determine the domain over which \( g(x, y) = \frac{x}{|x||y|} \) is continuous.
8. Determine the domain over which \( h(x, y, z) = \frac{x^2 - 1}{z\sqrt{y^2} - 1} \) is continuous.

Graphing Functions

1. Graph \( f(x, y) = 1 - x^2 + y^2 \).
2. Graph \( g(x, y) = x + y \).
3. Graph \( h(x, y) = \frac{x^2}{9} + \frac{y^2}{4} \).
4. Graph \( k(x, y) = |x| - |y| \).
5. Graph \( r(x, y) = \sin(x) \).

Partial Derivatives and Gradients

1. Let \( g(x, y) = x^3 - 4\log_7x^2 + \sin^{-1}(xy) \) and compute \( g_x \).
2. Let \( g(x, y) = \frac{x^2}{\sin(xy)} \) and compute \( g_y \).
3. Let \( g(x, y) = e^{xy^2} \) and compute \( \nabla g \).
4. Let \( g(x, y) = x\ln(y) - 4xy + x \) and compute \( g_x(1, 1) \) and \( g_y(1, 1) \).
5. Let \( g(x, y) = 5x^2y\sin(x - y) \) and compute \( g_y(\pi, 2\pi) \).
6. Let \( h(x, y, z) = \sqrt{2xz - 5y} + \cos^3(z\sin(x)) \) and compute \( \nabla h \).
7. Let \( h(x, y, z) = z^y \) and compute \( h_x(2, 3, 4) \).
8. Find \( f_{xx}, f_{zy} \), and \( f_{zxy} \) for \( f(x, y, z) = \frac{y}{x^3} - \sin(zy) - 3z^2 \).
Directional Derivatives

Remember: Directions vectors should be unit vectors.

1. Using the definition of directional derivative, compute the derivative of \( f(x,y) = x - y^2 \) at \((1,2)\) in the direction, \((1,1)\).

2. Find \( D_\overrightarrow{u} f(p) \) for \( f(x,y) = e^{xy} + 2x^2y^3 \); \( p = (3,1) \); \( \overrightarrow{u} \) is a unit vector parallel to \( \overrightarrow{v} = (-5,12) \).

3. Find \( D_\overrightarrow{u} f(p) \) for \( f(x,y) = e^{xy} + \ln(xy) \); \( p = (1,2) \); \( \overrightarrow{u} \) is a unit vector which makes an angle \(-\frac{\pi}{3}\) from the positive \(x\) axis.

4. Find \( D_\overrightarrow{u} f(p) \) for \( f(x,y,z) = x^2y - 4y^2z + xyz^2 \); \( p = (-2,1,-1) \); \( \overrightarrow{u} \) is a unit vector parallel to the vector \( \overrightarrow{AB} \) where \( A = (2,1,-5) \) and \( B = (-2,4,3) \).

Derivatives

For each of the following functions, compute the indicated “derivative” of the function. Because of the differing domains of the functions, the derivative could be a function (a partial derivative), a vector of functions (a gradient), or a matrix of functions (the ‘total’ derivative)!

1. \( f(x,y) = x^2y^3 \)

2. \( g(x,y) = \left(\frac{x^2}{y} - 3xy, \sin\left(\frac{x}{y}\right)\right) \)

3. \( h(x,y) = x^2 - e^{y\sqrt{z}} + \sinh(yz) \)

4. \( r(s,t,u) = (st, s^2tu, \sqrt{stu}, \ln(st^2u)) \)

Chain Rule

1. Let \( f(x,y) = 2x^2 - 7y \) and \( \overrightarrow{g}(t) = (\sin(t), \cos(t)) \). Compute the derivative of \( f \circ g \) in two ways. First compose \( f \) and \( g \) and take the derivative. Second, apply the chain rule. Verify that your solutions are the same.

2. Let \( f(x,y) = 4x^3y + e^{3y} + \frac{2}{x} \), \( x(t) = t^2 \), \( y(t) = 4t - 3 \), and \( g(t) = (x(t), y(t)) \). Compute \( (f \circ g)'(-1) \).

3. For each of the following problems, find \( \frac{dg}{dt} \) and evaluate at the given value of \( t \).
   - (a) \( g(x,y) = 3xy + e^3y^2 \) where \( x(t) = 4t^2 + t \), \( y(t) = 6 + 5t \), and \( t = 0 \)
   - (b) \( g(x,y,z) = x^3y + xz + \frac{x}{y-z} \) where \( x(t) = t^2 + 3 \), \( y(t) = 4t - t^2 \), \( z(t) = \cos(t - 3\pi) \), and \( t = 0 \)

4. Let \( f(x,y) = xy \ln(x) \) and \( \overrightarrow{g}(s,t) = (2st, t - s^3) \). State the domain \( f \), \( g \), and \( f \circ g \). Compute the gradient of \( (f \circ g) \).

5. For each of the following problems, find \( f_s \) and \( f_t \) (i.e. \( \frac{\partial f}{\partial s} \) and \( \frac{\partial f}{\partial t} \)).
   - (a) \( f(x,y) = 2x - y^2 \), \( x(s,t) = s \cos(t) \), and \( y(s,t) = (s + t)e^t \)
(b) \( f(x, y, z) = (x + 2y + 3z)^4 \) and \( x(s, t) = s + t, \ y(s, t) = s - t, \ z(s, t) = st \)

6. For each of the following problems, find \( \frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \) and \( \frac{\partial g}{\partial w} \) (i.e. \( g_u, g_v, \) and \( g_w \)).

(a) \( g(x, y) = (x + y) \ln(xy), \ x(u, v, w) = u + v - 3w, \) and \( y(u, v, w) = uv + 3w \)

(b) \( g(x, y, z) = yz + xz + xy, \ x(u, v, w) = u + v - 3vw, \ y(u, v, w) = v + w + 4uv, \) and \( z(u, v, w) = u + w - 5uv \)

7. For each of the following three problems, find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at the indicated point.

(a) \( z^3 - xy + 2yz + y^3 - 3 = 0; \ (1, 1, 1) \)

(b) \( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{49}{36}; \ (-1, 2, 3) \)

(c) \( ye^{zy} \cos(2xz) = 1; \ (\pi, 1, 4) \)

**Tangent Lines and Planes**

1. Find the (shortest) distance from the point \( (0, 1, 0) \) to the plane \( x + 2y + 3z = 4 \).

2. Find the equation of the tangent plane to the function \( z = x^2 - 4y^2 \) at \( (3, 1, 5) \).

3. For each of the following problems, find an equation of the tangent plane and equations of the normal line to the surface at the indicated point.

   (a) \( z + 1 = xe^y \cos(z); \ (1, 0, 0) \)

   (b) \( x^2 + 2y^2 + 3z^2 = 6; \ (1, -1, 1) \)

4. Find the tangent plane approximation of \( h(x, y) = x + x \ln(xy) \) when \( x = e \) and \( y = 1 \). Use this approximation to estimate \( h(2.7, 1.05) \).

5. Find the tangent line to the level curve of \( g(x, y) = e^{xy} + 3x^2 \sqrt{y} \) at \( p = (-2, 0) \).
Domains, Graphing, and Derivative Solutions

Domains of Functions

1. \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \} \)
2. \( \{(x, y) \in \mathbb{R}^2 : y \neq 0 \} \)
3. \( \{(x, y) \in \mathbb{R}^2 : |x| \neq |y| \} \)
4. all of \( \mathbb{R}^3 \) except where \( z \) equals the Pisot numbers
5. \( \{(x, y) \in \mathbb{R}^2 : |y| < 1 \} \)

Graphing Functions

1. a saddle centered at \((0, 0, 1)\)
2. a plane
3. squished paraboloid
4. planar saddle with a ‘point’ at \((0, 0, 0)\)
5. a three dimensional sine wave

Partial Derivatives and Gradients

1. \( g_x(x, y) = 3x^2 - \frac{8}{x \ln(7)} + \frac{y}{\sqrt{1-(xy)^2}} \)
2. \( g_y(x, y) = -\frac{x^2 \cos(xy)}{\sin^2(xy)} \)
3. \( \nabla g(x, y) = (y^2e^{xy^2}, 2xye^{xy^2}) \)
4. \( g_x(1, 1) = g_y(1, 1) = -3 \)
5. \( g_y(\pi, 2\pi) = 2\pi \xi^2 \)
6. \( \nabla h(x, y, z) = (\frac{z}{\sqrt{2xz-xy}} - 3z \cos(x) \cos^2(z \sin(x)) \sin(z \sin(x)), \frac{-5}{\sqrt{2xz-5y}}, \frac{x}{\sqrt{2xz-5y}} - 3 \sin(x) \cos^2(z \sin(x)) \sin(z \sin(x))) \)
7. \( h_x(2, 3, 4) \approx 17,035 \)
8. \( f_{xx} = \frac{12y}{x^2}, f_{xy} = -\cos(zy) + zy \sin(zy), f_{xzy} = 0 \)
Directional Derivatives

Remember: Directions vectors should be unit vectors.

1. \(-\frac{3}{\sqrt{2}}\)
2. \(\frac{1}{13}(588 + 31e^3)\)
3. \((1 - \frac{\sqrt{3}}{2})(e^2 + \frac{1}{2})\)
4. \(\frac{42}{\sqrt{89}}\)

Derivatives

1. \(\nabla f(x,y) = (2xy^3, 3x^2y^2)\)
2. \(Dg(x,y) = \left(\frac{2x}{y} - 3y, -\frac{x^2}{y^2} - 3x\right)\)
3. \(h(x,y) = (2x - y\sqrt{e^{xy\sqrt{z}}} - x\sqrt{e^{xy\sqrt{z}}} + z\cosh(yz))\)
4. \(Dr(s,t,u) = \left(\begin{array}{ccc}
t & s & 0 \\
\frac{2stu}{2\sqrt{stu}} & \frac{s^2u}{2\sqrt{stu}} & \frac{s^2t}{2\sqrt{stu}} \\
\frac{s^2u}{s^2u} & \frac{s^2t}{s^2u} & \frac{s^2u}{s^2u}
\end{array}\right)\)

Chain Rule

1. \((f \circ g)'(t) = \sin(t)(4\cos(t) + 7)\)
2. \((f \circ g)'(-1) \approx 188\)
3. (a) 84
4. \(\nabla (f \circ g) = ((2t^2 - 8s^2t)\ln(2st) + (2st^2 - 2s^4t)\frac{1}{t}, (4st - 2s^4)\ln(2st) + (2st^2 - 2s^4t)\frac{1}{t})\)
5. (a) \(f_s(s,t) = 2\cos(t) - 2(s+t)e^{2t}\) and \(f_t(s,t) = -2s\sin(t) - 2(s+t)e^{2t} - 2(s+t)^2e^{2t}\)

Tangent Lines and Planes

1. \(\frac{\sqrt{14}}{7}\)
2. \(6x - 8y - z = 5\)
3. (a) No solutions yet!
7.4 Optimization and Lagrange Multipliers Drill

Critical Points

1. For each of the following functions, find the critical points and determine if they are maxima, minima, or saddle points.
   (a) \( f(x, y) = 1 - x^2 - y^2 \)
   (b) \( g(x, y) = e^{-x} \sin y \)
   (c) \( F(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5 \)
   (d) \( g(x, y) = x^2 + xy + y^2 \)
   (e) \( z = 8x^3 - 24xy + y^3 \)

2. For each of the following functions, find the absolute extrema of the function on the given closed and bounded set \( R \) in \( \mathbb{R}^2 \).
   (a) \( f(x, y) = 2x^2 - y^2; \ R = \{(x, y) : x^2 + y^2 \leq 1\} \)
   (b) \( g(x, y) = x^2 + 3y^2 - 4x + 2y - 3; \ R = \{(x, y) : 0 \leq x \leq 3, -3 \leq y \leq 0\} \)

3. Find the direction at which the maximum rate of change of \( g(x, y) = \ln(xy) - 3x + 2y \) at \( p = (3, 2) \) will occur and find the maximum rate of change.

4. Find the direction in which the function \( f(x, y) = x^3 - y^5 \) increases the fastest at the point \( (2, 4) \).

5. For each of the following four problems, find all critical points of \( f \) and classify these critical points as relative maxima, relative minima, or saddle points using the second derivative test whenever possible.
   (a) \( f(x, y) = xy^2 - 6x^2 - 3y^2 \)
   (b) \( f(x, y) = \frac{9x}{x^2 + y^2 + 1} \)
   (c) \( g(x, y) = x^2 + y^3 + \frac{768}{x + y} \)

Optimization and Lagrange Multipliers

1. Find all extrema of \( f(x, y) = 1 - x^2 - y^2 \) subject to \( x + y = 1, x \geq 0, \) and \( y \geq 0 \).
2. Find all extrema of \( f(x, y) = 1 - x^2 - y^2 \) subject to \( x + y \leq 1, x \geq 0, \) and \( y \geq 0 \).
3. Find the absolute extrema of the function \( g(x, y) = 2\sin(x) + 5\cos(y) \) on the rectangular region \( R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 5\} \).
4. Find the minimum value of \( z = x^2 + y^2 \) subject to \( x + y = 24 \).
5. Find the extreme values of \( f(x, y) = 2x^2 + y^2 - y \) subject to \( x^2 + y^2 = 4 \) using Lagrange multipliers.
6. Find three positive numbers whose sum is 123 such that their product is as large as possible.
7. A container in \( \mathbb{R}^3 \) has the shape of a cube with each edge length 1. A (triangular) plate is placed in the container so that it intersects the cube in the plane \( x + y + z = 1 \). If the container is heated so that the temperature at each point is given by \( T(x, y, z) = 4 - 2x^2 - y^2 - z^2 \) in hundreds of degrees. What are the hottest and coldest points on the plate?
8. A company has three production plants, each manufacturing the same product. If plant A produces $x$ units at the cost of $$(x^2 + 2,000),$$ plant B produces $y$ units at the cost of $$(2y^2 + 3,000),$$ and plant C produces $z$ units at the cost of $$(z^2 + 4,000).$$ If there is an order for 11,000 units to be filled, determine how the production should be arranged among these three plants so that the total production cost can be minimized.
Optimization and Lagrange Multipliers Solutions

Critical Points

1. (a) \((0, 0, 1)\) is a max
   (b) none, why?
   (c) \((0, 1, 4)\) is a min
   (d) \((0, 0, 0)\) is a min

2. (a) \((\pm 1, 0, 2)\) are local minima and \((0, \pm 1, -1)\) are local maxima
   (b) all points to consider should be: \((2, -1/3, 0, 0), (3, 0), (3, -3), (0, -3), (2, 0), (0, -1/3),
       (3, -1/3), and (2, -3); (2, -1/3, -22/3) is a min; \((0, -3, 18)\) is a max

3. \((-8/3, 5/2)\) and \(\sqrt{481}/6\)

4. \((3, -320)\)
   (a) \((3, \pm 6, -54)\) are saddles and \((0, 0, 0)\) are all maxima
   (b) \((0, 1, 9/2)\) (D is long – use Maple!)
   (c) icky algebra – use Maple!

Optimization and Lagrange Multipliers

1. \((1/2, 1/2, 1/2)\) is the max, each of \((1, 0, 0)\) and \((0, 1, 0)\) is a min

2. \((0, 0, 1)\) is the max, each of \((1, 0, 0)\) and \((0, 1, 0)\) is a min

3. there is one saddle on the interior, but no extrema on the interior; the extrema occur on the boundaries at \((0, 0), (\pi/2, 5), (2, 0), and (\pi/2, 0)\)

4. \((12, 12, 288)\)

5. \((0, \pm 2)\) and \((\pm \sqrt{15}/2, 1/2)\)

6. \(x = 41, y = 41, z = 41\)
7.5 Integration Drill

Basics

1. Evaluate \( \int x^3y - 3x \, dx \) and \( \int_1^3 x^3y - 3x \, dy \)

2. Evaluate \( \int xy e^{xy} \, dx \) and \( \int_{-1}^1 xy e^{xy} \, dy \)

3. Compute the area of the region bounded by the parabola \( y = x^2 - 2 \) and the line \( y = x \) by first integrating with respect to \( x \) and then integrating with respect to \( y \).

4. Compute the area bounded by \( y = x^3 \) and \( y = 5x \) in four ways. (a) Single integral with respect to \( x \), (b) single integral w.r.t. \( y \), (c) double integral, \( dx \, dy \), and (d) double integral, \( dy \, dx \).

Regions

1. Sketch the region \( \phi = \pi/6 \) in spherical coordinates.

2. Sketch the region bounded by \( r = 1 \) and \( r = 2\sin(\theta) \) in polar coordinates.

3. Sketch the region bounded by \( x^2 + y^2 \leq 9 \) and write in polar coordinates.

4. Sketch the region between \( x^2 + y^2 = 25 \), \( x^2 + y^2 = 4 \), and \( x \geq 0 \) and write in polar coordinates.

5. Find the area of the region \( D \) bounded by \( y = \cos(x) \) and \( y = \sin(x) \) on the interval \( [0, \pi/4] \).

Integration

1. Compute \( \int_1^2 \int_0^3 x^2y^2 - 3xy^5 \, dy \, dx \) and \( \int_0^3 \int_0^2 x^2y^2 - 3xy^5 \, dx \, dy \). Are they equal? What theorem is this an example of?

2. Evaluate \( \int_0^1 \int_0^y e^2 \, dx \, dy \).

3. Compute \( \int_R x^2 e^{xy} \, dA \) where \( R = \{(x,y) : 0 \leq x \leq 3, 0 \leq y \leq 2\} \).

4. Compute \( \int_0^3 \int_0^2 \sqrt{2x+y} \, dy \, dx \).

5. Compute \( \int_R \frac{\ln(\sqrt{y})}{xy} \, dA \) where \( R = \{(x,y) : 1 \leq x \leq 4, 1 \leq y \leq e\} \).

6. Evaluate \( \int_D 160xy^3 \, dA \) where \( D \) is the region bounded by \( y = x^2 \) and \( y = \sqrt{x} \).

7. Sketch and find the volume of the solid bounded above by the plane \( z = y \) and below in the \( xy \)-plane by the part of the disk \( x^2 + y^2 \leq 1 \).

8. Sketch the region, \( D \), that is bounded by \( x = y^2 \) and \( x = 3 - 2y^2 \) and evaluate \( \int_D (y^2 - x) \, dA \)

9. Compute \( \int_0^2 \int_0^{\sin(x)} y \cos(x) \, dy \, dx \).
10. Determine the endpoints of integration for \( \int \int_S e^{xy} \, dA \) where \( S \) is the region bounded by \( y = \sqrt{x} \) and \( y = \frac{x}{9} \). Don’t integrate.

11. Determine the endpoints of integration for \( \int \int_S dA \) where \( S \) is bounded by \( x = y^2 + 4y \) and \( x = 3y + 2 \).

12. Determine the endpoints of integration for \( \int \int_S 2x \, dA \) where \( S \) is the region bounded by \( yx^2 = 1, y = x, x = 2, \) and \( y = 0 \).

13. Evaluate \( \int \int_D dA \) where \( D \) is the region bounded by the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)

**Polar Coordinate Integration**

1. Sketch the region \( D \) between \( r = \cos \left( \frac{\theta}{2} \right) \) and \( x^2 + y^2 = 1 \) with \( 0 \leq \theta \leq \pi \). Evaluate \( \int_D 1 \, dA \).

2. Find the volume of the solid bounded by the cone \( \phi = \frac{\pi}{6} \) and the sphere \( \rho = 4 \).

3. Consider \( \int_R x^2 + y^2 + 1 \, dA \) where \( R = \{ (x, y) : x \geq 0, 9 \leq x^2 + y^2 \leq 16 \} \). Write the integral in both rectangular and polar coordinates. Compute each to verify your answer.

**Coordinate Transformations**

1. Find the Jacobian for the transformation: \( x = u^2 + v^2 + w, y = uv - v, \) and \( z = \ln(w) - \frac{v}{u} \)

2. Find the Jacobian for the transformation: \( x = r \cos \theta, y = r \sin \theta, \) and \( z = z \).

3. Find the Jacobian for the transformation: \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, \) and \( z = \rho \cos \phi \).

4. Evaluate \( \int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x-y}{2} \, dx \, dy \) using \( u = \frac{2x-y}{2} \) and \( v = \frac{y}{2} \).

5. Use the transformation \( x = \frac{u}{v} \) and \( y = v \) to rewrite (but not evaluate) the double integral \( \int \int \sqrt{xy^3} \, dx \, dy \) over the region in the plane bounded by the \( x \)-axis, the \( y \)-axis, and the lines \( y = -2x + 2 \) and \( x + y = 7 \).

6. Compute \( \int_0^1 \int_0^{y^2} (1-y) \sin \left( \frac{x}{y} \right) \, dx \, dy \) using \( u = \frac{x}{y} \) and \( v = 1 - y \).

7. Write an integral in rectangular coordinates that represents the area enclosed by the ellipse \( \frac{x^2}{16} + \frac{y^2}{49} = 1 \). Now, compute this integral by using the transformation \( x = 4u \) and \( y = 7v \).

8. Find the volume of the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) using the transformation \( x = au, y = bv, z = cw \).

9. Evaluate the triple integral \( \int_0^6 \int_0^8 \int_0^{x+y+4} \frac{2x-y}{2} + \frac{y}{6} + \frac{z}{3} \, dx \, dy \, dz \) using the transformation \( u = \frac{2x-y}{2}, v = \frac{y}{2}, \) and \( w = \frac{z}{3} \).
Triple Integrals, Cylindrical, and Spherical Coordinates

1. Sketch and find the volume of the solid formed by \( f(x, y) = 4x + 2y \) above the region in the \( xy\)-plane bounded by \( x = 2, x = 4, y = -x, y = x^2 \).

2. Fill in the blanks:
   \[
   \int_0^1 \int_{-x}^0 \int_{-x}^0 f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \int_0^{1-x-2y} f(x, y, z) \, dz \, dy \, dx
   \]
   \[
   = \int_0^1 \int_{-x}^0 f(x, y, z) \, dy \, dz \, dx
   \]
   \[
   = \int_0^1 \int_{-x}^0 f(x, y, z) \, dx \, dz \, dy
   \]

3. \[
   \int_0^1 \int_{-x}^0 \int_{-x}^0 f(z, y, x) \, dz \, dy \, dx = \int_0^1 \int_{-x}^0 f(x, y, z) \, dx \, dz \, dy = \int_{-x}^0 \int_0^1 f(x, y, z) \, dx \, dz \, dy
   \]

4. \[
   \int_0^1 \int_{-x}^0 \int_{-x}^0 f(z, y, x) \, dz \, dy \, dx = \int_0^1 \int_{-x}^0 f(x, y, z) \, dx \, dz \, dy = \int_{-x}^0 \int_0^1 f(x, y, z) \, dx \, dz \, dy
   \]

5. Find the volume of the solid bounded by \( x^2 + y^2 + z = 8 \) and \( z = 4 \).

6. Evaluate \[
   \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z \, \sqrt{4-x^2-y^2} \, dz \, dy \, dx
   \]
   by converting to (a) cylindrical coordinates; (b) spherical coordinates.

Application - Center of Mass

1. Suppose \( \delta(x, y) = x + y \) is the density function of a thin sheet of material bounded by the curve \( x^2 = 4y \) and \( x + y = 8 \). Find its total mass. Find its first moments. Find its center of mass. Find its second moments.
Integration Solutions

Basics

1. \( \frac{1}{4}x^4y - \frac{3}{2}x^2 \) and \( 4x^3 - 6x \)
2. \( e^{xy}(x - \frac{1}{y}) \) and \( e^{x}(1 - \frac{1}{x}) + e^{-x}(1 + \frac{1}{x}) \)
3. \( 9/2 \)
4. \( 25/4 \)

Regions

1. it’s a cone
2. it’s the intersection of two circles
3. \( r \leq 3 \)
4. \( 2 \leq r \leq 5 \) and \( \theta \in [-\pi/2, \pi/2] \)
5. \( \sqrt{2} - 1 \)

Integration

1. \( -705, \) Fubini’s Theorem
2. \( \frac{1}{2}(e - 1) \)
3. \( -\frac{1}{4}(17 - 5e^6) \)
4. \( \approx 11.62 \)
5. \( \approx 0.346 \)
6. 6
7. \( \frac{2}{3} \)
8. \( \frac{24}{5} \)
9. \( \frac{1}{6}\sin^3(2) \)
10. \( \int_0^{81} \int_{\sqrt{2}}^9 e^{xy} \, dA \)
11. \( 4.5 \)
12. \( \frac{2}{3} + 2\ln(2) \)
13. \( 6\pi \)

Polar Coordinate Integration

1. \( \frac{\pi}{4} \)
2. $-\frac{64(\sqrt{3}-2)\pi}{3}$

3. $\frac{189\pi}{4}$

Coordinate Transformations

1. $-2u(\frac{1}{w}-1) - 2v + \frac{1}{u}(2v^2 - w) + v/u^2$

2. $r$

3. $\rho^2\sin(\phi)$

4. $2$

5. $\int_0^7 \int_{7v-v^2}^{v-v^2/2} \sqrt{uv}dudv$

6.

7. area of ellipse is $\pi ab$ in this case $28\pi$ (just like circle)

8. volume of ellipsoid is $4/3\pi abc$ (just like sphere!)

Triple Integrals, Cylindrical, and Spherical Coordinates

1. $2472/5$

2. $\int_0^1 \int_0^{1-x} \int_0^{\frac{1-x}{z}} f(x,y,z)dy
dz
dx$

3. $\int_0^1 \int_0^{1-x} \int_0^{\frac{1-x}{z}} f(x,y,z)dy
dz
dx$

4.

5. $8\pi$
7.6 Line Integrals, Flux, Divergence, and Gauss’ Theorem Drill

Note: In this section, we write the functions and integrals using many different notations. If you are unsure about the meaning of a notation, please ask!

Vector Fields, Curl, and Divergence

1. Sketch the vector field, \( \vec{f}(x,y) = x^2 \hat{i} + \hat{j} \).
2. Sketch the vector field, \( \vec{g}(x,y) = (x,-y) \).
3. Sketch the vector field, \( \vec{h}(x,y,z) = y\hat{j} \).
4. Compute the divergence and curl of the vector field, \( \vec{f}(x,y,z) = (y^2z, x^3 + z + y, \cos(xyz)) \).
5. Find \( g \) satisfying \( \nabla g = F \) if it exists. \( F(x,y) = (ye^{xy} + 2x) \hat{i} + (xe^{xy} - 2y) \hat{j} \)
6. Find \( g \) satisfying \( \nabla g = F \) if it exists. \( F(x,y) = (e^x \sin(y), e^x \cos(y)) \)
7. Is \( \vec{f}(x,y) = (2xy, x^2) \) a conservative vector field? If so, find a potential for it.
8. Is \( \vec{h}(x,y) = (y\cos(x), \sin(x)) \) a conservative vector field? If so, find a potential for it.
9. Is \( \vec{g}(x,y,z) = (e^x \sin y + yz) \hat{i} + (xz + y) \hat{j} + (e^x \cos z + xy + z^2) \hat{k} \) a conservative vector field? If so, find a potential for it.
10. Is \( \vec{g}(x,y) = 2x\hat{i} + y\hat{j} \) a conservative vector field? If so, find a potential for it.
11. Show that \( \vec{F}(x,y,z) = x\hat{i} + y\hat{j} + 2z\hat{k} \) is conservative and find a function \( f \) such that \( \vec{F} = \nabla f \).

Line Integrals over Scalar Fields

1. Let \( f(x,y) = x+y \) and \( \vec{c} \) be the unit circle in \( \mathbb{R}^2 \). Evaluate \( \int_{\vec{c}} f \, ds \). Recall that \( ds \) means to evaluate with respect to the arc length.
2. Evaluate \( \int_{\vec{c}} \sqrt{xy + 2y + 2} \, ds \) with \( \vec{c} \) the line segment from \((0,1)\) to \((0,-1)\).
3. Evaluate \( \int_{\vec{c}} (x - y + z - 2) \, ds \) where \( \vec{c} \) is the line segment from \((0,1,1)\) to \((1,0,1)\).

Line Integrals over Vector Fields

1. Compute \( \int_{\vec{c}} f \cdot dr \) where \( f(x,y) = (y,x^2) \) and \( \vec{c}(t) = (4-t,4t-t^2) \) for \( 0 \leq t \leq 3 \).
2. Compute \( \int_{\vec{c}} f \cdot dr \) where \( f(x,y) = (y,x^2) \) and \( \vec{c}(t) = (t,4t-t^2) \) for \( 1 \leq t \leq 4 \).
3. Compute \( \int_{\vec{c}} F \cdot dr \) where \( F(x,y) = (-\frac{1}{2}x,-\frac{1}{2}y, \frac{1}{4}) \) and \( \vec{c}(t) = (\cos(t),\sin(t),t) \), \( 1 \leq t \leq 4 \).

Divergence Theorem and Green’s Theorem

1. Verify the divergence theorem for the flow \( f(x,y) = (0,y) \) over the circle, \( x^2 + y^2 = 5 \).
2. Let \( f(x,y) = (u(x,y), v(x,y)) = (-x^2,xy^2) \) and \( \vec{c} = \{(x,y) : x^2 + y^2 = 9 \} \) and \( D \) be the region bounded by \( \vec{c} \). Verify Green’s Theorem by evaluating both \( \int_{\vec{c}} f(\vec{x}) \, d\vec{x} \) and \( \int_D v_x (x,y) - u_y (x,y) \, dA \).
3. Verify Green’s Theorem where \( f(x,y) = (4xy, y^2) \) and \( \mathbf{c} \) is the curve \( y = x^3 \) from the \((0, 0)\) to \((2, 8)\) and the line segment from \((2, 8)\) to \((0, 0)\).

4. If you want more practice on verifying Green’s and Gauss’ theorems, then note that each problem that asks you to verify Gauss’ theorem could have asked you to verify Green’s theorem and vice-versa. You won’t need solutions because you are computing both sides of the equation and they must be equal if all your integration is correct.

**Divergence Theorem in Three Dimensions**

1. Verify the divergence theorem for \( f(x,y,z) = (xy, z, x+y) \) over the region in the first octant bounded by \( y = 4, z = 4 - x, z = 0, y = 0, \) and \( x = 0. \)

2. Verify the divergence theorem for \( f(x,y,z) = (2x, -2y, z^2) \) over the region \([0,3] \times [0,3] \times [0,3]. \)
Line Integrals, Flux, Divergence, and Gauss’ Theorem Solutions

Vector Fields, Curl, and Divergence

1. no sketch
2. no sketch
3. no sketch
4. no solution
5. \( g(x, y) = e^{xy} + x^2 - y^2 \)
6. \( g(x, y) = e^x \sin(y) \)
7. yes, \( g(x, y) = x^2y \)
8. yes, \( g(x, y) = y \sin(x) \)
9. yes, \( g(x, y, z) = e^x \sin(z) + xyz + \frac{1}{2}y^2 + \frac{1}{2}z^3 \)
10. no solution
11. no solution

Line Integrals over Scalar Fields

1. 0
2. \( 8/3 \)
3. \( -\sqrt{2} \)

Line Integrals over Vector Fields

1. \( 69/2 \)
2. \( -69/2 \)
3. \( 3\pi/4 \)

Divergence Theorem and Green’s Theorem

1. If both sides are equal, you probably got it right!
2. \( 81\pi/2 \)
3. \( -256/15 \)
4. no solution

Divergence Theorem in Three Dimensions

1. no solution
2. no solution