Analysis

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To the Student

Think for a moment about how you pick up a new skill, like swimming, bicycle riding or violin playing, for example. How would a year-long lecture course in any of these help you acquire the skill? It would certainly be worthwhile to see someone doing the activity before you start trying to do it yourself. Occasional advice along the way might also be appreciated, but each of these activities is something you must try to do by yourself. Each very likely requires some restructuring of one’s brain. Such a process takes some time.

Somewhere along the line a persistent myth developed that learning mathematics had to be different. Twelve academic years, five days a week, of nearly constant lecturing on how to do problems and how to understand mathematical ideas has become normal practice in getting students ready for college level mathematics. Commonly, in college, this persists for at least a year but maybe four years (if a student elects to major in mathematics). Then this is followed often by several years of lecturing at the graduate level. Then, the idea seems to be, someone is so filled with knowledge that they are finally ready to create some new mathematics, to do research. How does this sound to you?

The present notes is an attempt to break this cycle. You will be given problems if this material is used as a basis for a class. If you find this material on your own, find someone to whom you can present your work. You will likely find the going to be rather slow at first but then the pace will pick up as you better grasp what it means to prove a theorem. What you will be doing, either as part of a class or on your own, is becoming a mathematician.

Quite a few research mathematicians have taken courses based on theorem sequences as are to follow. A good many, including myself, attribute their initial and continuing interest in mathematics to the foundation they received from such courses.
Chapter 1

Theorem Sequence

**Definition 1:** The statement that $S$ is a *segment* means that there are points $a$ and $b$ such that $S$ is the set of all points between $a$ and $b$.

**Definition 2:** The statement that $I$ is an *interval* means that there are points $a$ and $b$ such that $I$ is the set consisting of $a$, $b$, and all points between $a$ and $b$.

**Definition 3:** Suppose $M$ is a point collection. The statement that the point $p$ is a *limit point* of $M$ means that every segment containing $p$ contains a point of $M$ different from $p$.

**Definition 4:** Suppose $M$ is a point collection. The statement that the point $p$ is a *boundary point* of $M$ means that every segment containing $p$ contains a point of $M$ and a point not in $M$.

**Theorem 5:** If $a$ and $b$ are two points, then $a$ is a limit point of the interval $[a,b]$.

**Theorem 6:** Suppose $M$ is a point collection consisting of exactly three points. Then $M$ has no limit point.

**Theorem 7:** If $H$ and $K$ are two segments which have a point in common, then the common part of $H$ and $K$ is a segment.

**Definition 8:** Suppose $p$ is a point and $p_1, p_2, p_3, \ldots$ is a point sequence. The statement that $p$ is a *sequential limit point* of $p_1, p_2, p_3, \ldots$ means that if $S$ is a segment containing $p$, then there is a positive integer $N$ such that $p_n$ is in $S$ for every integer $n$ greater than $N$.

**Theorem 9:** No sequence has two sequential limit points.

**Definition 10:** If $M$ is a point collection, then the statement that $M$ is *bounded above* means that there is a point $p$ such that no point of $M$ is to the
right of \( p \). The statement that \( M \) is bounded below means that there is a point \( p \) such that no point of \( M \) is to the left of \( p \). The statement that \( M \) is bounded means that it is bounded above and bounded below.

**Axiom 11:** If \( M \) is a point collection which is bounded above, then \( M \) has a least upper bound. If \( M \) is a point collection which is bounded below, then \( M \) has a greatest lower bound.

**Theorem 12:** If \( p_1, p_2, p_3, \ldots \) is an increasing sequence that is bounded above, then \( p_1, p_2, p_3, \ldots \) has a sequential limit point.

**Definition 13:** The statement that the point collection \( M \) is closed means that if \( p \) is a limit point of \( M \), then \( p \) is in \( M \).

**Theorem 14:** Suppose \( M \) is a point collection which has a limit point, and \( K \) is the set to which a point belongs if and only if it is a limit point of \( M \). Then \( K \) is closed.

**Theorem 15:** Suppose each of \( H \) and \( K \) is a point collection, and \( p \) is a limit point of the union of \( H \) and \( K \). Then \( p \) is a limit point of \( H \) or a limit point of \( K \).

**Definition 16:** The statement that the point collection sequence \( M_1, M_2, M_3, \ldots \) is nested means that if \( n \) is a positive integer, then \( M_{n+1} \) is a subset of \( M_n \).

**Theorem 17:** There is a nested sequence of segments which have no common point.

**Theorem 18:** Every infinite and bounded point collection has a limit point.

**Definition 19:** The statement that the collection \( G \) of point collections covers the point collection \( M \) means that if \( p \) is in \( M \), then some member of \( G \) contains \( p \).

**Theorem 20:** If \( M \) is an interval, and \( G \) is a collection of segments which covers \( M \), then some finite subcollection of \( G \) covers \( M \).

**Definition 21:** The statement that the point sequence of \( p_1, p_2, p_3, \ldots \) is a Cauchy sequence means that if \( \varepsilon \) is a positive number, then there is a positive integer \( N \) such that \( |p_n - p_N| < \varepsilon \) for every positive integer \( n \) greater than \( N \).

**Theorem 22:** Every sequence with a sequential limit is a Cauchy sequence.

**Theorem 23:** If \( p \) is a limit point of the point collection \( M \), then there is a sequence \( p_1, p_2, p_3, \ldots \) of distinct points of \( M \) which has sequential limit point \( p \).

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**Theorem 24:** If each of $H$ and $K$ is a closed point collection and $H$ and $K$ have a point in common, then the common part of $H$ and $K$ is closed.

**Theorem 25:** If $I_1, I_2, I_3, \ldots$ is a nested sequence of intervals, then $I_1, I_2, I_3, \ldots$ have a point in common.

**Theorem 26:** If $M_1, M_2, M_3, \ldots$ is a nested sequence of closed and bounded point collections, then $M_1, M_2, M_3, \ldots$ have a point in common.

**Theorem 27:** Every Cauchy sequence has a sequential limit point.

**Definition 28:** The statement that $g$ is a *simple graph* means that $g$ is a point collection in the plane such that no vertical line contains two points of $g$.

**Definition 29:** The statement that the simple graph $g$ is *continuous* at the point $p$ of $g$ means that if $\alpha$ and $\beta$ are two horizontal lines with $p$ between them, then there are two vertical lines $h$ and $k$ with $p$ between them such that every point of $g$ between $h$ and $k$ is also between $\alpha$ and $\beta$.

**Theorem 30:** Suppose $g$ is the simple graph consisting of all points $(x, x^2)$ for all numbers $x$. Then $g$ is continuous at the point $(1, 1)$.

**Theorem 31:** Suppose that $g$ is the simple graph consisting of all points $(x, \frac{1}{x})$ for all numbers $x > 0$. Then $g$ is continuous at each of its points.

**Theorem 32:** Suppose $g$ is the simple graph consisting of all points $(x, x^2 + \frac{1}{x})$ for all numbers $x \neq 0$. Then $g$ is continuous at each of its points.

**Theorem 33:** Suppose $g$ is an increasing continuous simple graph with domain an interval (or segment), and $L$ is a horizontal line such that some point of $g$ is above $L$ and some point of $g$ is below $L$. Then some point of $g$ is on $L$.

**Theorem 34:** If $g$ is a continuous simple graph with domain an interval, then some horizontal line is above $g$.

**Theorem 35:** If $g$ is a continuous simple graph with domain an interval, then the range of $g$ is a point or an interval.

**Theorem 36:** Suppose $g$ is a continuous simple graph with domain an interval (or segment), and $L$ is a horizontal line such that some point of $g$ is above $L$ and some point of $g$ is below $L$. Then some point of $g$ is on $L$. 
Theorem 37: Suppose $g$ is a continuous simple graph with domain an interval. Then there is a point of $g$ such that no other point of $g$ is above it.

Theorem 38: Suppose that $p_1, p_2, p_3, \ldots$ is a bounded sequence. Then some subsequence of this sequence has a sequential limit point.

Note: From here on, the term “function” is used in place of “simple graph”.

Definition 39: Suppose $f$ is a function, and $c$ is in the domain of $f$. The statement that $f$ is differentiable at $c$ means that

i) $c$ is in the domain of $f$ and is a limit point of the domain of $f$;

ii) there is a number $d$ such that if $\varepsilon > 0$, there is $\delta > 0$ such that if $x$ is in the domain of $f$ and $0 < |x - c| < \delta$, then

$$\left| d - \frac{(f(x) - f(c))}{(x - c)} \right| < \varepsilon$$

Theorem 40: If $f$ is a function and $c$ is a member of the domain of $f$ at which $f$ is differentiable, then there is only one number $d$ such that (ii) in the above definition holds (and it is called $f'(c)$).

Theorem 41: Suppose $f$ is a function, and $c$ is a member of the domain of $f$ at which $f$ is differentiable. Then $f$ is continuous at $c$.

Theorem 42: Suppose $f$ is a function whose domain includes the segment $(a, b)$, $c$ is a member of $(a, b)$ at which $f$ is differentiable, and $f'(c) > 0$. Then there is a segment $S$ containing $c$ such that

i) if $x$ is in $S$ and $x < c$, then $f(x) < f(c)$; and

ii) if $x$ is in $S$ and $x > c$, the $f(x) > f(c)$.

Theorem 43: Suppose $f$ is a function whose domain includes the segment $(a, b)$, and $c$ is a member of $(a, b)$ at which $f$ is differentiable. Suppose also that if $x$ is in $(a, b)$, then $f(x) < f(c)$. Then $f'(c) = 0$.

Theorem 44: Suppose $[a, b]$ is an interval and $f$ is a continuous function with domain $[a, b]$ such that $f(a) = 0 = f(b)$ and $f$ is differentiable at each member of $(a, b)$. There is number $c$ in $(a, b)$ such that $f'(c) = 0$.

Definition 45: Suppose $f$ is a function, and $M$ is a subset of the domain of $f$. The statement that $f$ is uniformly continuous on $M$ means that if $\varepsilon > 0$, then there is $\delta > 0$ such that if $x$ and $y$ are in $M$ and $|y - x| < \delta$, then $|f(x) - f(y)| < \varepsilon$. 
Theorem 46: Suppose $f$ is a continuous function whose domain includes the interval $[a, b]$. Then $f$ is uniformly continuous on $[a, b]$.

Definition 47: Suppose $a < b$, and $f$ is a continuous function whose domain includes $[a, b]$. The statement that $U$ is an upper sum for $f$ on $[a, b]$ means that there is a positive integer $n$, an increasing sequence $t_0, t_1, \ldots, t_n$, and a nondecreasing sequence $s_1, s_2, \ldots, s_n$ such that

i) $t_0 = a, t_n = b$;

ii) $s_i$ is in $[t_{i-1}, t_i]$, and $f(s_i) \geq f(x)$ for all $x$ in $[t_{i-1}, t_i], i = 1, 2, \ldots, n$; and

iii) $U = \sum_{i=1}^{n} f(s_i)(t_i - t_{i-1})$.

Lower sums are defined similarly.

Definition 48: Suppose that $f$ is a continuous function whose domain includes the interval $[a, b]$. The statement that $f$ is integrable from $a$ to $b$ means that there is one and only one number which exceeds no upper sum (for $f$ on $[a, b]$) and is exceeded by no lower sum for $f$ on $[a, b]$.

The unique number in the above definition is denoted by $\int_{a}^{b} f$. If $c > d$ and the function $f$ is integrable from $d$ to $c$, then $\int_{d}^{c} f$ is by definition $-\int_{c}^{d} f$. Also by definition $\int_{c}^{c} f = 0$.

Theorem 49: Suppose $f$ is a nondecreasing function whose domain includes the interval $[a, b]$. Then $f$ is integrable from $a$ to $b$.

Theorem 50: Suppose $a < b$, and $f$ is a continuous function whose domain includes $[a, b]$. Then every lower sum for $f$ on $[a, b]$ is less than or equal to every upper sum for $f$ on $[a, b]$.

Theorem 51: If $f$ is a continuous function whose domain includes the interval $[a, b]$, then $f$ is integrable from $a$ to $b$.

Theorem 52: Suppose $f$ is a continuous function whose domain includes the interval $[a, b]$, and $c$ is in $[a, b]$. Then

$$\int_{a}^{c} f + \int_{c}^{b} f = \int_{a}^{b} f$$

Theorem 53: Suppose each of $f$ and $g$ is a continuous function whose domain includes the interval $[a, b]$. Then

$$\int_{a}^{b} f + \int_{a}^{b} g = \int_{a}^{b} (f + g)$$
Theorem 54: Suppose $f$ is a continuous function whose domain includes the interval $[a, b]$. There is a number $c$ in $[a, b]$ such that

$$\int_{a}^{b} f = f(c)(b - a)$$

Theorem 55: Suppose $a < b$, each of $f$ and $g$ is a function whose domain includes the segment $(a, b)$, and each of $\alpha$ and $\beta$ is a number. If $c$ is in $(a, b)$, and each of $f$ and $g$ is differentiable at $c$, then $\alpha f + \beta g$ is differentiable at $c$, and

$$(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c)$$

Theorem 56: Suppose $f$ is a continuous function with domain the interval $[a, b]$, $c$ is in $[a, b]$, and $g$ is the function with domain $[a, b]$ such that

$$g(x) = \int_{c}^{x} f \text{ for all } x \text{ in } [a, b]$$

Then $g' = f$.

Theorem 57: Suppose $f$ is a continuous function whose domain includes the interval $[a, b]$, and $F$ is a function such that $F'(x) = f(x)$ for all $x$ in $[a, b]$. Then

$$\int_{a}^{b} f = F(b) - F(a)$$

Theorem 58: Suppose $f$ is a function whose domain includes the interval $[a, b]$, and $c$ is a number. Then

$$\int_{a}^{b} cf = c \int_{a}^{b} f$$

Definition 59: The statement that the point collection $S$ in the plane is a region means that there is a positive number $r$ and a point $p$ in the plane such that $S$ is the collection of all points in the plane which are distant from $p$ by an amount less than $r$.

Definition 60: Suppose $M$ is a point collection in the plane, and $p$ is a point in the plane. The statement that $p$ is a limit point of $M$ means that every region containing $p$ contains a point of $M$ different from $p$.

Theorem 61: Every infinite and bounded point collection in the plane has a limit point.
Definition 62: Suppose $M$ is a number collection, and each of $f, f_1, f_2, \ldots$ is a function whose domain includes $M$. The statement that $f_1, f_2, \ldots$ converges uniformly to $f$ on $M$ means that if $\varepsilon > 0$, there is a positive integer $N$ such that if $n$ is an integer greater than $N$, then

$$|f(x) - f_n(x)| < \varepsilon \text{ for all } x \in M$$

Theorem 63: Suppose $[a, b]$ is an interval, each of $f, f_1, f_2, \ldots$ is a continuous function with domain $[a, b]$, and $f_1, f_2, \ldots$ converges uniformly to $f$ on $[a, b]$. If each of $f_1, f_2, \ldots$ is continuous, then $f$ is continuous.

Theorem 64: Suppose $[a, b]$ is an interval, each of $f, f_1, f_2, \ldots$ is a continuous function with domain $[a, b]$, and $f_1, f_2, \ldots$ converges uniformly to $f$ on $[a, b]$. Then

$$\int_a^b f_1, \int_a^b f_2, \ldots \text{ converges to } \int_a^b f$$

Theorem 65: Suppose each of $[a, b]$ and $[c, d]$ is an interval, and $f$ is a continuous function with domain $[a, b] \times [c, d]$. Then $f$ is uniformly continuous on $[a, b] \times [c, d]$.

Definition 66: The statement that the point collection $M$ is perfect means that every point of $M$ is a limit point of $M$.

Definition 67: A point collection $M$ is countable if there is a sequence $p_1, p_2, p_3, \ldots$ such that for every $m$ in $M$, there is an integer $i$ such that $m = p_i$.

Theorem 68: No closed and countable number collection is perfect.

Theorem 69: Suppose $M$ is a closed and bounded point collection in the plane, and $G$ is a collection of regions covering $M$. Then some finite subcollection of $G$ covers $M$.

Theorem 70: Suppose $f$ is a continuous function with domain $[a, b] \times [c, d]$ and range in $R$. Suppose also that $h$ is the function with domain $[a, b]$ such that

$$h(x) = \int_c^d f(x, y) \, dy \text{ for all } x \in [a, b]$$

Then $h$ is continuous.

Theorem 71: Suppose $f$ is a continuous function with domain $[a, b] \times [c, d]$ and range in $R$. Then $f$ is integrable on $[a, b] \times [c, d]$.
Theorem 72: If $f$ is a continuous function with domain $[a, b] \times [c, d]$ and range in $\mathbb{R}$, then
\[
\int_c^d \int_a^b f(x, y) \, dx \, dy = \int_{[a,b] \times [c,d]} f
\]

Theorem 73: Suppose $f$ is a function with domain $[a, b] \times [c, d]$, such that each of the partial derivatives $f_{1,2}$ and $f_{2,1}$ exists and is continuous on $[a, b] \times [c, d]$. Then $f_{1,2} = f_{2,1}$

Theorem 74: There is a closed bounded perfect number collection which contains no interval.

Theorem 75: Suppose $f$ is a continuous function with domain the interval $[a, b]$. There is a function $F$ such that $F' = f$.

Theorem 76: Suppose that each of $u$ and $v$ is a function with domain $[a, b] \times [c, d]$ such that the partial derivatives $u_1, u_2, v_1, v_2$ exist and are continuous on $[a, b] \times [c, d]$. If $u_2 = v_1$, there is a function $F$ on $[a, b] \times [c, d]$ such that $F_1 = u, F_2 = v$.

Definition 77: Suppose $[a, b]$ is an interval. A partition of $[a, b]$ is a finite ordered sequence, $t_0, t_1, t_2, \ldots, t_n$ such that $t_0 = a, t_n = b$, and $t_{i-1} < t_i$ for all $i = 1, 2, \ldots, n$.

Definition 78: Suppose $f$ is a function with domain the interval $[a, b]$. The statement that the graph of $f$ has length means that there is a number $L$ such that if $t_0, t_1, t_2, \ldots, t_n$ is a partition from $a$ to $b$, then
\[
\sum_{i=1}^n [(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2]^{1/2} \leq L
\]

The least such number $L$ is called the length of the graph of $f$.

Theorem 79: Suppose $f$ is a function with domain the interval $[a, b]$, and $f'$ is continuous on $[a, b]$. Then the graph of $f$ has length. Moreover, if $g(x) = (1 + f'(x)^2)^{1/2}$ for all $x$ in $[a, b]$, then the length of the graph of $f$ is $\int_a^b g$.

Theorem 80: Suppose $f$ is a nondecreasing function on $[a, b]$. There is at most a countable subset of $[a, b]$ on which $f$ is not continuous.

Theorem 81: If $M$ is an uncountable set of positive numbers, there is a positive number $\varepsilon$ such that uncountably many members of $M$ are greater than $\varepsilon$. 
**Definition 82:** Suppose $f$ is a function, and $y$ is in the domain of $f$. The statement that $f$ has a **right limit** at $y$ means that

i) $y$ is a limit point of the set of all points in the domain of $f$ which are to the right of $y$; and

ii) there is a number $L$ such that if $\varepsilon > 0$, then there is $\delta > 0$ such that if $x$ is in the domain of $f$ and $y < x < y + \delta$, then $|f(x) - L| < \varepsilon$.

Such a number $L$ is called the right limit of $f$ at $y$, and is denoted by $f(y+)$. Similar statements hold for left limits.

**Theorem 83:** Suppose $f$ is a nondecreasing function with domain the segment $(a, b)$. Then if $x$ is in $(a, b)$, $f$ has right and left limits at $x$.

**Theorem 84:** Suppose $f$ is a function, and $c$ is in the domain of $f$, such that $f$ has left and right limits at $c$ and $f(c+) = f(c) = f(c-)$. Then $f$ is continuous at $c$.

**Theorem 85:** Suppose $f$ is a continuous function whose domain includes the interval $[a, b]$. Then

$$\left| \int_a^b f \right| \leq \int_a^b |f| \text{ if } a < b$$

**Theorem 86:** Suppose $m$ is a nondecreasing function whose domain includes the interval $[a, b]$. There is a unique number $w$ such that if $t_0, t_1, \cdots, t_n$ is a partition from $a$ to $b$, then

$$\sum_{i=1}^{n} t_{i-1} [m(t_i) - m(t_{i-1})] \leq w \leq \sum_{i=1}^{n} t_i [m(t_i) - m(t_{i-1})]$$

**Theorem 87:** Suppose $h$ is a function whose domain includes all positive numbers. The following two statements are equivalent:

i) If $x_1, x_2, \cdots$ is an unbounded increasing sequence of positive numbers, then $h(x_1), h(x_2), \cdots$ converges to $L$; and

ii) $h(x) \to L$ as $x \to \infty$.

**Theorem 88:** Suppose $[a, b]$ is an interval, $c$ is in $[a, b]$, and each of $g_1, g_2, \ldots$ is a continuous function such that

$$g_n(x) = \int_c^x g_{n-1}, \text{ for all } n = 1, 2, \cdots$$

Then $g_1, g_2, \cdots$ converges uniformly to the zero function on $[a, b]$. 

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Theorem 89: If $K > 0$, there is $L > 0$ such that $K^n/n! \leq L2^{-n}$, for all $n = 1, 2, \ldots$.

Theorem 90: Suppose that each of $f$ and $g$ is a function, and $c$ is a number at which both $f$ and $g$ are differentiable. If $c$ is a limit point of the intersection of the domains of $f$ and $g$, then $fg$ is differentiable at $c$ and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

Definition 91: The statement that the point collection $M$ has length 0 means that if $\varepsilon > 0$, there is a sequence $S_1, S_2, \ldots$ of segments covering $M$ such that $|S_1| + \cdots + |S_n| < \varepsilon$ for all positive integers $n$.

Theorem 92: Suppose $M$ is the set of all rational numbers in $[0, 1]$. $M$ has length 0.

Theorem 93: Suppose $n$ is a positive integer, $[a, b]$ is an interval, and $f$ is a function such that each of $f, f', f'', \ldots, f^{(n+1)}$ is continuous and has domain $[a, b]$. Then

$$f(x) = \sum_{i=0}^{n} f^{(i)}(a)(x-a)^i/i! + (-1)^n \int_{a}^{x} f^{(n+1)}(t)(t-a)^n/n! \, dt$$

for all $x$ in $[a, b]$.

Theorem 94: Suppose that $a_1, a_2, \ldots$ is a number sequence, and $r$ is in $(0, 1)$ such that $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all positive integers $n$. Then $a_1 + a_2 + \cdots$ converges.

Theorem 95: Suppose that $a_1, a_2, \ldots$ is a decreasing sequence of positive numbers with sequential limit 0. Then

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Theorem 96: Suppose that $f$ is a continuous function with domain the interval $[a, b]$, and $c$ is in $[a, b]$. If

$$f(x) = \int_{c}^{x} f$$

for all $x$ in $[a, b]$, then $f(x) = 0$ for all $x$ in $[a, b]$.

Theorem 97: Suppose $c$ is a number, and each of $f$ and $g$ is a function such that $g$ is differentiable at $c$, $f$ is differentiable at $g(c)$, and $c$ is a limit point of the domain of $f(g)$. Then $f(g)$ is differentiable at $c$ and

$$(f(g))'(c) = f'(g(c))f'(c)$$
Theorem 98: Suppose that $f$ is a function whose domain includes the interval $[a, b]$, and $c$ is in $[a, b]$. If the domain of each of $f', f'', \cdots$ includes $[a, b]$, and there is a number $M$ such that $|f^{(n)}(x)| \leq M$ for all $x$ in $[a, b]$, $n = 1, 2, \cdots$, then

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(c)(x - c)^i / i!$$

for all $x$ in $[a, b]$.

Theorem 99: Suppose $f$ is a function whose domain includes the interval $[a, b]$, and $c$ is in $(a, b)$. If the domain of each of each of $f', f'', \cdots$ includes $[a, b]$ and there are positive numbers $M$ and $\rho$ such that

$$|f^{(n)}(x)| \leq M \rho^n$$

for all $x$ in $[a, b]$, $n = 1, 2, \cdots$, then there is $\delta > 0$ such that if $|x - c| < \delta$, and $x$ is in $[a, b]$, then

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(c)(x - c)^i / i!$$

Theorem 100: Suppose $a_1, a_2, \ldots$ is a number sequence, and

$|a_1| + |a_2| + \cdots$ converges. Suppose also that $n_1, n_2, \ldots$ is a sequence of positive integers such that

i) if $i$ is a positive integer then $i = n_j$ for some positive integer $j$;  
ii) if $i$ and $j$ are two positive integers, then $n_i \neq n_j$.

Then $a_{n_1} + a_{n_2} + \cdots$ converges and equals $a_1 + a_2 + \cdots$. 