Linear Point Set Theory

A Vehicle for Mathematical Metamorphosis

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Contents

Preface iii
Acknowledgments v
To the Instructor vi
To the Student x
1 Introduction 1
2 Consequences of Axiom 1 4
3 Consequences of Axioms 1 and 2 7
4 Consequences of Axioms 1, 2 and 3 13
5 Consequences of Axioms 1, 2, 3 and 4 17
A Motel $\infty$ 18
B Induction 19
C Countability 22
Preface

Many years ago, when I began my teaching career, the predominating teaching style was the lecture method. Being trained under H.S. Wall and R. L. Moore at the University of Texas in Austin, Texas, over my career, I taught many of my classes using a learner-centered style of teaching. Both Wall and Moore taught in such a manner with Moore’s style eventually becoming known as the Texas Method or, merely, the Moore Method. Of course, for many good reasons and out of necessity, I also taught many classes using the lecture method. Sadly, I was a member of a pedagogical minority.

After some years, the educational community began realizing that they should be engaging their students; however, the general complaint was that, apparently, no knew one how to effectively engage students, or so we were told. But, yet, there were people who knew how! The students of H. S. Wall and R.L. Moore and their academic descendants had been engaging their students for many decades. These notes represent such an endeavor. My hope is that you will enjoy teaching from them as much as I have. More importantly, your students will enjoy the course and become transformed; that is, they will become real mathematicians.

This brings me to the title of these notes. The bare-bones content is linear point set theory: the spirit is the quickening of the mathematical soul. They are the best vehicle I know of for taking students from a state of not knowing how to do mathematics beyond algorithmic thinking to a state where they have become almost totally transformed into a new intellectual point of view, a mathematical metamorphosis, if you will. The approach in these notes is clean, simple and uncluttered with the needless notation and cognitive noise found in today’s courses. The material builds continuously in increasing levels of sophistication almost in a step-by-step fashion rather than starting a new genre every few days as done by contemporaneous transition courses.

Finally, at the end of your course, you will find that many students can now do proofs, use language concisely and use their imaginations to discover worlds they have never seen. Your colleagues may ask you the following question:

“Do you really expect students to prove those theorems on their own?”
Hopefully, after you have used these notes, you can give the following answer:

“Yes, with patience and skill, students can do significant mathematics. The road is not easy, but, it is certainly worth it! You will be surprised that ordinary students can do extraordinarily things, if you give them the freedom.”

Enjoy!
Acknowledgments

A smaller subset of these notes fell into my hands when I began my teaching career. The original author or authors are unknown to me. The seed for these notes was planted at that time. Moreover, Foundations of Point Set Theory by R. L. Moore and the experience of being in Moore’s classes have informed my design of this course and many more courses like this that I have taught.

Special thanks go to all my many students over the decades who have learned from these notes and who have helped to improve them. They are in my memory all the time: they are a part of me. They might be surprised to learn that even from my early days of teaching that I think about them from time to time.

I express my appreciation to Dr. Joanne Baker, Dr. Dale Daniel, Jason Montgomery, and the referee for locating various errors and making many constructive and thoughtful suggestions. Those errors that remain are, of course, my responsibility.

Finally, Alaine Fay, my wife of many years, has heard me talk about this course, about the students and has inspired my teaching and the design of my courses over the years. Although she has never taken a Moore Method class, she may know more about the method than many who have actually taken such a course. Would it be that all mathematics teachers have a partner in life who has played such a substantial and supportive role.
To the Instructor

These few pages represent a set of notes that I used for over thirty years in a course that was positioned in the curriculum to transition majors in mathematics to the acquisition of good proof techniques and the development of imagination. For the most part, this was a successful course. Individual problems or theorems vary from moderately difficult to challenging. Individual success, of course, is dependent on student attitude, work ethic and latent mathematical talent. The teacher of a course such as this one will have to work harder than he or she would if teaching a traditional transition course; however, that teacher will have more fun and the experience will be more profitable for both students and teacher. The intent of these notes is to cause a veritable metamorphosis of students from a culture of algorithmic thinking to a new culture, a mathematical culture.

This was a course with a junior number. We expected sophomores and, sometimes, first semester freshmen to take the course; however, I would discourage the latter who, although capable, might better benefit if they delayed taking the course for a semester or a year. This was our “bread and butter” course for two reasons:

- The course became a magnet for potential math majors. It had quite a reputation even though a variety of people taught the course. One could find majors such as from English, philosophy, or classics enrolled in this course. Some succeeded quite well even adding mathematics as a major.

- Because the course was not burdened with coverage of specific information, teacher and student could focus on one thing, students learning how to do proofs without distractions.

There are three big factors that make this course work and greatly enhance the possibility of the metamorphosis I mentioned earlier.

- Set the bar high but within the grasp of the students. It is most important that the challenges be real and worthy of attack. There may be other methods of teaching and other curriculum materials that may place information, even beautiful information, in the students’ minds but there is still another very important factor. It is that the joy of proving a very hard theorem on your own is a thing to be treasured. Meeting
a challenge develops the mathematical spirit and creates mathematicians. This set of notes will give that opportunity to students if they will put forth the effort and believe they can be successful.

• The proof of each theorem contains a key. The teacher protects that key while the students are the ones who attempt its discovery. There can be much discussion about language, ideas or techniques but do not divulge the core idea behind the proof; that is for the students to discover on their own. I have seen teaching moments literally stolen from students by someone saying too much at the wrong time. Therefore, it is imperative that teachers not give hints, even subtle or hidden hints, that give insight into the key or core proof idea. Keys to proofs must be protected. It may seem extreme to a product of today’s culture but R.L. Moore \(^1\) did not want students to even chat about proofs of theorems outside class for fear that some student would hear an inerrant clue that exposes a key to a proof. I agree with Professor Moore.

• Adjust the level and speed if necessary. There may be times when the course is not working at the macro level or even the micro level. Make adjustments. Think about the class chemistry. What are the student expectations that are inhibiting them that have not been addressed? Is the material too abstract or too concrete or too easy? How much the students grow has a lot to do with the caliber of the challenge.

Consider the following:

• I expected students to dedicate themselves to this course. It was the one time we did not have to concern ourselves with what would be remembered later – other courses covered the requisite math major material. It is important to get the entire class engaged.

• Pick a classroom that barely holds the students. It is important that the students not get lost in the classroom!

• Class sizes of approximately 10 - 15 work well. With smaller classes, you run the risk of not getting a rich involvement of students and interplay between the students.

• Attempt to orient your students to this new approach.

• Repeatedly remind the students what you are doing and why.

• Have a change of pace once in a while. For example:
  
  – Do some enrichment that connects with the course such as the Motel Problem you will find in Appendix A.

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\(^1\) R. L. Moore was a teacher-scholar of the twentieth century. He was a world class mathematician and the father of a teaching method that bears his name. A good source is *R. L. Moore: Mathematician and Teacher* by John Parker.
– Take some time to review the theorems that have been covered. Here is a format I have used: One of the students volunteers to be the stenographer for the class. He or she writes a line of a proof on the board that is posed by other students. I call on the students one by one, to give the first line of a proof or the next line of the proof. When we finish the proof, each participant earns some credit. The stenographer earns credit as well. This drill is good for class chemistry and gives the class a change of pace. This assures that students have at least one correct proof in their notes. It also allows discussions on the finer points of writing proofs. Students are allowed to pass without participation and without penalty. Students are not allowed to look at their notes during the exercise.

• After some time, take the opportunity to talk about how one discovers a proof. I used to tell my students, “You are not in this course to learn how to think; you are here to learn how YOU think.”

• Hand out essays on creativity such as *Mathematical Creation* by Henri Poincaré. Anecdotes about R. L. Moore or some other mathematician are good at just the right time. I have been known to show the documentary film, “Challenge in the Classroom” a film about Moore’s teaching.²

• Many students will believe that they will be graded on the percentage of proofs they create. Such is not the case in my classes. Dissuade the students from thinking along these lines. It is their development that is most important. Evaluate students on how well they have mastered process.

• Because this material is more challenging than the usual transition course, I discuss the part fear plays in their performance. I tell them that everyone is afraid of something but some people let fear control them and others control fear.

• Piet Hein, a Danish mathematician, produced a set of little books called *Grooks*. Each book was a compilation of one page drawings or cartoons with a short poem underneath. I used to read these to the students in one class period. The ones I selected are *Problems, T.T.T.*, and *Simply Assisting God*.

• The role of exercises, problems and theorems is as follows: The theorems are where the action of the course resides and represent the main results of the course. With occasional exceptions, it is expected that only students present the theorems and the problems. The problems

²This documentary can now be found in the publication division of the Mathematical Association of America.
represent minor results whose importance is to prepare the students for proofs of the theorems. It is suggested that all problems and theorems be undertaken except as noted in the next bullet. The exercises are optional and may be skipped; however, working through them will better prepare the students. The exercises may be done by students, teachers or by collaboration. Some lectures based on the exercises may help speed up the course.

- In lieu of covering all theorems, a class that is able to cover the material through the significant results – Theorems 36 through 37, or in the vicinity thereof – has done an adequate job. The quality is strongly dependent on how deeply and carefully the material is covered in class. Sometimes limited coverage is better than covering a lot of material quickly. This is a judgment call by the teacher.

- One way to speed up the course is to eliminate coverage of theorems from Definition 15 to Theorem 34. In this way, the class may cover more than stated in the previous bullet.
To the Student

These notes represent a very different culture of learning. To be successful, you will have to re-orient your mind and expectations of what a learner-centered course should look like. The burden is now on you. You will create knowledge. Your teacher will act as a coach and an arbiter of what constitutes a good proof. He or she will encourage you and will exhort you to work hard and to work smart.

These notes are in the spirit of R. L. Moore, the father of the American school of point set topology. In his prime, he was considered to be one of the top ten mathematicians in the world and was considered to be one of the most effective teachers in collegiate mathematics in the first half of the twentieth century. This course is essentially taught the way R. L. Moore taught his courses for over fifty years and in the rich tradition of many of his mathematical descendants over many decades.

*Linear Point Set Theory: A Vehicle for Mathematical Metamorphosis* is designed to inculcate in you those mental processes that characterize mathematical thought. A successful completion of these notes will develop your powers of deduction and imagination. You will learn to use language precisely and concisely. Settling conjectures, creating counterexamples and conceptualizing an abstract definition will stretch your imagination.

The usual format of a course using these notes is not primarily lecture although I have been known to lecture when the need arises. There are no surprises about the way people learn. Vince Lombardi, the coach of the Green Bay Packers in their heyday, said that football is nothing more than blocking, tackling, and running—the fundamentals. I stress the fundamentals of mathematics—logic and imagination. Class members present much of the material. Most class periods are a sort of laboratory where student presentations are critiqued by the class. You learn mathematics the best way I know—by doing rather than by watching. My experience is that students who do well in this course develop an enthusiasm and a sense of accomplishment quite analogous to that of climbing a mountain. There is no substitute for doing proofs or solving problems on your own, which you know to be challenging, especially if you know you received little or no help.

To communicate the importance of a student doing proofs or solving problems on his or her own, imagine a Monarch butterfly emerging from its chrysalis, in its pupal (immature) state. The process of emergence strength-
ens the butterfly’s wings so that it may be able to fly and feed itself. If it
does not endure this process, it will die. I remember my own struggles and
transformation into a mathematician. My hope is that in time, you should re-
alize that you can become a mathematician as well. If for some strange, odd
reason, you do not succeed, you will not die as in my butterfly metaphor.
There are more important things in life. Shake it off and move on! Who
knows – you may come back to these notes or a course like this and have a
rousing success. The passage of time makes a difference!

Charles A. Coppin  Journal of Inquiry-Based Learning in Mathematics
Chapter 1

Introduction

There is probably no other science which presents such different appearances to one who cultivates it and one who does not, as mathematics. To [the noncultivator] it is ancient, venerable, and complete; a body of dry, irrefutable, unambiguous reasoning. To the mathematician, on the other hand, his science is yet in purple bloom of vigorous youth, everywhere stretching out after the “attainable but unattained”, and full of the excitement of nascent thoughts; its logic is beset with ambiguities, and its analytic processes, like Bunyan’s road, have a quagmire on one side and a deep ditch on the other, and branch off into innumerable by-paths that end in a wilderness.


Preliminary Notions. The following ideas may be useful throughout this study.

- The notions of set and collection are synonymous. When we state “a set of sets”, we would be better to state “a collection of sets.” By $x \in M$, we mean “$x$ is a member of $M$” or “$x$ is in $M$” and, of course, $x \notin M$ means “$x$ is not in $M$” or “$x$ is not a member of $M$.” A set $A$ is said to be a subset of a set $B$ (written “$A \subseteq B$”), if each member of $A$ is a member of $B$. If $A$ is a subset of $B$ but $B$ contains members not in $A$, then $A$ is said to be a proper subset of $B$ (sometimes written “$A \subset B$”). When we state that the set $A$ is equal to the set $B$ (written “$A = B$”), we mean that $A \subseteq B$ and $B \subseteq A$. By $A \cup B$ (written “$A$ union $B$”), we mean the set to which a point belongs if that point belongs to $A$ or $B$. When $A$ and $B$ share a common point, by $A \cap B$ (written “$A$ intersection $B$”), we mean the set to which a point belongs if that point belongs to $A$ and $B$. When $G$ is a collection of sets, by $\bigcup G$ or $\bigcup_{g \in G} g$ we mean the set to which a point belongs if it belongs to at least one member of $G$ and, similarly, when a point belongs to each member

\[1\] If the reader is not familiar with proving two sets equal, the instructor of the class should supply that experience to anyone in the class who is a novice with such things.
of \( G \), by \( \cap_{g \in G} \) we mean the set to which a point belongs if it belongs to each member of \( G \). When each of \( A \) and \( B \) is a set, by \( A \setminus B \) we mean the set to which a point belongs if it belongs to \( A \) but does not belong to \( B \). When \( A \subseteq U \), by \( A^c \), we mean \( U \setminus A \); however, \( U \) must be the universal set, the set of all points. Importantly, de Morgan’s Laws will play an important role in our studies. They are

\[
( \bigcup_{g \in G} g)^c = \bigcap_{g \in G} g^c
\]

and

\[
( \bigcap_{g \in G} g)^c = \bigcup_{g \in G} g^c^2.
\]

- We assume the fact that the set of positive integers has the **Well Ordering Property**; that is, that any nonempty set of positive integers has a least member. We prefer application of this principle over any other methods where proof by mathematical induction is required. Although the Well Ordering Principle is somewhat harder to apply than other approaches, it is natural and primitive, and requires deeper understanding than the other methods. See Appendix B for further development.

- A set \( M \) is said to be **finite** if there is a terminating sequence \( a_1, a_2, \ldots, a_m \) of distinct terms that are the elements of that set; that is,

\[
M = \{a_1, a_2, \ldots, a_m\}
\]

where it is understood that \( m \) is a positive integer. A set \( M \) is said to be **infinite** if there is a nonterminating sequence \( a_1, a_2, \ldots \) of distinct terms that are the elements of the set; that is,

\[
\{a_1, a_2, \ldots\} \subseteq M
\]

We assume without proof that any subset of a finite set is a finite set and that any set that contains an infinite set is itself infinite. These results can be proved from what has been given in this section. A set \( M \) is said to be **countable** if there is a sequence (terminating or not) whose terms are the elements of the set. Of course, a set is said to be **uncountable** if it is not countable. See Appendix C for more development of ideas on countability and uncountability.

- By \( = \), we mean **logical identity**; thus, when we say \( x = y \), we mean \( x \) and \( y \) are different names for the same object. \( Tom = Harry \) means \( Tom \) and \( Harry \) are the same person with the name \( Harry \) and the name \( Tom \).

- Students should realize that the proof of any theorem is based on the axioms given to the reader at the point in these notes where the theorem is stated. Some statements that are theorems may not be theorems if stated before a certain axiom.

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\footnote{Some review or instruction concerning the truth of de Morgan’s Laws and their proofs may be required.}
Axiom Systems.

An axiom system has two components:

- A set of special words which we officially assume have no meaning. The words are sometimes called “undefined terms”, “primitives” or just “technical terms.”

- A set of statements called axioms which we assume to be true concerning the “undefined terms.”

An interpretation of an axiom system is a set of meanings for the undefined terms. These meanings are assigned without any regard for whether the axioms become true statements about the undefined terms. A model for an axiom system is an interpretation for the axiom system in which the axioms are true statements about the undefined terms.
Chapter 2

Consequences of Axiom 1

Undefined Terms. In the spirit of the Axiomatic Method, we leave undefined the word point and the expression the point x precedes the point y. This undefined expression will be written $x < y$. In these notes, all sets have at least one element. By “point x follows the point y” we mean $y < x$. By “$x \leq y$”, we mean that “$x < y$ or $x = y$.” In case a misunderstanding persists concerning the relationship between “point”, “point set” and $S$, let it be clear that unless there is a stipulation to the contrary

- “point” always means member of $S$.
- “point set” means a subset of $S$.
- $S$ is a set of points.

Axiom 1. $S$ has the following properties:

a. $x < y$ or $y < x$ for any two different points $x$ and $y$.

b. $x < y$ implies $x \neq y$.

c. $x < y$ and $y < z$ implies $x < z$ for points $x$, $y$, and $z$.

Problem 1. Restate Axiom 1, in plain English without using so called “mathematical symbols.”

Exercise 1. Prove that the “or” in Axiom 1a is an exclusive “or”\(^1\) when Axiom 1a, 1b and 1c are taken into account; that is, prove that $x < y$ and $y < x$ cannot both be true.

Exercise 2. Suppose by point we mean a human being. What interpretation could you give to the word precedes so that Axiom 1 is satisfied?

Exercise 3. HFD Corporation, the world’s largest corporation, hires literally thousands of people each year. Suppose by point we mean an employee of

\(^1\)In the case of “inclusive or”, “$p$ or $q$” is true if $p$ is true, $q$ is true or both are true. The meaning of “exclusive or” is as follows. The statement “$p$ or $q$” is true only if $q$ is true and $p$ is false or $p$ is true and $q$ is false.
HFD. By \( x \) precedes \( y \) we mean that employee \( x \) was hired by HFD on a day before the day employee \( y \) was hired. Are Axioms 1a, 1b and 1c all satisfied? Explain.

**Exercise 4.** Give a model of Axiom 1 where \( \text{point} \) means integer and \( x \) precedes \( y \) is defined in such a way that 0 precedes all other integers.

**Exercise 5.** Which of the following interpretations are models for Axiom 1? Explain.

(a) (Integer Interpretation) By \( \text{point} \) we mean integer with the normal meaning of \( \text{precedes} \).

(b) (Gap Interpretation) By \( \text{point} \) we mean a negative number, 0, or a number greater than 1 with normal meaning of \( \text{precedes} \) on the number line.

(c) By \( \text{point} \), we mean an ordered number pair \((a, b)\) where \( a \leq 0 \) and \( b = 0 \) or \( a > 0 \) and \( b = a \) or \( b = -a \). By \((x, y)\) precedes \((u, v)\), we mean \( x < u \).

(d) By \( \text{point} \), we mean a member of the open interval \((a, b)\) with the normal meaning of \( \text{precedes} \).

(e) By \( \text{point} \), we mean a member of the closed interval \([a, b]\) with the normal meaning of \( \text{precedes} \).

(f) (Lexicographic Interpretation) By \( \text{point} \) we mean ordered number pair. By \( \text{precedes} \) we mean the following. If \( x = (a, b) \) precedes \( y = (c, d) \) we mean that \( a < c \) or if \( a = c \), then \( b < d \).

(h) By \( \text{point} \) we mean an ordered pair on the unit circle. By \( x \) precedes \( y \) we mean that \( x \) sits on the unit circle counterclockwise to the point \( y \) on the unit circle.

**Definition 1.** The point \( c \) is said to be a **first point** of the point set \( M \) if and only if both of the following two conditions are met:

(a) The point \( c \) is a member of the point set \( M \).

(b) No point of \( M \) precedes \( c \).

Define **last point** similarly.

**Problem 2.** Explain why the assumption that a point set has two first (last) points leads to a contradiction.

When the reader requires more on mathematical induction, see Appendix B for examples of mathematical induction and practice problems.

**Theorem 1.** If \( M \) is a finite point set, then \( M \) has a first point and a last point.
Theorem 2. If $M$ is a finite point set, then $M$ can be written as $\{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$ for some positive integer $n$.

Exercise 6. Give a model for the present axiom system such that $S$ has a first point and a last point.

Definition 2. The point $z$ is said to be between points $x$ and $y$ if and only if $x$ precedes $z$ and $z$ precedes $y$ or vice versa.

Definition 3. The point set $R$ is said to be a region if and only if there are points $x$ and $y$ such that $R$ is the set of all points between $x$ and $y$. The points $x$ and $y$ are known as end points of $R$.

Exercise 7. Prove that the end points of a region do not belong to the region.

Exercise 8. Create a model for Axiom 1 wherein some point $x$ is not contained in any region.
Chapter 3

Consequences of Axioms 1 and 2

Axiom 2. S does not have a first point and does not have a last point.

Exercise 9. Explain why S is not a region.

Exercise 10. Which of the interpretations of Exercise 5 are models of Axioms 1 and 2?

Theorem 3. Each point belongs to a region.

Exercise 11. If one assumes the statement of Theorem 3 as a substitute for Axiom 2, can one prove the statement that is now Axiom 2 as a theorem?

Theorem 4. Prove that \( R_1 \cap R_2 \) is a region if each of \( R_1 \) and \( R_2 \) is a region containing one or more points in common. Moreover, if \( G \) is a finite collection of regions containing one or more points in common, then \( \bigcap_{g \in G} g \) is a region.

Note. From Theorem 4, we know that if \( x \) is a member of a region \( R_1 \) and a member of a region \( R_2 \), then there is a region \( R_3 \) containing \( x \) which is a subset of \( R_1 \cap R_2 \). Of course, this property applies to any finite collection of regions containing a point \( x \).

Definition 4. Two sets are said to be disjoint or mutually exclusive if and only if they have no point in common. The collection of sets \( G \) is said to be mutually disjoint or mutually exclusive if and only if each pair of sets in \( G \) are disjoint.

Example 1. The collection \( \{(1/(2n + 1), 1/2n) : n = 1, 2, 3, \ldots\} \) of open intervals of numbers is mutually exclusive.

Exercise 12. Consider a model of Axioms 1 and 2 where point means an integer and precedes means the ordinary meaning given for “<” when speaking of the integers. Give a mutually exclusive collection \( G \) of regions each containing just one integer.
Theorem 5. Suppose \( x \) and \( y \) are different points. Then there is a region containing \( x \) but not \( y \) and vice versa. Moreover, there are two disjoint regions, one containing \( x \) and the other containing \( y \).

Definition 5. A point \( p \) is said to be a limit point of a point set \( M \) if and only if each region containing \( p \) contains a point of \( M \) other than \( p \).

Exercise 13. Complete. The statement a point \( p \) is not a limit point of a point set \( M \) means ....

Exercise 14. Using the number line as a model, prove that \( p \) is a limit point of the given set \( M \) in each example below:

(a) \( p = 0 \) and \( M = (0, 1] \).
(b) \( p = .5 \) and \( M = [0, 1] \).
(c) \( p = 1 \) and \( M \) is the set of all real numbers.
(d) \( p = 0 \) and \( M = \{1, 1/2, 1/3, \ldots\} \).

Exercise 15. In reference to the lexicographic model of Exercise 5, is \( (1, 1) \) a limit point of the set \( M \), which consists of all points preceding \( (1, 1) \)?

Exercise 16. Using the set of real numbers as a model, which of the following are true and which are not true?

(a) Any infinite number set \( M \) has a limit point.
(b) Any subset of the open interval \((a, b)\) has a limit point.
(c) All limit points of a given set belong to the set itself.

Exercise 17. In reference to the Gap Model of Exercise 5, is \( 0 \) a limit point of \( A = \{x : x > 1\} \)? Explain.

Problem 3. Suppose each of \( H \) and \( K \) is a point set and \( p \) is a limit point of \( H \). Satisfy yourself that we do not have sufficient conditions to prove that \( p \) is a limit point of \( K \) as well. Would the condition that \( K \) is a subset of \( H \) suffice? What condition would be sufficient?

Theorem 6. If the point \( p \) is not a limit point of the point set \( H \) nor of the point set \( K \), then \( p \) is not a limit point of \( H \cup K \).

Problem 4. Give the contrapositive\(^1\) of Theorem 6.

Theorem 7. Any limit point of the union of a finite collection \( G \) of point sets is a limit point of at least one member of \( G \).

Corollary 8. Finite point sets do not have limit points.

\(^1\)A contrapositive of a statement \( p \Rightarrow q \) is the equivalent statement \( \neg q \Rightarrow \neg p \).
Theorem 9. If the point \( p \) is a limit point of \( H \cup K \) where \( H \) is a point set and \( K \) is a finite point set, then \( p \) is a limit point of \( H \).

Theorem 10. Each region containing a limit point \( p \) of the point set \( M \) contains infinitely many points of \( M \).

Definition 6. Suppose \( p_1, p_2, p_3, \ldots \) is a nonterminating sequence of points. Then a \( k \)-tail of \( p_1, p_2, p_3, \ldots \) is the nonterminating sequence \( p_k, p_{k+1}, p_{k+2}, \ldots \) where \( k \) is some positive integer. When there is no need to refer to the integer \( k \), we will merely use the word tail instead of \( k \)-tail.

Definition 7. A nonterminating sequence of points \( p_1, p_2, p_3, \ldots \) is said to converge to a point \( L \) if and only if each region \( R \) containing \( L \) contains a tail of the nonterminating sequence.

Exercise 18. Fill in the blank: A nonterminating sequence of points \( p_1, p_2, p_3, \ldots \) is said to converge to a point \( L \) in \( S \) if and only if each region \( R \) containing \( L \) contains all but \underline{massive} many points of the nonterminating sequence.

Question 1.

- Does each nonterminating sequence have a first or a last point?
- Does each bounded nonterminating sequence have a first or a last point?
- Does each convergent nonterminating sequence have a first or a last point?

Definition 8. A nonterminating sequence of points \( p_1, p_2, p_3, \ldots \) is said to be a nondecreasing sequence if and only if \( p_n \leq p_{n+1} \) for each positive integer \( n \). Nonincreasing is defined similarly. A nonterminating sequence is said to be monotonic if and only if it is nonincreasing or it is nondecreasing. The nonterminating sequence of points \( p_1, p_2, p_3, \ldots \) is said to be increasing if and only if \( p_n < p_{n+1} \) for each positive integer \( n \). Decreasing is defined similarly.

Theorem 11. No nonterminating sequence of points converges to each of two different points.

Theorem 12. If \( p_1, p_2, p_3, \ldots \) is a nonterminating sequence of distinct points converging to the point \( L \), then \( L \) is the only limit point of the set \( \{p_1, p_2, p_3, \ldots \} \).

Theorem 13. If \( p_1, p_2, p_3, \ldots \) is a nonterminating sequence of points convergent to a point, then some region contains the entire nonterminating sequence.

Definition 9. The point set \( M \) is said to be closed if there is no limit point of \( M \) that does not belong to \( M \).
**Notation 1.** Suppose $H$ is a point set. If $H$ has a limit point, then $H'$ denotes the set of all limit points of $H$. Also, $\overline{H}$ (read “$H$ bar”) denotes $H$ together with all of its limit points and is called the closure of $H$ (not to be confused with “closed” below). Note that $\overline{H} = H \cup H'$.

**Theorem 14.** Suppose $M$ is a point set. If $p$ is a limit point of $M'$, then $p$ is a limit point of $M$.

**Problem 5.** $\overline{H}$ is closed for each point set $H$.

**Exercise 19.** Suppose $M$ is a point set. Prove or disprove that

a. $M$ is closed if and only if $M' \subseteq M$.

b. $M$ is closed if and only if $\overline{M} = M$.

c. $(\overline{M})' = \overline{M}$.

**Definition 10.** The point set $I$ is said to be an interval (denoted as interval $xy$ or $[x,y]$) if and only if there are points $x$ and $y$ such that $x < y$ and $I$ consists of $x$, $y$, and any point that is between $x$ and $y$.

**Exercise 20.** Each interval is closed.

**Theorem 15.**

(a) $S$ is a closed point set.

(b) $\bigcup_{g \in G} g$ is closed for each finite collection $G$ of closed sets.

(c) $\bigcap_{g \in G} g$ is closed for each collection $G$ of closed sets having a common point.

**Exercise 21.** Give an example of a collection $G$ of closed sets of real numbers such that $\bigcup_{g \in G} g$ is not closed.

**Definition 11.** The point $x$ is said to be an interior point of a point set $M$ if and only if there is a region $R$ containing $x$ and $R \subseteq M$. A point set $H$ is said to be open if and only if each point of $H$ is an interior point of $H$.

**Exercise 22.** The set of all points that precede a point $x$ is an open set. Moreover, the set of all points that follow the point $x$ is an open set.

**Exercise 23.** Each region is an open point set.

**Theorem 16.** $M^c$ is closed for each open point set $M$ that is a proper subset of $S$.

**Theorem 17.** $M^c$ is open for each closed point set $M$ that is a proper subset of $S$. 
Exercise 24. Give an example of a number set that is not open and not closed.

Exercise 25. Give a model of the Axioms 1 and 2 in which there is a proper subset that is both open and closed.

Theorem 18. Apply de Morgan’s Laws to Theorem 15 to prove the following:

(a) $S$ is an open point set.

(b) $\bigcap_{g \in G} g$ is open for each finite collection $G$ of open sets having a common point.

(c) $\bigcup_{g \in G} g$ is open for each collection $G$ of open sets.

Problem 6. Does there exist a subset of the real numbers that contains no interval, is closed, and each point of the set is a limit point of the set?

Definition 12. Two point sets are said to be mutually separated if and only if they have no point in common and neither of them contains a limit point of the other. In other words, two point sets $H$ and $K$ are mutually separated if and only if $\overline{H}$ and $K$ are disjoint, and $H$ and $\overline{K}$ are disjoint.

Definition 13. A point set is said to be disconnected if and only if it can be expressed as the union of two mutually separated point sets. Otherwise, it is said to be connected. In other words, a point set is connected if and only if it is not the union of two mutually separated point sets.

Problem 7. The singleton $\{p\}$ is connected for each point $p$.

Theorem 19. If $H$ and $K$ are two mutually separated point sets and $A$ and $B$ are point sets such that $A \subseteq H$ and $B \subseteq K$, then $A$ and $B$ are mutually separated point sets.

Theorem 20. If $H$ and $K$ are two mutually separated point sets and $M$ is a connected subset of $H \cup K$, then $M$ is a subset of one of the sets $H$ and $K$.

Corollary 21. $\bigcup_{g \in G} g$ is connected for each collection $G$ of connected point sets all containing a common connected point set $C$.

Corollary 22. $M \cup \{p\}$ is a connected point set for each connected point set $M$ and $p$, a limit point of $M$.

Corollary 23. $M \cup Q$ is connected for each connected point set $M$ and $Q$, a set of limit points of $M$.

Exercise 26. Must $C$ in Corollary 21 be connected for the conclusion to hold?
Theorem 24. Each point of a non-degenerate connected point set is a limit point of that point set.

Theorem 25. Suppose the point set $M$ is not connected. Then $M$ can be expressed as the union of two point sets $A$ and $B$ with the following properties:

(a) $A$ and $B$ are mutually separated.

(b) Each point of $A$ precedes each point of $B$. 

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Chapter 4

Consequences of Axioms 1, 2 and 3

Axiom 3. $S$ is connected.

Theorem 26. $S$ contains infinitely many points and each point is a limit point of $S$.

Theorem 27. If $S$ is the union of two point sets $H$ and $K$ such that each point of $H$ precedes each point of $K$, then $H$ has a last point or $K$ has a first point but not both.

Exercise 27. Prove that all regions and intervals are connected.

Exercise 28. Suppose $M$ is a connected subset of $S$. Is $M$ a region or interval or something else?

Theorem 28. No proper subset of $S$ is both open and closed.

Corollary 29. Each pair of points has a point between them.

Corollary 30. Each point of a region is a limit point of that region.

Corollary 31. Each end point of a region is a limit point of that region.

Corollary 32. No region has a first (last) point.

Question 2. Is Theorem 28 equivalent to Axiom 3?

Definition 14. A point set $M$ is said to have an upper bound if and only if there is some point $U$ (called an upper bound of $M$) such that $x \leq U$ for each $x$ in $M$. Lower bound is defined similarly. Moreover, $M$ is said to be bounded if and only if $M$ has both an upper bound and a lower bound. Of course, in general, if a point is a subset of a region, it is bounded.

Definition 15. The point $L$ is said to be a least upper bound of the point set $M$ if and only if $L$ is an upper bound of $M$ and no upper bound of $M$ precedes $L$. Greatest lower bound is defined similarly.

Theorem 33. If each of $L_1$ and $L_2$ is a least upper bound of the point set $M$, then $L_1 = L_2$. A similar statement is true for greatest lower bounds.
Exercise 29. If \( L \) is the least upper bound of the point set \( M \) that does not contain \( L \), then \( L \) is a limit point of \( M \).

Theorem 34. If \( M \) is a point set with an upper bound, then \( M \) has a least upper bound. Also, if \( M \) is a point set with a lower bound, then \( M \) has a greatest lower bound.

Theorem 35. Each increasing nonterminating sequence of points with an upper bound converges to a point.

Definition 16. A collection \( G \) of sets is said to cover a point set \( M \) if and only if each point of \( M \) is in at least one set of \( G \), that is, \( M \) is a subset of \( \bigcup_{g \in G} g \). If some subcollection \( G' \) of \( G \) covers \( M \), then \( G' \) is said to be a subcover of \( G \). If \( G \) is a collection of regions, then \( G \) is said to be a region cover. If \( G \) is a collection of open sets, then \( G \) is said to be an open cover.

Problem 8. Prove. A point set has a region cover if and only if it has an open cover.

Exercise 30. Suppose \( G \) is a collection of regions where each point of the interval \( ab \) is contained in at most finitely many members of \( G \). Prove that a finite subcollection of \( G \) covers \( ab \).

Exercise 31. Suppose \( G \) is a collection of regions that covers the interval \( ab \).

a. Consider the collection \( G' \) of regions in \( G \) which contain \( a \) and not \( b \). Prove that the set of right hand end points of members of \( G' \) has a least upper bound \( L \) less than or equal to \( b \).

b. Prove that the interval \( aL \) is covered by a finite subcollection of \( G \).

Exercise 32. Suppose \( G \) is an infinite collection of regions covering the interval \( ab \). Let \( M \) be the set of all points \( x \) in the interval \( ab \) which are not the point \( a \) and finitely many members of \( G \) cover the interval \( ax \).

a. Is \( M \) not empty? Prove your answer.

b. Does \( b \) belong to \( M \)? Prove your answer.

Theorem 36. If \( G \) is a collection of regions covering the interval \( ab \), then there is some finite subcollection of \( G \) that covers the interval \( ab \).

Definition 17. A point set \( M \) is said to be compact if and only if each open cover of \( M \) contains a finite subcover of \( M \).

Exercise 33. Each interval \( ab \) is compact.

Exercise 34. Each closed and bounded point set is compact.

Theorem 37. Each bounded infinite point set has a limit point.
**Exercise 35.** Each closed and bounded point set contains a first point and a last point.

**Theorem 38.** If $M_1, M_2, M_3, \ldots$ is a nonterminating sequence of closed and bounded point sets such that for each $n$, $M_n$ contains $M_{n+1}$, then the sets have a point in common. Furthermore, the set of all points common to these sets is closed.

**Theorem 39.** If $M$ is an infinite set of points, then there is an increasing or decreasing sequence of points in $M$.

**Definition 18.** A point set $M$ is said to be **perfect** if and only if it has the following two properties:

(a) $M$ is closed

(b) Each point of $M$ is a limit point of $M$.

**Theorem 40.** Each perfect point set is uncountable.

**Exercise 36.** Each region, interval and $S$ are uncountable.

**Exercise 37.** The set of numbers is uncountable.

**Theorem 41.** Each point set without any limit points is countable.

**Problem 9.** Give the contrapositive of Theorem 41.

**Definition 19.** A point set $M$ is said to be **nowhere dense** if and only if each region contains a region that does not intersect $M$.

**Exercise 38.** Each point set that is a subset of a nowhere dense set is itself a nowhere dense point set.

**Exercise 39.** $H \cup K$ is nowhere dense for each nowhere dense set $H$ and each nowhere dense set $K$.

**Theorem 42.** No region is the union of a countable number of nowhere dense point sets.

**Theorem 43.** Suppose $M = \bigcup_{n=1}^{\infty} g_n$ where $g_n$ is closed for $n = 1, 2, \ldots$ and each point of $g_n$ is a limit point of $M - g_n$ for $n = 1, 2, \ldots$. Then $M$ is not closed.

**Theorem 44.** No closed point set $M$ is the union of a countable number of point sets such that each of them is nowhere dense in $M$.

**Definition 20.** A point set $M$ is said to be **everywhere dense** if and only if each point is a point of $M$ or is a limit point of $M$.

**Exercise 40.** If the point set $M$ is everywhere dense, then each region contains a point of $M$. 
Exercise 41. If the point set $M$ is everywhere dense, then $M$ fails to be nowhere dense.

Theorem 45. If the point set $M$ is nowhere dense, then $M^c$ is everywhere dense.

Theorem 46. If $G$ is a countable collection of open sets each of which is everywhere dense, the intersection of the elements of $G$ is everywhere dense.
Chapter 5

Consequences of Axioms 1, 2, 3 and 4

Axiom 4. $S$ is separable; that is, $S$ contains a countable subset whose closure is $S$.

Exercise 42. Is the Lexicographic interpretation a model for Axioms 1, 2, 3 and 4?

Theorem 47. There do not exist uncountably many mutually exclusive regions.

Theorem 48. There exists an increasing unbounded nonterminating sequence.

Theorem 49. If $p$ is a point, there exists an increasing nonterminating sequence converging to $p$.

Theorem 50. If $p$ is a limit point of the point set $M$, there exists a nonterminating sequence of distinct points in $M$ converging to $p$.

Theorem 51. There exists a countable collection $F$ of regions such that if $R$ is a region and $p$ is a point of $R$, then some region of $F$ contains $p$ and lies in $R$.

Theorem 52. If $G$ is a collection of regions covering the point set $M$, then some countable subcollection of $G$ covers $M$.

Theorem 53. Each uncountable point set contains a limit point of itself.

Theorem 54. There does not exist a collection of mutually exclusive intervals covering $S$. 
Appendix A

Motel $\infty$

You have heard of Motel 6, but have you heard of Motel $\infty$? There is one just outside Euclid, Texas on highway $\pi$. One day last year on my way to a mathematics convention in San Antonio, I needed a place to spend the night. It was almost midnight and I was tired from driving and working on a proof of Poincare’s conjecture.

This motel was most amazing! It had the same number of rooms as there are positive integers. In other words, each room had as its number a positive integer and each positive integer was the number of a room. As you might expect in the usual parlance of the assignment of room numbers, no two positive integers were used more than once as a number and no room had more than one number. Moreover, each room had a capacity of just one and, to make matters worse, each room was filled to capacity.

Here is my question. Did Motel $\infty$ have a vacancy? Could they give me a room?
Appendix B

Induction

Throughout Linear Point Set Theory, we will assume a primitive notion of the arithmetic and order properties of the numbers but, for this supplement, we will not assume any properties of the positive integers. We define the positive integers below as if we never had seen them before.

**Definition 1.** The number set $M$ is said to be **inductive** if and only if

(a) 1 belongs to $M$ and  
(b) if $x$ belongs to $M$, then $x + 1$ belongs to $M$.

**Definition 2.** A number $x$ is a positive integer if and only if $x$ belongs to each inductive set. We will denote the set of positive integers by $Z^+$.  

**Theorem 1.** 1 is a positive integer.  

**Theorem 2.** If $x$ is a positive integer, then $x + 1$ is a positive integer.  

**Theorem 3.** If $M$ is an inductive subset of $Z^+$, then $M = Z^+$.  

The standard method for application of Theorem 3 is called mathematical induction. This method is the one most often taught, the easiest to use and the least understood. The context is normally of some mathematical truth or statement which one wishes to prove true for each positive integer. It is not always clear what the statement is that one seeks to prove. This the first obstacle which must be overcome. Each problem is unique. There are no real guidelines except for those couched in algebraic terms such as Theorem 8 and Theorem 9 stated below. In these cases, mathematical induction is a reasonably tractable proposition. A description is as follows:

**Mathematical Induction.**

**Step 1** If not already done, state the problem or theorem at hand in the form,  

"$S(n)$ is true for each positive integer $n$ where $S(n)$ is a statement in terms of $n$.”
Step 2 Prove \( S(n) \) is true for \( n = 1 \).

Step 3 Assuming \( S(n) \) is true for \( n = k \), prove that \( S(n) \) is true for \( n = k + 1 \).

**Explanation of the method.** Define the truth set, \( T = \{ n : S(n) \text{ is true and } n \in \mathbb{Z}^+ \} \). Our goal is to show that \( T = \mathbb{Z}^+ \). Once this is done we will have shown that \( S(n) \) is true for each positive integer \( n \). All we need to do is show that \( T \) is an inductive subset of \( \mathbb{Z}^+ \). Clearly, \( T \) is already a subset of \( \mathbb{Z}^+ \). Thus, we merely have to show that \( T \) is an inductive set. In Steps 2 and 3, we show that \( S(n) \) is true for \( n = 1 \) and that \( S(n + 1) \) is true when \( S(n) \) is assumed to be true and, therefore, \( T = \mathbb{Z}^+ \). If the student is taught only the steps above, he or she may not always be clear as to why these operations are being performed. What has been done is that the truth set, \( T \), is shown to be inductive. Then, by the above corollary, \( T = \mathbb{Z}^+ \). Thus, \( S(n) \) is true for each positive integer \( n \). For an application of this method, see the proof of the next theorem.

**Theorem 4.** If \( k \) is a positive integer, then \( 1 \leq k \).

*Proof.* Suppose \( S(n) \) is the statement, “\( 1 \leq n \).” Let \( T = \{ n : S(n) \text{ is true and } n \in \mathbb{Z}^+ \} \), that is, \( T = \{ n : 1 \leq n \text{ and } n \text{ is a positive integer} \} \). Now, we must show that \( T \) is an inductive set.

Clearly, 1 belongs to \( T \).

Suppose \( k \) belongs to \( T \). Then, \( 1 \leq k \) and \( k \) is a positive integer. Adding 1 to each side of the inequality, we have that \( 1 + 1 \leq k + 1 \). Thus, \( 1 \leq k + 1 \). From, Theorem 2, since \( k \) is a positive integer, \( k + 1 \) is a positive integer. Thus, \( k + 1 \) belongs to \( T \).

By Definition 1, \( T \) is inductive subset of \( \mathbb{Z}^+ \).

By Theorem 3, \( T = \mathbb{Z}^+ \).

\( S(n) \) is true for each positive integer \( n \) or, its equivalent, \( 1 \leq k \) is true for each positive integer \( k \). \( \square \)

**Theorem 5.** If each of \( x \) and \( y \) is a positive integer, then \( x + y \) is a positive integer.

**Theorem 6.** If each of \( x \) and \( y \) is a positive integer with \( x > y \), then \( x - y \) is a positive integer.

**Theorem 7.** (Well Ordering Principle). Every nonempty subset of the positive integers has a least member.

You are now free to use any arithmetical properties of positive integers you wish.

We can now construct the following notation:

**Definition 3.** Define 2 as \( 1 + 1 \), 3 as \( 2 + 1 \), ....
Prove theorems 8 and 9 using the Well Ordering Principle.

**Theorem 8.** \((n + 1)! > 2n + 3\) for \(n > 5\).

**Theorem 9.** Prove \(1 + 2 + \cdots + n = n(n + 1)/2\) for each positive integer \(n\).

**Definition 4.** A positive integer \(n\) is said to be even if and only if \(n = 2k\) for some positive integer \(k\) and \(n\) is said to be odd if and only if \(n = 2k − 1\) for some positive integer \(k\).

**Theorem 10.** Each positive integer is odd or even but not both.

**Theorem 11.** (The Division Algorithm). If each of \(a\) and \(b\) is a positive integer with \(a \geq b\), then there is exactly one positive integer \(q\) and exactly one nonnegative integer \(r\) such that \(0 \leq r < b\) and \(a = bq + r\).

**Definition 5.** If \(A\) is a set, then the set of all subsets of \(A\) is said to be the power set of \(A\) and is denoted by \(\mathcal{P}(A)\). For example, if \(A = \{1, 2\}\), then \(\mathcal{P}(A) = \{\{1\}, \{2\}, \{1, 2\}\}\).

**Problem 10.** If \(n\) is a positive integer and \(A\) is a set with \(n\) elements, make a guess as to how many elements are in \(\mathcal{P}(A)\). Prove your guess.

**Definition 6.** If there is some positive integer \(n\) such that \(M\) is the set of distinct objects \(\{a_1, a_2, \ldots, a_n\}\), then \(M\) is said to be finite.

**Theorem 12.** Every finite number set has a first member and a last member.

**Theorem 13.** If \(M\) is a finite number set consisting of \(n\) members, then there exists numbers \(a_1, a_2, \ldots, a_n\) of \(M\) such that \(a_1 < a_2 < \cdots < a_n\).

**Problem 11.** By experimentation, determine the number of distinct ways of filling \(n\) slots with \(n\) given different objects, one object per slot. Then, by the Well Ordering Principle, prove your answer.

**Problem 12.** (Hanoi’s Tower) Suppose you have three pegs and on one of these you have piled up \(n\) rings of decreasing radius if one proceeds from the bottom ring to the top one. The problem is to move these \(n\) rings from the given peg to another peg, in such a way that no larger ring is ever placed over a smaller one. A move is the shifting of one ring from one peg to another. By experimentation, determine a formula for the number of moves required in terms of \(n\). Then, prove your answer by the Well Ordering Principle.

**Problem 13.** A set of lines in the plane is said to be in general position if no two lines are parallel and no three lines intersect at point. Determine by guessing or by some other means the number of pieces into which \(n\) lines cut the plane. Then, by the Well Ordering Principle, prove your answer.
Appendix C

Countability

The following are general statements concerning sets.

**Definition 1.** A set $M$ is said to be countable if $M$ is either finite or $M = p_1, p_2, p_3, \ldots$ where $p_1, p_2, p_3, \ldots$ is a nonterminating sequence of distinct elements.

**Example 1.** The following are countable sets—that is to say, these sets can be written as the terms of a nonterminating sequence:

- The set of all positive integers.
- Any collection of mutually disjoint open intervals of real numbers.
- The cartesian product of any two infinite countable sets.

**Problem 1.** Show that the set $M = \{(m, n) : m, n \text{ are positive integers}\}$ is countable. Start by drawing a graph of $M$.

**Problem 2.** Show that the set of all lines with integer slopes and integer $y$-intercepts is countable.

**Problem 3.** We know that the set of positive integers is countable. Show that any subset of the positive integers is countable.

**Problem 4.** Show that the set of rational numbers is countable.

**Problem 5.** Show that the union of two disjoint, infinite, countable sets is countable.

**Problem 6.** Show that the union of two infinite, countable sets is countable.

**Problem 7.** Show that the union of countably many countable sets is countable.

**Definition 2.** A set $M$ is said to be uncountable if $M$ is not countable.

**Problem 8.** Show that the assumption that the set of all sequences of 0’s and 1’s is countable leads to a contradiction.
Bibliography


