Foundations of Calculus
Properties of the Real Numbers, Functions and Continuity
A Transition Course

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To the Student

This course will likely be different from any course you have taken. First of all, there is no textbook for the course. These notes will serve in place of a book. Instead of reading (and probably memorizing) the proofs that some author has presented, you will be expected to produce your own. If you have never had a course in which you were expected to produce your own proofs, do not worry because your instructor does not expect that you have. Instead, you are going to discover some mathematics for yourself. Along the way you will find out that, indeed, you can do this and you will get a taste of the personal satisfaction and the thrill of doing so. You do not have to bring a lot with you for this—mostly a willingness to work hard even in the face of frustration. Your instructor will set the rules for how you do this, but it is my opinion that you will get the most from your study if you do most of this on your own. My recommendations to my classes was to limit their discussions with other students in the class to the meaning of the statements of the definitions and theorems. Any work on proofs should be your own without assistance from outside sources—books, other students, or the internet. Of course, seeking guidance from your instructor when you are stuck is highly recommended, but do not be surprised if your questions are met with questions.

These notes attempt to create for you a world in which you work to discover insights and construct your proofs. In order to make this as transparent as possible, an attempt has been made to create for you a place where you clearly understand what you have to work with. Consequently, these notes define for you a mathematical system in which you will work. This system consists of two undefined terms, some defined terms, some assumptions about these terms (the axioms), and some statements of theorems that you are to prove. Accompanying the statements of theorems will be some questions and problems. Often these questions and problems can lead you to a deeper understanding of this system and even allow you to expand it through some theorems you find on your own.

Why do we begin with undefined terms? I am confident that you have gone to a dictionary to look up a word you do not know. Have you ever done this only to encounter in the definition another word you do not know?
Upon seeking a meaning for this word you encounter yet another word you
do not know. This process may continue for a while when the latest word
you seek has in its definition the term you began with. What you have
encountered somewhere in this process is one of the “undefined” terms used
by the compilers of the dictionary. A brief reflection on what happened leads
us to realize that, when one defines words, some words must be assumed to
be known to the users of the dictionary. In our system, there are two such
terms that are outside the common, everyday use of the English language—
point and the phrase to the left of. Intuitively, we are assuming that the
elements of the set under consideration (our points) are strung out in some
linear fashion so that, if two of them are considered, one of them appears
to precede the other. All other terms in the notes are ultimately defined
using these undefined terms. There are initially six axioms five of which
cement the intuitive notion of to the left of. Ultimately, your proofs should
rely only on the definitions of the terms and these six axioms although, of
course, once a theorem has been proved its statement may be used in any
argument you make for the truth of an ensuing theorem.

Sets underlie virtually all of mathematics and that is certainly the case
for these notes. In the preceding paragraph that term was used even though
it is neither a defined nor an undefined term. You will need to use some
intuitive set theory as you proceed through the notes. If you think of a set
as a collection of objects and you are able to form new sets such as the set
of objects common to a collection of sets or the union of a collection of
sets (collection is used synonymously with the term set to avoid awkward
language like a set of sets), you should be able to get along just fine in
the course. One anomaly that quickly arises in this “intuitive” set theory
is whether there can be a set without any objects in it, a so-called empty
set. If you talk to a child about sets in this intuitive way as a collection of
objects, they might look quizzically at you if you then talk about an empty
set. Although an empty set may be a convenience at times, throughout these
notes the term “point set” will refer to a set that contains at least one point.
As a consequence, if, in the proof of a theorem, you state that something is
a point set, it will be incumbent on you to show that it contains at least one
point in order to proceed talking about it.

Little words make a big difference and in these notes we use all words
carefully. You know the difference between ‘a’ and ‘the’. To underscore
the difference, consider the following: Your phone rings and you answer it.
In a short period of time you realize the person on the line did not intend to
call your number. Do you say to them, “You must have the wrong number”?
Most people would, but do you mean to tell them that yours was the only
number they did not intend to call? Perhaps, one should say, “You have
a wrong number”. Language is our vehicle for communicating our ideas.
To be sure we are not misunderstood we must use our language carefully. Prepositions are important; the use of the plural is important; each word is important! Be careful.

Finally, we make a remark about models for our mathematical system. Our intuitive “model” for this system is, of course, the number line in the sense that we interpret ‘point’ to be ‘real number’ and ‘to the left of’ to be ‘is less than’. With these interpretations of our undefined terms, it is easy to believe that all of the axioms (with the possible exception of the first one) are satisfied. However, there are other, equally valid interpretations of the undefined terms for which our axiom system holds. One really nice feature of our approach is that any proof given that relies only on the axioms and definitions yields a theorem that holds true in any model of our system. Even though the set of real numbers model is convenient for drawing pictures and thinking about approaches to proofs, be sure you rely only on axioms and definitions and not on any special properties of the real numbers in constructing proofs (such as forming \((P + Q)/2\) to get a point between \(P\) and \(Q\) instead of citing Axiom 2.)

That is enough talk from me. It is time to begin this adventure. Enjoy!!
Chapter 1

BASIC PROPERTIES OF POINT SETS

Undefined terms: point, to the left of

Definition 1. Suppose $P$ is a point and $M$ is a point set. The statement that $P$ is a leftmost point of $M$ means $P$ is a point of $M$ and if $x$ is a point of $M$ then $x$ is not to the left of $P$.

Problem 2. Formulate a meaning for the statement that $P$ is a rightmost point of $M$. (Note that the phrase ‘to the right of’ is not defined. Try to formulate the meaning without using ‘to the right of’.)

We now assume that we have a set whose elements are called points and there is a meaning of the phrase ‘to the left of’ so that all six of the following axioms hold.

AXIOM 1. If $S_1$ and $S_2$ are point sets such that (1) if $x$ is a point then $x$ is in $S_1$ or $x$ is in $S_2$ and (2) if $x$ is a point of $S_1$ and $y$ is a point of $S_2$ then $x$ is to the left of $y$, then $S_1$ has a rightmost point or $S_2$ has a leftmost point.

AXIOM 2. If each of $P$ and $Q$ is a point and $P$ is to the left of $Q$ then there exists a point $x$ such that $P$ is to the left of $x$ and $x$ is to the left of $Q$.

AXIOM 3. If $P$ and $Q$ are points then $P$ is to the left of $Q$ or $Q$ is to the left of $P$.

AXIOM 4. If $P$, $Q$, and $R$ are points, $P$ is to the left of $Q$ and $Q$ is to the left of $R$, then $P$ is to the left of $R$.

AXIOM 5. If $P$ is a point then $P$ is not to the left of $P$.

AXIOM 6. If $P$ is a point then there exist points $Q$ and $R$ such that $Q$ is to the left of $P$ and $P$ is to the left of $R$. 

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Question 3. Can a point set have two leftmost points?

Problem 4. Show that, under the hypothesis of Axiom 1, if $S_1$ has a rightmost point then $S_2$ does not have a leftmost point.

Problem 5. Complete the following statement: The point $P$ is not the leftmost point of the point set $M$ means . . . . (This is called the bare denial (or negation) of the statement that $P$ is the leftmost point of $M$.)

Theorem 6. If $M$ is a point set and $B$ is a point to the left of every point of $M$ then $M$ has a leftmost point or there is a rightmost of all the points to the left of every point of $M$.

Definition 7. If $P$ and $Q$ are points, the statement that $P$ is to the right of $Q$ means $Q$ is to the left of $P$.

Definition 8. If $P$ and $Q$ are points and $R$ is a point, the statement that $R$ is between $P$ and $Q$ means $R$ is to the right of $P$ and to the left of $Q$ or $R$ is to the right of $Q$ and to the left of $P$.

Definition 9. Suppose $A$ and $B$ are two points and $A$ is to the left of $B$. By the segment $AB$, [denoted $(A,B)$], is meant the point set to which the point $x$ belongs if and only if $x$ is between $A$ and $B$. The statement that the point set $s$ is a segment means there exist points $A$ and $B$ such that $s$ is $(A,B)$.

Definition 10. Suppose $P$ is a point and $M$ is a point set. The statement that $P$ is a limit point of $M$ means if $s$ is a segment containing $P$ then $s$ contains a point of $M$ distinct from $P$.

Problem 11. Write the bare denial of the statement that the point $P$ is a limit point of the point set $M$.

Question 12. Suppose $A$ is a point and $M$ is a point set whose only member is $A$. Is $A$ a limit point of $M$? If $x$ is to the left of $A$, is $x$ a limit point of $M$? If $x$ is to the right of $A$ is $x$ a limit point of $M$? Does $M$ have a limit point?

Question 13. Suppose $A$ and $B$ are two points and $M$ is a point set whose only members are $A$ and $B$. Does $M$ have a limit point?

Question 14. If $M$ is a segment, does $M$ have a limit point?

Definition 15. A point set $M$ is called an interval provided there exist two points $A$ and $B$ with $A$ to the left of $B$ and such that $x$ belongs to $M$ if and only if $x$ is in the segment $(A,B)$ or $x$ is $A$ or $x$ is $B$. In this case $M$ is called the interval $AB$ and is denoted $[A,B]$.

Question 16. If $M$ is an interval, does $M$ have a limit point?

Question 17. Is there a point set with only one limit point?
Problem 18. Show that if each of \(s_1\) and \(s_2\) is a segment containing the point \(P\), then there is a segment \(s\) containing \(P\) such that \(s\) is a subset of \(s_1\) and \(s\) is a subset of \(s_2\).

Theorem 19. If \(M\) is a point set without a leftmost point and there is a point to the left of every point of \(M\), then \(M\) has a limit point.

Definition 20. The statement that a set \(M\) is finite means there is a positive integer \(n\) such that \(M\) contains only \(n\) elements.

We will use without proof the following property of the set of positive integers:

If \(K\) is a set of positive integers, \(K\) has a least element.

Theorem 21. If \(M\) is a finite point set then \(M\) does not have a limit point.

Question 22. Is there an infinite point set that does not have a limit point?

Theorem 23. If \(M\) is an infinite subset of a segment then \(M\) has a limit point.

Question 24. Does each finite point set have a leftmost point?

Question 25. Suppose \(s_1, s_2, s_3, \ldots\) is a sequence of segments such that \(s_2\) is a subset of \(s_1\), \(s_3\) is a subset of \(s_2\), \ldots. Is there a point that belongs to every term of the sequence \(s_1, s_2, s_3, \ldots\)?

Theorem 26. Suppose \(I_1, I_2, I_3, \ldots\) is a sequence of intervals such that \(I_{n+1}\) is a subset of \(I_n\) for each positive integer \(n\). Then there is a point that belongs to every interval in the sequence \(I_1, I_2, I_3, \ldots\). Further, if the common part of all the intervals in the sequence \(I_1, I_2, I_3, \ldots\) contains two points, the common part is an interval. Moreover, if \(C\) is the common part and \(s\) is a segment containing \(C\), then there is a positive integer \(n\) such that \(I_n\) is a subset of \(s\).

Question 27. Is there a sequence \(r_1, r_2, r_3, \ldots\) of numbers such that \(\{r_1, r_2, r_3, \ldots\}\) is the set of rational numbers in \([0, 1]\)?

Question 28. Is there a sequence \(x_1, x_2, x_3, \ldots\) of points of the interval \([A, B]\) such that \(\{x_1, x_2, x_3, \ldots\}\) is \([A, B]\)?

Definition 29. The statement that the point sets \(H\) and \(K\) are mutually exclusive means no point belongs to both \(H\) and \(K\).

Definition 30. The statement that \(H\) and \(K\) are mutually separated means \(H\) and \(K\) are mutually exclusive and neither contains a limit point of the other.

Question 31. Is the interval \([A, B]\) the union of two mutually separated point sets?
Notation: If $M$ is a point set, $\overline{M}$ denotes the set to which the point $P$ belongs if and only if $P$ is a point of $M$ or $P$ is a limit point of $M$.

**Definition 32.** The statement that the point set $M$ is closed means if $x$ is a limit point of $M$ then $x$ is a point of $M$.

**Problem 33.** If the point set $A$ is a subset of the point set $B$ then $\overline{A}$ is a subset of $\overline{B}$.

**Problem 34.** If $M$ is a point set, $\langle M \rangle = \overline{M}$.

**Problem 35.** If $M$ is a point set, $\overline{M}$ is closed.

**Problem 36.** If each of $H$ and $K$ is a point set, $\overline{H \cup K} = \overline{H} \cup \overline{K}$.

**Question 37.** Is every finite point set closed?

**Question 38.** Can “$\cup$” in Problem 36 be replaced by “$\cap$”?

**Definition 39.** The statement that the collection $\mathcal{G}$ of sets covers the set $M$ means if $x$ is in $M$ then some element of $\mathcal{G}$ contains $x$.

**Question 40.** (Four Questions)

a. Does there exist a collection $\mathcal{G}$ of segments covering the segment $(A,B)$ such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover $(A,B)$?

b. Does there exist a collection $\mathcal{G}$ of intervals covering the segment $(A,B)$ such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover $(A,B)$?

c. Does there exist a collection $\mathcal{G}$ of segments covering the interval $[A,B]$ such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover $[A,B]$?

d. Does there exist a collection $\mathcal{G}$ of intervals covering the interval $[A,B]$ such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover $[A,B]$?

**Question 41.** Does there exist a closed point set $M$ such that every point of $M$ is a limit point of $M$ and $M$ contains no interval?

**Theorem 42.** The interval $[A,B]$ is not the union of two mutually separated point sets.

**Theorem 43.** If $\mathcal{G}$ is a collection of segments covering the interval $[A,B]$, then there is a finite subcollection $\mathcal{H}$ of $\mathcal{G}$ that covers $[A,B]$.

**Question 44.** Does Theorem 43 remain true if the interval $[A,B]$ is replaced by a closed subset of $[A,B]$?
**Question 45.** Let $M$ be a point set with the property that if $\mathcal{G}$ is a collection of segments covering $M$ then some finite subcollection of $\mathcal{G}$ covers $M$. Is it true that $M$ is closed?
Chapter 2

FUNCTIONS

Definition 46. A function is a set of ordered pairs such that no two pairs in the set have the same first term. If $f$ is a function and $M$ is the set of first terms of pairs in $f$, then $f$ is said to be a function defined on $M$.

Notation: If $f$ is a function and the pair $(x, y)$ is in $f$, we sometimes write $y = f(x)$.

Some Examples

1: $f_1$ is the set to which the pair $(x, y)$ of points belongs if and only if $x$ is $y$.

2: Suppose $P$ is a point and $Q_1$ and $Q_2$ are two points. Denote by $f_2$ the set to which the pair $(x, y)$ of points belongs if and only if it is true that if $x$ is to the left of $P$ then $y$ is $Q_1$ and $y$ is $Q_2$ otherwise.

3: Suppose $P$ is a point. Denote by $f_3$ the set to which the pair $(x, y)$ of points belongs if and only if $y$ is $P$.

Problem 47. Show that each of $f_1, f_2,$ and $f_3$ is a function.

Definition 48. Suppose $f$ is a function defined on the point set $M$, $P$ is a point of $M$ and the set of second terms of pairs in $f$ is a point set. The statement that $f$ is continuous at $P$ means if $t$ is a segment containing $f(P)$ then there is a segment $s$ containing $P$ such that if $x$ is a point of $M$ in $s$ then $f(x)$ is in $t$.

Problem 49. Check the definition of $f$ is continuous at $P$ for each of the examples $f_1, f_2,$ and $f_3$ at several points.

Problem 50. Write the bare denial of the definition of $f$ is continuous at $P$. 
Theorem 51. Suppose $M$ is a point set, $P$ is a point of $M$ and $P$ is not a limit point of $M$. If $f$ is a function defined on $M$, then $f$ is continuous at $P$.

Theorem 52. Suppose $M$ is a point set, $P$ is a point of $M$ and $f$ is a function defined on $M$. If $f$ is continuous at $P$ and $Q$ is a point to the left of $f(P)$, then there is a segment $s$ containing $P$ such that if $x$ is a point of $M$ in $s$ then $Q$ is to the left of $f(x)$.

Problem 53. Suppose $f$ is a function defined on the interval $[A, B]$ such that if $P$ is a point of $[A, B]$ then $f$ is continuous at $P$. Let $M_Q$ denote the set to which the point $x$ of $[A, B]$ belongs if and only if $f(x)$ is not to the left of $Q$. Show that for each point $Q$ such that $Q$ is to the left of $f(t)$ for some point $t$ of $[A, B]$, the set $M_Q$ is a closed point set.

Theorem 54. Suppose $f$ is a function defined on the interval $[A, B]$, $f(A)$ is not $f(B)$, and if $P$ is a point of $[A, B]$ then $f$ is continuous at $P$. If $Q$ is a point between $f(A)$ and $f(B)$ then there is a point $C$ between $A$ and $B$ such that $f(C) = Q$.

Theorem 55. Suppose $f$ is a function defined on the interval $[A, B]$ such that if $P$ is a point of $[A, B]$ then $f$ is continuous at $P$. Then there exist points $C$ and $D$ such that if $x$ is a point of $[A, B]$, then $f(x)$ is in the segment $(C, D)$.

Definition 56. Suppose $f$ is a function defined on the point set $M$, $P$ is a limit point of $M$ and $L$ is a point. The statement that $f$ has limit $L$ at $P$ means if $t$ is a segment containing $L$ then there is a segment $s$ containing $P$ such that if $x$ is a point of $M$ in $s$ and $x$ is distinct from $P$ then $f(x)$ is in $t$.

Theorem 57. Suppose $M$ is a point set, $f$ is a function defined on $M$ and $P$ is a point of $M$ that is a limit point of $M$. Then $f$ is continuous at $P$ if and only if $f$ has limit $f(P)$ at $P$.

Theorem 58. Suppose $M$ is a point set, $f$ is a function defined on $M$ and $P$ is a limit point of $M$ and $K$ and $L$ are points. If $f$ has limit $L$ at $P$ then $f$ does not have limit $K$ at $P$.

Definition 59. A collection $\mathcal{G}$ of point sets is called monotonic provided it is true that if $g$ and $h$ are in $\mathcal{G}$ then $h$ is a subset of $g$ or $g$ is a subset of $h$.

Problem 60. Show that if $\mathcal{G}$ is a monotonic collection of closed subsets of the interval $[A, B]$, then there is a point that belongs to every set in $\mathcal{G}$.

Definition 61. The statement that the point set $M$ has the finite covering property means if $\mathcal{G}$ is a collection of segments covering $M$ then there is a finite subcollection $\mathcal{H}$ of $\mathcal{G}$ that covers $M$.

Problem 62. Show that if the point set $M$ has the finite covering property then $M$ is closed.

Problem 63. Show that if $M$ is a closed subset of the interval $[A, B]$ then $M$ has the finite covering property.
Definition 64. If $f$ is a function defined on the point set $M$, denote by $f[M]$ the set of all second terms of pairs in $f$.

Definition 65. If $f$ is a function defined on the point set $M$ and $f$ is continuous at each point of $M$, we say $f$ is **continuous** on $M$.

Theorem 66. If $f$ is continuous on the interval $M = [A, B]$ and $f[M]$ contains two points, then $f[M]$ is an interval.

Problem 67. Show that if $P$ and $Q$ are points, there exist mutually exclusive segments $s_P$ and $s_Q$ containing $P$ and $Q$, respectively.

Definition 68. A set is said to be **open** if it is the union of a collection of segments.

Problem 69. Show that if $P$ is a point and $K$ is a closed set not containing $P$, then there exist mutually exclusive open sets $O_P$ and $O_K$ containing $P$ and $K$, respectively.

Problem 70. Show that if $H$ and $K$ are mutually exclusive closed points sets, then there exist mutually exclusive open sets $O_H$ and $O_K$ containing $H$ and $K$, respectively.

Problem 71. Show that if $H$ and $K$ are mutually separated point sets, then there exist mutually exclusive open sets $O_H$ and $O_K$ containing $H$ and $K$, respectively.
Chapter 3

AXIOM 7 AND ITS CONSEQUENCES

Definition 72. A sequence is a function defined on the set of positive integers.

If \( r \) is a sequence and \( i \) is a positive integer, we often denote \( r(i) \) by \( r_i \) (as we have been doing).

AXIOM 7. There exists a sequence of points \( r_1, r_2, r_3, \ldots \) such that if \( A \) and \( B \) are points then there is a positive integer \( i \) such that \( r_i \) is between \( A \) and \( B \).

Problem 73. Show that if \( A \) and \( B \) are points there is a point \( x \) between \( A \) and \( B \) such that \( x \) does not belong to \( \{ r_1, r_2, r_3, \ldots \} \).

Theorem 74. If \( x \) is a point then \( x \) is a limit point of \( \{ r_1, r_2, r_3, \ldots \} \).

Theorem 75. If \( P \) is a point, there is a sequence \( y_1, y_2, y_3, \ldots \) of points of \( \{ r_1, r_2, r_3, \ldots \} \) such that \( y_1 \) is to the left of \( y_2 \), \( y_2 \) is to the left of \( y_3, \ldots \) and \( P \) is a limit point of \( \{ y_1, y_2, y_3, \ldots \} \).

Theorem 76. If \( P \) is a point, there is a sequence \( s_1, s_2, s_3, \ldots \) of segments containing \( P \) such that \( s_2 \) is a subset of \( s_1 \), \( s_3 \) is a subset of \( s_2, \ldots \) and \( P \) is the only point common to \( s_1, s_2, s_3, \ldots \).

Theorem 77. There exists a sequence \( s_1, s_2, s_3, \ldots \) of segments such that if \( s \) is a segment and \( P \) is a point of \( s \) then there is a segment \( s_i \) in the sequence \( s_1, s_2, s_3, \ldots \) that contains \( P \) and is a subset of \( s \).

Theorem 78. Suppose \( P \) is a point, \( M \) is a point set and \( s_1, s_2, s_3, \ldots \) is a sequence of segments as in Theorem 77. Then, \( P \) is a limit point of \( M \) if and only if it is true that if \( s_i \) is a segment in the sequence \( s_1, s_2, s_3, \ldots \) and \( s_i \) contains \( P \) then \( s_i \) contains a point of \( M \) distinct from \( P \).
Definition 79. The statement that a set X is countable means there is a sequence \( x_1, x_2, x_3, \ldots \) such that \( X = \{x_1, x_2, x_3, \ldots \} \). If a set is not countable we say that it is uncountable.

Theorem 80. If \( M \) is an uncountable point set then some point of \( M \) is a limit point of \( M \).

Definition 81. Suppose \( x_1, x_2, x_3, \ldots \) is a sequence of points and \( x \) is a point. The statement that \( x_1, x_2, x_3, \ldots \) converges to \( x \) means if \( s \) is a segment containing \( x \) there is a positive integer \( N \) such that if \( n \geq N \) then \( x_n \) is in \( s \).

Problem 82. Guess what. Yes, write a bare denial for Definition 81.

Theorem 83. Suppose \( x_1, x_2, x_3, \ldots \) is a sequence of points that converges to the point \( x \) and \( y \) is a point different from \( x \). Then, \( x_1, x_2, x_3, \ldots \) does not converge to \( y \).

Theorem 84. Suppose the sequence \( x_1, x_2, x_3, \ldots \) converges to the point \( x \) and the sequence \( y_1, y_2, y_3, \ldots \) converges to \( x \). If \( z_1, z_2, z_3, \ldots \) is a sequence of points such that, for each positive integer \( i \), \( z_i \) is between \( x_i \) and \( y_i \), then \( z_1, z_2, z_3, \ldots \) converges to \( x \).

Definition 85. If \( x_1, x_2, x_3, \ldots \) is a sequence, the statement that \( x_1, x_2, x_3, \ldots \) converges means there is a point \( x \) such that \( x_1, x_2, x_3, \ldots \) converges to \( x \).

Question 86. If \( x \) is a point and \( x_1, x_2, x_3, \ldots \) is a sequence of points such that, for each positive integer \( i \), \( x_i = x \), does the sequence \( x_1, x_2, x_3, \ldots \) converge?

Theorem 87. Suppose \( x_1, x_2, x_3, \ldots \) is a sequence such that, for each positive integer \( i \), \( x_i \) is to the left of \( x_{i+1} \). If there is a point \( B \) such that, for each positive integer \( i \), \( x_i \) is to the left of \( B \) then \( x_1, x_2, x_3, \ldots \) converges.

Theorem 88. If the point \( P \) is a limit point of the point set \( M \) then there is a sequence of points of \( M \) that converges to \( P \).

Theorem 89. If \( M \) is a point set, the point \( P \) belongs to \( M \) if and only if there is a sequence of points of \( M \) that converges to \( P \).

Theorem 90. Suppose \( M \) is a point set, \( f \) is a function defined on \( M \) and \( P \) is a point of \( M \). Then, \( f \) is continuous at \( P \) if and only if it is true that if \( x_1, x_2, x_3, \ldots \) is a sequence of points of \( M \) that converges to \( P \) then \( f(x_1), f(x_2), f(x_3), \ldots \) converges to \( f(P) \).

Theorem 91. Suppose \( [A, B] \) is an interval and \( f \) is a function defined on \( [A, B] \). Then, \( f \) is continuous on \( [A, B] \) if and only if for each subset \( M \) of \( [A, B] \), \( f[M] \) is a subset of \( f[M] \).
Chapter 4

SIMPLE GRAPHS

For the remainder of the course, the over-riding assumption is that $S$ is the set of real numbers. We will use the word “point” in two different senses. Most of the time, point will mean “ordered number pair”. However, at times (and it should be clear from context) point and real number may be used synonymously.

Definition 92. A simple graph $f$ is a set of points such that if $h$ is a vertical line that intersects $f$ then $h$ contains only one point of $f$.

Definition 93. The statement that the simple graph $f$ has property $U$ at the point $P$ of $f$ means if $l$ is a horizontal line with $P$ below it, there exist vertical lines $h$ and $k$ with $P$ between them such that if $Q$ is a point of $f$ between $h$ and $k$ then $Q$ is below $l$.

Definition 94. If $f$ is a simple graph, by the $x$-projection of $f$ is meant the set of first terms of pairs in $f$.

Theorem 95. Suppose $f$ is a simple graph with $x$-projection an interval and $f$ has property $U$ at each point. If $l$ is a horizontal line and $M = \{z \mid z$ is the $x$-projection of a point of $f$ that is on or above $l\}$ then $M$ is closed.

Definition 96. If $f$ is a simple graph, the statement that the point $P$ of $f$ is a high point of $f$ means if $Q$ is a point of $f$ then $Q$ is not above the horizontal line passing through $P$.

Theorem 97. If $f$ is a simple graph with $x$-projection an interval and $f$ has property $U$ at each point then $f$ has a high point.

Definition 98. The statement that the simple graph $f$ has property $L$ at $P$ means if $l$ is a horizontal line with $P$ above it, then there exist vertical lines $h$ and $k$ with $P$ between them such that if $Q$ is a point of $f$ between $h$ and $k$ then $Q$ is above $l$.

Problem 99. State and prove theorems analogous to the two stated above for simple graphs with property $L$. 

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Theorem 100. Suppose $f$ is a simple graph and let $P$ be a point of $f$. Then $f$ has both property $U$ and property $L$ at $P$ if and only if $f$ is continuous at the $x$-projection of $P$.

Definition 101. Suppose $f$ is a simple graph and $P$ is a point of $f$ such that the $x$-projection of $P$ is a limit point of the $x$-projection of $f$. The statement that $f$ has slope $m$ at $P$ means if $A$ is a line containing $P$ with slope greater than $m$ and $B$ is a line containing $P$ with slope less than $m$ then there exist vertical lines $h$ and $k$ with $P$ between them such that if $Q$ is a point of $f$ distinct from $P$ between $h$ and $k$ then $Q$ is between $A$ and $B$.

Problem 102. Show that if $f$ has slope $m_1$ at $P$ and $f$ has slope $m_2$ at $P$ then $m_1 = m_2$.

Definition 103. If $f$ has slope $m$ at $P$ and $l$ is the line through $P$ with slope $m$, then $l$ is called the tangent line to $f$ at $P$.

Theorem 104. Suppose $f$ is a simple graph with slope $m$ at $P$. If $l$ is a line containing $P$ such that no point of $f$ is above $l$ then $l$ is the tangent line to $f$ at $P$.

Theorem 105. If the simple graph $f$ has slope $m$ at $P$ then $f$ has property $U$ and property $L$ at $P$. 

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Journal of Inquiry-Based Learning in Mathematics