Foundations of Calculus
Properties of the Real Numbers, Functions and Continuity

A Transition Course

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## To the Student

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To the Instructor

I first taught a course from notes similar to these in 1965 at the University of Houston. We had decided no longer to make an issue of some of the theorems in our elementary calculus courses. Specifically, we decided to omit a formal proof of the fact that a continuous function on an interval achieves a maximum value on the interval. Instead, we would present plausibility arguments for these theorems and leave the proofs for future courses. We instituted a course, envisioned as a course primarily for our majors, intended to introduce the students to proofs and made proving the maximum value theorem a major objective of the course. I understood that Moore had taught a course at the University of Texas which used “point” and “to the left of” as its undefined terms and the Dedekind axiom for completeness of the space. Though I did not have direct access to notes from Moore’s course, I pieced together the first version of these notes based on those undefined terms, the Dedekind axiom and much input from colleagues at Houston.

0.1 Class Usage and Grading

From 1965 through 2002, I taught a three semester-hour course from these notes at least once per year and, many years, every semester. Prerequisites for the course were always two semesters of calculus although I would have welcomed a bright student who had not had even a first calculus course. When the course was completed, my students often asked why this course did not precede their calculus sequence. Class sizes normally ranged from ten to twenty-five students. Fewer than five students can place extraordinary burdens on the students; more than thirty can make it quite difficult to find enough class time for everyone to get to the board often enough. My preference was to offer the course in a three class meetings per week setting. Undergraduates at the sophomore level often have difficulty with the longer time between class meetings dictated by a two class meetings per week schedule.

Grading a course like this is difficult. It is invariably subjective. Recalling how an instructor’s patience had benefitted me in my first experience in a course of this sort, I always tried to err on the side of the student in
determining grades for such a course. I evolved into giving two in-class exams plus a final exam in the course. I graded these carefully—more carefully than the coarse ‘A’, ‘B’, ‘C’, ‘D’, ‘F’ mark (sometimes accompanied by a ‘-‘ but almost never a ‘+‘) that I put on the front of each paper. All of the exams had a similar format: the first problem was to write a few of the definitions, the second problem was to write a few negations of definitions, the third problem was to give one or two examples (either as presented or a minor variation on what was presented in class), and the last problem or two consisted of providing proofs of theorems or portions of theorems from the course (I tried to design these so that understanding trumped memorization). Approximate weights I gave to the exam components were roughly 20% for the definitions and negations, 30% for examples, and 50% for the proofs. When I handed back the first exam, I always told them that the grade on that first exam could never hurt their final grade in the course provided they showed improvement on the subsequent exams and did well in writing and class participation. Near the end of the term, I assigned a writing grade and a class participation grade (both of these were, admittedly, subjective grades in the form of letter grades) that I shared with each student on the last day of classes. Final course grades were determined by looking for a consensus among these letter grades with the greatest weight going to class participation (no participation = ‘B’ or lower) and the final exam. As might be expected, a good grade for course participation was invariably accompanied by good grades on the exams as well as the writing. I rarely had a student who completed the course get a failing grade. Most of the time students heading for failure simply dropped the class along the way.

0.2 Things I Cover on the First Day

I did not hand this out nor did I repeat it verbatim but I more or less said everything written in this section during the first class meeting. I normally read over this just before I went to class on the first day to remind myself of the things I wanted to be sure I covered. Some of this is repeated in the notes to the student.

This course will likely be different from most of the courses you will take at the university. You noticed, if you went to the bookstore for a textbook, that there is no text for the course. In this course, we are going to start learning to prove theorems exploring what mathematics is all about—proofs of theorems. Mathematics is not a spectator sport, so you will be expected to do mathematics—not listen to me talk about mathematics. Although some of you may have had a course similar to this one, it will not be assumed that any of you have any skills in proving theorems. But, if you are not willing
to work at the task of proving theorems, drop now and do something else with your semester and my time. However, you will find that in all the upper level mathematics courses, your instructor will assume that you already know what a proof is and how to construct proofs for yourself. Taking those courses with that assumption being false is risky for you.

In this course we will try to be very careful with the English language. Why? Theorems and their proofs have to be stated in our language, but there can be no ambiguity in our use of the language. Thus, when there is a need for the singular we will use the singular; when there is a need for the plural, we will use the plural; when a preposition is called for, we will use it; when a or the is used we will try not to misuse it. As an example of a misuse of the, consider what happens when you answer a telephone and it soon becomes obvious that the caller did not intend to call your number. What do you say? “You have the wrong number.” Now, do you mean to tell the caller that, of all the numbers they could have dialed, yours was the only one that the person did not intend to call? Perhaps, one should say “You have a wrong number.”

Let’s talk a little about mathematical systems. A mathematical system consists of undefined terms, axioms, definitions, examples and theorems. We have to have undefined terms as a sort of common ground from which we can begin. In looking up a word you do not know in the dictionary, has the following ever happened to you? The word you look up is defined using a word you do not know so you look up that word. In turn, in its definition there is a word you do not know so you look it up. Once again you encounter a word you do not know. After this process continues for a time or two more, suddenly the word you started out looking up is used in a definition. Compilers of dictionaries, you see, have to have some “primitive” words that they may assume that every reader "knows” so that they can define the others. When the scenario described occurs, somewhere along the way you are encountering one of their primitives which you do not know. For similar reasons undefined terms are required in any mathematical system. We have to have some common ground from which we can begin–our undefined terms. Just because we take some terms as undefined does not mean we do not have some intuitive idea about them and their meaning. It is simply that if we try to define them we soon see that there other terms we are assuming that everyone “knows.” In this course our undefined terms are “point” and “to the left of.” In some sense this list is not complete, since I will assume we all have in common some of the language of set theory and, technically, at least “set” should be an undefined term. But, this is not a course in set theory and you all should have seen and heard enough of its language and ideas to get along in here just fine.

The “axioms” are the assumptions we make about the terms we have not
defined, together with assumptions we make about some terms we define. The undefined terms and the axioms are the building blocks upon which
the course is constructed. It consists of new language we build up through
definitions and a body of statements we can prove—the theorems. Of course,
in mathematics, it is the theorems along with their proofs in which we are
most interested.

It is my opinion that mathematics is best learned by doing—not by listen-
ing to or watching someone else do it. In here, you will get the opportunity
to do some mathematics. If you take advantage of this opportunity, you will
find the pleasure and attraction that mathematics has, and, perhaps, you too
will become hooked on it as I, as well as my colleagues in the mathematics
department, have. Mathematics is both fun and frustrating, but you have to
do it to experience the sheer exhilaration it affords. Early in the course you
will probably experience some of its frustration. This is quite natural. All I
ask is that you keep trying the problems, questions and theorems that I give
you. If you do, sooner or later, somehow, somewhere, someway a little light
will go on and mathematics will never be the same for you again. I do not
mean by this that suddenly you will never have trouble with it again. I still
have trouble with it. Mathematics requires thought, originality, understand-
ing, cleverness, and other intangibles. These things cannot be taught nor are
they easy to discover in yourself. It will always require work, work, and
more work. It is my hope that, after this course, you will at least begin to be
able to recognize what has to be done, know something of how to go about
doing it, and recognize when you have done it.

There are a few rules I want us to agree on from the beginning. (1) Do
not look in books or on the internet to try to find the proofs of the theorems
or answers to the questions raised in this class. Do not discuss the course
with others on campus who may have taken the course. Remember we are
in here to learn to do mathematics, not to see what some author can do.
(2) You should discuss the course with classmates outside of class, but do
not collaborate with others in proving the theorems. Part of your grade
will be derived from class participation and no more than one person will
get “credit” for the proof of any one theorem. (3) In class, feel free to ask
questions when the proofs being presented, but let’s not make a “community
project” out of any of the theorems. That is, everyone should refrain from
making suggestions to the presenter at the board. Such suggestions not only
can cause them to lose their train of thought but also no more than one
person will get “credit” for any one theorem.

Grades in the course will be based on participation in class. Theorems
will be presented by the students at the board and the proofs will be defended
to the class. I will not prove any of the theorems for you. Some will be
reluctant to go to the board, but let me assure you now that no one has yet
died while they were presenting a proof at the board in my class. If you are worried about making a mistake in front of the class, don’t. The only people who will not make a mistake in class are those who do not go to the board (and even that is a mistake). However, for those of you who for one reason or another do not get “credit” for class participation, you can still earn up to a ‘B’ by doing well on the exams in class and on the writing assignments. There will be two one-hour exams plus a comprehensive final. Exams will cover definitions, examples, and theorems we have done in class but will not necessarily be limited to material covered in class, although that material will make up the bulk of any in-class examination. At the regularly scheduled time for the final exam, there will be a final exam covering the entire course. If you work on the theorems and present some of the proofs in class, these exams will be easy for you (and will be almost unnecessary). In addition to examinations, writing will give you a second means of earning a grade. Several times during the term, after a proof has been presented, I will ask you to “write it up.” This written work will be turned in and I will read it. Normally, at the next class meeting you will get this written work back with either a check mark on it or “Rewrite” written on it. If you are asked to rewrite it, you should rework your proof and turn it in again for reading. This process may be repeated several times before you get a check mark and, therefore, credit for the assignment. This process affords you an excellent opportunity to learn how to communicate mathematics and should prepare you well for taking the examinations in addition to increasing your understanding of the course material.

0.3 Specific Comments on Selected Items

From the beginning I encourage the students to draw pictures on a line. I tell them we will assume without proof that the set of all real numbers satisfies our axiom system. Of course, Axioms 1 through 6 are not quite sufficient to characterize the set of real numbers since there are models for that axiom system that are not separable such as the lexicographic square and the long line (making sure not to have a first or last point in the model). This deficiency in the axiom system makes some of the proofs somewhat different from those found in textbooks covering the material from the notes but I find this actually to be an advantage of the notes.

The heart of this course consists of the first two chapters. Some semesters my classes barely got through a proof of the maximum value theorem. A couple of directions I have taken classes that got through this theorem with time to spare are included in the notes. This material is in the chapters, “Simple Graphs” and “Consequences of Axiom 7.” My inclination is
to do the material on Axiom 7 but the approach to functions taken by some classes lends itself naturally to the material on simple graphs.

Writing has become a very important part of the course I teach from these notes. I take up and read proofs of almost all of the numbered theorems. Normally, early in the course, there is an extensive rewriting effort required of the students. By the end of the term, most students can write without having to do a rewrite or at least not more than one rewrite. I initiated the writing component to address the question of what the students who do not participate in the work at the board get out of the course. The added benefits are not limited to that, however. For example, most students can write better exams for having the practice writing in the course.

I normally talk a good bit on the first day. The gist of what I say is found back in Section 0.2. On the first day, after handing out the first sheet of notes and giving the students a few minutes to look them over, you can begin by asking for solutions to Problem 2. Although Problem 2 is stated before the axioms, it should be answered in light of the upcoming axioms. Normally, the students will not give a preamble statement when they propose a solution. This allows you to ask such questions as “what is \( P \)” and “what is \( M \)” when they propose a solution. Sometimes, they will call it the rightmost point of \( M \) which allows a discussion of the use of “a” vs. “the”. At some point, ‘testing’ the definition should arise. A good place to do this is with a single-point point set. Other issues include that in saying “if \( x \) is a point of \( M \)” we mean that no point of \( M \) is to the left of \( P \). That is, when we say “if \( x \) is a point of \( M \)” we are saying something about all the points of \( M \). These discussions normally spill over into the second class meeting.

Opportunities abound in the notes for student discovery of theorems not explicitly stated in the notes. It is useful to know that if the point \( P \) is to the left of the point \( Q \) then \( Q \) is not to the left of \( P \). Students will use this without recognizing that it needs proof. When asked about why this is true, they usually quickly come up with a proof. The result can now be called a theorem and named after the student who proves it. Another opportunity for extracting a theorem from presentations will arise after the introduction of limit points. After they have proved once or twice as a matter of course in proving some theorems that if \( P \) is a limit point of \( H \) and \( H \) is a subset of \( M \) then \( P \) is a limit point of \( M \). This statement can be labeled as a theorem and it, too, can be named after the student who “discovers” it. We insert it into the theorem sequence with a number like \( x.1 \).

Axiom 1 is quite complicated. Its meaning should be discussed. The students should be asked to identify its hypothesis and its conclusion. Some
off hand remark should be made about how it is used but this should be done casually. The use of “or” in mathematics should be discussed. Discussion of the other axioms should highlight the fact the phrase “P and Q are points” implies P and Q are different. To allow that P and Q might be the same, we say “if each of P and Q is a point”.

In the notes, when I use the term “point set” I mean the set is not empty. You should make sure your students understand this or make adjustments in the notes accordingly.

I think the most elegant solution to Problem 2 is “P is a point of M and if x is a point of M then P is not to the left of x.” However, I think one should accept any correct solution offered by a student.

Problem 11 begins a critical process in the course: getting the students to negate statements. They have real problems with this, especially early on. This first problem of that sort allows highlighting that, in negating a statement involving “and”, it becomes a statement involving “or”. This is not a bad place to point out that negating a negation should return to the original statement. In order to help all students understand that denying a statement involving “and” produces a statement involving “or”, I use the following analogy: I select two pieces of chalk and put one in my right hand and one in my left. I ask if it is true that I have a piece of chalk in my left hand and I have a piece of chalk in my right hand. They all agree that it is. Then, I put one piece back on the chalk tray and repeat the question and ask why it is now false. I often move the chalk to the other hand and ask again if it is true. After putting down both pieces of chalk I ask if it is true that I have a piece of chalk in my left hand or I have a piece of chalk in my right hand. Some such analogy seems to help them understand negating “and” and “or” statements.

When students attempt a negation, I write their words (exactly as they say them) on the board or I have them go to the board and write. I add amendments as the discussion proceeds. Additional attempts are written leaving the old ones written on the board. At the end of the discussion, I put a check mark by the correct negation. I usually strike the incorrect ones or put an X by them.

Another analogy that is useful to get students to understand formal statements is the following: Ask each student to take a coin from their pocket or purse (some may not get a coin but this is okay). It helps if the instructor takes a coin. Then, ask them how they could decide whether the following statements are true. (1) If x is a person in this room then x has a coin in their hand. (2) There is a person in this room who does not have a coin in their
hand. You can begin the discussion of the truth of (1) by showing the coin in the instructor’s hand and asking if that is enough evidence to decide.

Classroom discussion probably has become commonplace by the time you ask for a proof of Theorem 6. At that point, it is a good idea to state some ground rules that can prevent a presentation of a proof from becoming a “class project” in which the class members give “help” to the student at the board. There are a number of things wrong with this, not the least of which is that the student at the board can lose his or her train of thought for their own proof in the face of suggestions from the class.

In proving Theorem 6, they need to use Axiom 1. The hypothesis of Axiom 1 has two parts. To illustrate that both conditions must be met to use Axiom 1, I sometimes use a “soda machine” analogy. In order to get a soft drink from a 50 cent machine (this analogy is getting out of date due to inflation, but when I began using these notes one could get a soft drink for 10 cents), one must insert two quarters. If both are not supplied, one will not get a drink.

In proving Theorem 6, there is a tendency for the students to use the phrase “the point $x$ is to the left of the set $M$” or “the set $X$ is to the left of the set $Y$.” You may not be able to get them to avoid this but I usually ask them to say precisely what they mean when they use such a phrase the first time, pointing out that it is not defined. Usually, I will suggest that we will have enough definitions in the course and it is better to avoid more.

In proofs, students will sometimes assume the conclusion of the theorem in the middle of an argument. If no one in the class objects, I always stop the argument at this point and observe there is nothing left to prove if they assume the conclusion. I will ask them if they can get along without the assumption and try to determine if they have some idea about how to prove the theorem.

After Theorem 6 is proved, you should mention that the analog to Theorem 6 “on the other side” also holds and we will assume this without additional proof.

In order to lead up to an inductive proof that finite point sets do not have limit points, I try to get the students to show that the point set in Question 13 does not have a limit point by using only the fact that a one-point point set does not have a limit point as established in Question 12. I hope that with some help they discover that if the point $P$ is not a limit point of the point set $H$ and $P$ is not a limit point of the point set $K$ then $P$ is not a limit point
of the point set $H \cup K$. This is yet another opportunity for them to discover and prove a theorem on their own.

Before the class begins working on Question 17, I usually announce that we will assume, without proof, that the set of real numbers with “to the left of” meaning “is less than” is a model for our axiom system. This allows them a familiar setting in which to describe an example answering Question 17 as well as pointing out that the theorems we prove hold for the reals. Of course, every model for the axiom system contains a set with only one limit point, and, if you prefer it, you can insist that they answer this question without resorting to the reals. However, I find it advantageous to allow them to look for examples within the model for the axiom system that inspired the system in the first place.

You must take care not to try to force your own method of proof of a theorem on the student at the board. On the other hand, forcing the student to be careful with their language may, initially, require some carefully posed leading questions.

As the course progresses, I try to get the students to see that to begin to understand a definition one must (1) know the statement of the definition, (2) be able to write a bare denial of the definition, (3) get some examples both of something satisfying the definition and of something not satisfying the definition and (4) prove some theorems with the term in the hypothesis as well as some with the term in the conclusion. All of these aspects are explored in examining the definition of limit point of a set, continuity, and the other definitions central to the course.

As a means of assisting students in understanding the use of definitions, I often mention the analogy of playing a game. For example, the limit point game as played by the definition goes this way. Someone claims that a certain point is a limit point of a set. If this were a game, their opponent would give them a segment containing the point. In order to win the game, they would have to be able to produce in the given segment a point of the set distinct from the point in question. They quickly see that a strategy for the opponent to win the game is to be able to produce a segment containing the point so that the first person cannot find a point of the set in it distinct from the point in question. Throughout the course, the game analogy can be recalled when the students are struggling with a definition or its negation.

When the students answer the question of whether there is a set with only one limit point, I usually follow this with the question of whether there is a point set with only two limit points.
After Theorem 19 has been proved, it is useful to discuss that the proof identifies a specific point that is a limit point of the set. There are times when it can be useful to identify that point as the one coming from the proof. I usually suggest that they add a sentence to the statement of Theorem 19: “Moreover, the rightmost of all the points to the left of every point of \( M \) is a limit point of \( M \).”

Induction is required to prove Theorem 21 and to answer Question 24. I try to have them thinking about both of these at about the same time. In Question 12 they are led to show that a one-point point set has no limit point. Question 13 gets that a two-point point set has no limit point. Usually they will do this on a case by case basis. Trying to get Theorem 21 done this way, of course, requires an inductive proof that a finite set can be ordered from left to right which leads naturally to Question 24. Question 24 often gets answered to the students’ satisfaction without a formal inductive argument by getting an infinite 'decreasing' sequence of points in any set without a leftmost point. Rather than forcing them to use induction to construct the sequence, I will often give them a proof by induction on the number of points in the set. I use as the Axiom of Induction that every set of positive integers has a least member for two reasons: (1) it is not the standard textbook form of the axiom and (2) every student believes it is true. I then suggest that they try to emulate the argument to prove Theorem 21. At some appropriate point when Theorem 21 and Question 24 are completed, I try to discuss the usual Axiom of Induction. Most of them have seen it somewhere and may now be less mystified by it. The course can get bogged down by induction. I avoid this at all costs. This includes, as I mention above, possibly giving them a proof by induction that finite sets have leftmost points.

As mentioned earlier, one of the most difficult things to do when a student is presenting an argument is to make sure your questions are not designed to lead them to a proof you have in mind. Rather, if you ask leading questions, they need to lead in the general direction the student is headed provided their direction is correct. Of course, if it is not correct, the questions need to lead to their seeing that the approach that they are using is not going to work.

A definition of “infinite” is not in the current notes. Perhaps it should be. However, I usually state that a point set is infinite provided it is not finite (or that a non-empty set is infinite provided it is not finite).

At some time in the semester, you may have no one ready to present anything. When this occurs, I usually do one of two things. I introduce something new that is coming up or, if I think the class may be slacking off, I may dismiss class early letting some displeasure show.
Regarding the material on functions, I normally get them started on functions by drawing pictures of two parallel lines with the domain of the function lying in one of the lines and the range in the other rather than using the traditional Cartesian product. This is certainly quite a different approach to functions from that of most of their earlier experience with functions in calculus. This separation allows them to think more easily (and perhaps more clearly) about the definition of continuity and certainly bridges the gap for drawing pictures in later courses such as complex variables or topology. In addition, I will draw "standard" pictures of the examples from time to time to be sure they make the connection with functions from their background.

At some point, you will get a student who, in proving a theorem, provides a number of extraneous statements that are not directly to the point. I mention that it is always good practice to keep in mind where one is going in a proof. After someone has successfully argued a theorem involving extraneous diversions, I often use the following analogy: When I was about twelve years old, my father bought a tractor. Although I was too young to drive a car, I knew I was not too young to drive a tractor. My father decided to let me drive it, but he believed I should be doing something useful while I was driving. So, he took me out to a two-acre field and told me to use a turning plow on the tractor to plow the field. He stressed the importance of plowing a straight furrow and told me to get started. I started out down the field trying ever so carefully to steer straight. When I got to the other end of the field and stopped, he told me to look back at my work. The furrow was very crooked! I told him I had tried very hard to plow straight. He told me that there is a secret to plowing a straight furrow. I should pick out a tree or a fence post at the far end of the field or beyond and I should drive toward it. When I did that going back, the furrow was straight. Keeping the conclusion of the theorem in mind allows you to plow straight for it from the hypothesis.

When I get to the question of the uncountability of the set of real numbers, I will give hints. First I try to determine whether someone has heard the answer or how they think it comes out. I also try to determine if anyone is working on the question and what they are trying. If no one is working on the question but the class thinks the set is uncountable, I suggest they suppose \( x_1, x_2, x_3, \ldots \) is a sequence such that the set of terms of the sequence is \([0, 1]\). I then put \( x_1 \) on a picture and ask if they can get an interval lying in \([0, 1]\) which does not contain \( x_1 \). Sometimes this leads them to discover a proof that depends only on Theorem 26. Such a proof was first shown to me by a student in a class taught from these notes many years ago at the University of Houston.

On the Four Questions problem, I usually accept an answer on \((0, 1)\) or...
[0, 1] for those questions for which an example provides an answer to the question. Students tend to think about this question in terms of sequences and it is impossible to do this in general with sequences due to the lack of separability. An answer is possible for any segment or interval without using sequences. For example, the collection $\mathcal{G}$ of all segments of the form $(x, B)$ for $x$ in $(A, B)$ covers the segment $(A, B)$ and no finite subcollection of $\mathcal{G}$ covers $(A, B)$. I normally tell them at least one question of the four has an answer of “yes” and at least one of “no.”

I normally lead into Question 41 with several questions beginning with something simple like: ‘Is there a closed set that is not an interval?’ or ‘Is there a closed point set that contains no interval?’

For Theorem 43, I normally have to give a hint or a sequence of hints leading to their defining the set of all points $x$ of $[A, B]$ so that $[A, x]$ can be covered by a finite subcollection of $\mathcal{G}$ so then their objective is to show that $B$ is in this set. I normally do this in a sequence of hints. First I draw $[A, B]$ and a segment in $\mathcal{G}$ containing $A$. I choose a point $x$ in that segment and in $(A, B)$ and state that $x$ is interesting. I then allow them a class meeting or two to think about this. If that does not get them going, I draw the same picture and indicate the interval $[A, x]$ and state that the interval $[A, x]$ is interesting. Later, if necessary, I define $M$ to be $\{x \in [A, B] \mid [A, x]$ can be covered by a finite subcollection of $\mathcal{G}\}$ and suggest that they try to prove $B$ is in $M$.

Be prepared to spend some time allowing them to get a formulation of the bare denial of ‘$f$ is continuous at $P$’. I think that those students who are successful in doing this on their own have really begun to understand how to negate a statement. I have them test their ideas against the examples in the notes. After they get this, I normally have them test their new understanding by commenting that they should write a bare denial of their bare denial to see if they get the original definition back.

Theorems 55 and 66 sometimes require hints. Sometimes this can be as simple as suggesting they try to use Theorem 43. Other classes require more hints. By the time you get to this portion of the course and beyond, you should know your students well enough and have sufficient experience in this method of teaching to decide on an appropriate course of action. Just resist the urge to prove the theorems for them. This not only spoils their fun, it can prevent them from making a personal break-through which could be all-important in their future study of mathematics. Besides, one beauty of a course like this is the lack of a necessity to cover material at all costs. Just be sure you make it clear to the class that you are always willing to listen to any proof of any theorem from the course at any time even after the course is over (and mean it!).

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I normally reproduce the notes for the classes. I do not hand them out in one package. Rather, I pass them out about one page at a time. A suggested breakup for the handouts is: (1) From the beginning up through Theorem 6, (2) From the end of handout 1 through Theorem 19, (3) From the end of handout 2 through Question 27, (4) From the end of handout 3 through Question 41, (5) From the end of handout 4 through Theorem 52, (6) Problem 53 and Theorems 54 and 55, (7) From the end of handout 6 through Problem 62, (8) From the end of handout 7 through Problem 70, (9) From the end of handout 8 through Theorem 80.

At some point give them the rest of the material (no course should end with nothing left to be done).
To the Student

This course will likely be different from any course you have taken. First of all, there is no textbook for the course. These notes will serve in place of a book. Instead of reading (and probably memorizing) the proofs that some author has presented, you will be expected to produce your own. If you have never had a course in which you were expected to produce your own proofs, do not worry because your instructor does not expect that you have. Instead, you are going to discover some mathematics for yourself. Along the way you will find out that, indeed, you can do this and you will get a taste of the personal satisfaction and the thrill of doing so. You do not have to bring a lot with you for this—mostly a willingness to work hard even in the face of frustration. Your instructor will set the rules for how you do this, but it is my opinion that you will get the most from your study if you do most of this on your own. My recommendations to my classes was to limit their discussions with other students in the class to the meaning of the statements of the definitions and theorems. Any work on proofs should be your own without assistance from outside sources—books, other students, or the internet. Of course, seeking guidance from your instructor when you are stuck is highly recommended, but do not be surprised if your questions are met with questions.

These notes attempt to create for you a world in which you work to discover insights and construct your proofs. In order to make this as transparent as possible, an attempt has been made to create for you a place where you clearly understand what you have to work with. Consequently, these notes define for you a mathematical system in which you will work. This system consists of two undefined terms, some defined terms, some assumptions about these terms (the axioms), and some statements of theorems that you are to prove. Accompanying the statements of theorems will be some questions and problems. Often these questions and problems can lead you to a deeper understanding of this system and even allow you to expand it through some theorems you find on your own.

Why do we begin with undefined terms? I am confident that you have gone to a dictionary to look up a word you do not know. Have you ever done this only to encounter in the definition another word you do not know?
Upon seeking a meaning for this word you encounter yet another word you do not know. This process may continue for a while when the latest word you seek has in its definition the term you began with. What you have encountered somewhere in this process is one of the “undefined” terms used by the compilers of the dictionary. A brief reflection on what happened leads us to realize that, when one defines words, some words must be assumed to be known to the users of the dictionary. In our system, there are two such terms that are outside the common, everyday use of the English language—point and the phrase to the left of. Intuitively, we are assuming that the elements of the set under consideration (our points) are strung out in some linear fashion so that, if two of them are considered, one of them appears to precede the other. All other terms in the notes are ultimately defined using these undefined terms. There are initially six axioms five of which cement the intuitive notion of to the left of. Ultimately, your proofs should rely only on the definitions of the terms and these six axioms although, of course, once a theorem has been proved its statement may be used in any argument you make for the truth of an ensuing theorem.

Sets underlie virtually all of mathematics and that is certainly the case for these notes. In the preceding paragraph that term was used even though it is neither a defined nor an undefined term. You will need to use some intuitive set theory as you proceed through the notes. If you think of a set as a collection of objects and you are able to form new sets such as the set of objects common to a collection of sets or the union of a collection of sets (collection is used synonymously with the term set to avoid awkward language like a set of sets), you should be able to get along just fine in the course. One anomaly that quickly arises in this “intuitive” set theory is whether there can be a set without any objects in it, a so-called empty set. If you talk to a child about sets in this intuitive way as a collection of objects, they might look quizzically at you if you then talk about an empty set. Although an empty set may be a convenience at times, throughout these notes the term “point set” will refer to a set that contains at least one point. As a consequence, if, in the proof of a theorem, you state that something is a point set, it will be incumbent on you to show that it contains at least one point in order to proceed talking about it.

Little words make a big difference and in these notes we use all words carefully. You know the difference between ‘a’ and ‘the’. To underscore the difference, consider the following: Your phone rings and you answer it. In a short period of time you realize the person on the line did not intend to call your number. Do you say to them, “You must have the wrong number”? Most people would, but do you mean to tell them that yours was the only number they did not intend to call? Perhaps, one should say, “You have a wrong number”. Language is our vehicle for communicating our ideas.
To be sure we are not misunderstood we must use our language carefully. Prepositions are important; the use of the plural is important; each word is important! Be careful.

Finally, we make a remark about models for our mathematical system. Our intuitive “model” for this system is, of course, the number line in the sense that we interpret ‘point’ to be ‘real number’ and ‘to the left of’ to be ‘is less than’. With these interpretations of our undefined terms, it is easy to believe that all of the axioms (with the possible exception of the first one) are satisfied. However, there are other, equally valid interpretations of the undefined terms for which our axiom system holds. One really nice feature of our approach is that any proof given that relies only on the axioms and definitions yields a theorem that holds true in any model of our system. Even though the set of real numbers model is convenient for drawing pictures and thinking about approaches to proofs, be sure you rely only on axioms and definitions and not on any special properties of the real numbers in constructing proofs (such as forming \((P + Q)/2\) to get a point between \(P\) and \(Q\) instead of citing Axiom 2.)

That is enough talk from me. It is time to begin this adventure. Enjoy!!
Chapter 1

BASIC PROPERTIES OF POINT SETS

Undefined terms: point, to the left of

Definition 1. Suppose $P$ is a point and $M$ is a point set. The statement that $P$ is a leftmost point of $M$ means $P$ is a point of $M$ and if $x$ is a point of $M$ then $x$ is not to the left of $P$.

Problem 2. Formulate a meaning for the statement that $P$ is a rightmost point of $M$. (Note that the phrase ‘to the right of’ is not defined. Try to formulate the meaning without using ‘to the right of’.)

We now assume that we have a set whose elements are called points and there is a meaning of the phrase ‘to the left of’ so that all six of the following axioms hold.

AXIOM 1. If $S_1$ and $S_2$ are point sets such that (1) if $x$ is a point then $x$ is in $S_1$ or $x$ is in $S_2$ and (2) if $x$ is a point of $S_1$ and $y$ is a point of $S_2$ then $x$ is to the left of $y$, then $S_1$ has a rightmost point or $S_2$ has a leftmost point.

AXIOM 2. If each of $P$ and $Q$ is a point and $P$ is to the left of $Q$ then there exists a point $x$ such that $P$ is to the left of $x$ and $x$ is to the left of $Q$.

AXIOM 3. If $P$ and $Q$ are points then $P$ is to the left of $Q$ or $Q$ is to the left of $P$.

AXIOM 4. If $P$, $Q$, and $R$ are points, $P$ is to the left of $Q$ and $Q$ is to the left of $R$, then $P$ is to the left of $R$.

AXIOM 5. If $P$ is a point then $P$ is not to the left of $P$.

AXIOM 6. If $P$ is a point then there exist points $Q$ and $R$ such that $Q$ is to the left of $P$ and $P$ is to the left of $R$. 
Question 3. Can a point set have two leftmost points?

Problem 4. Show that, under the hypothesis of Axiom 1, if \( S_1 \) has a rightmost point then \( S_2 \) does not have a leftmost point.

Problem 5. Complete the following statement: The point \( P \) is not the leftmost point of the point set \( M \) means ....
(This is called the bare denial (or negation) of the statement that \( P \) is the leftmost point of \( M \).)

Theorem 6. If \( M \) is a point set and \( B \) is a point to the left of every point of \( M \) then \( M \) has a leftmost point or there is a rightmost of all the points to the left of every point of \( M \).

Definition 7. If \( P \) and \( Q \) are points, the statement that \( P \) is to the right of \( Q \) means \( Q \) is to the left of \( P \).

Definition 8. If \( P \) and \( Q \) are points and \( R \) is a point, the statement that \( R \) is between \( P \) and \( Q \) means \( R \) is to the right of \( P \) and to the left of \( Q \) or \( R \) is to the right of \( Q \) and to the left of \( P \).

Definition 9. Suppose \( A \) and \( B \) are two points and \( A \) is to the left of \( B \). By the segment \( AB \), [denoted \( (A,B) \)], is meant the point set to which the point \( x \) belongs if and only if \( x \) is between \( A \) and \( B \). The statement that the point set \( s \) is a segment means there exist points \( A \) and \( B \) such that \( s \) is \( (A,B) \).

Definition 10. Suppose \( P \) is a point and \( M \) is a point set. The statement that \( P \) is a limit point of \( M \) means if \( s \) is a segment containing \( P \) then \( s \) contains a point of \( M \) distinct from \( P \).

Problem 11. Write the bare denial of the statement that the point \( P \) is a limit point of the point set \( M \).

Question 12. Suppose \( A \) is a point and \( M \) is a point set whose only member is \( A \). Is \( A \) a limit point of \( M \)? If \( x \) is to the left of \( A \), is \( x \) a limit point of \( M \)? If \( x \) is to the right of \( A \) is \( x \) a limit point of \( M \)? Does \( M \) have a limit point?

Question 13. Suppose \( A \) and \( B \) are two points and \( M \) is a point set whose only members are \( A \) and \( B \). Does \( M \) have a limit point?

Question 14. If \( M \) is a segment, does \( M \) have a limit point?

Definition 15. A point set \( M \) is called an interval provided there exist two points \( A \) and \( B \) with \( A \) to the left of \( B \) and such that \( x \) belongs to \( M \) if and only if \( x \) is in the segment \( (A,B) \) or \( x \) is \( A \) or \( x \) is \( B \). In this case \( M \) is called the interval \( AB \) and is denoted \( [A,B] \).

Question 16. If \( M \) is an interval, does \( M \) have a limit point?

Question 17. Is there a point set with only one limit point?
Problem 18. Show that if each of $s_1$ and $s_2$ is a segment containing the point $P$, then there is a segment $s$ containing $P$ such that $s$ is a subset of $s_1$ and $s$ is a subset of $s_2$.

Theorem 19. If $M$ is a point set without a leftmost point and there is a point to the left of every point of $M$, then $M$ has a limit point.

Definition 20. The statement that a set $M$ is finite means there is a positive integer $n$ such that $M$ contains only $n$ elements.

We will use without proof the following property of the set of positive integers:

If $K$ is a set of positive integers, $K$ has a least element.

Theorem 21. If $M$ is a finite point set then $M$ does not have a limit point.

Question 22. Is there an infinite point set that does not have a limit point?

Theorem 23. If $M$ is an infinite subset of a segment then $M$ has a limit point.

Question 24. Does each finite point set have a leftmost point?

Question 25. Suppose $s_1$, $s_2$, $s_3$, . . . is a sequence of segments such that $s_2$ is a subset of $s_1$, $s_3$ is a subset of $s_2$, . . . . Is there a point that belongs to every term of the sequence $s_1$, $s_2$, $s_3$, . . . ?

Theorem 26. Suppose $I_1$, $I_2$, $I_3$, . . . is a sequence of intervals such that $I_{n+1}$ is a subset of $I_n$ for each positive integer $n$. Then there is a point that belongs to every interval in the sequence $I_1$, $I_2$, $I_3$, . . . . Further, if the common part of all the intervals in the sequence $I_1$, $I_2$, $I_3$, . . . contains two points, the common part is an interval. Moreover, if $C$ is the common part and $s$ is a segment containing $C$, then there is a positive integer $n$ such that $I_n$ is a subset of $s$.

Question 27. Is there a sequence $r_1$, $r_2$, $r_3$, . . . of numbers such that $\{r_1, r_2, r_3, \ldots\}$ is the set of rational numbers in $[0, 1]$?

Question 28. Is there a sequence $x_1$, $x_2$, $x_3$, . . . of points of the interval $[A, B]$ such that $\{x_1, x_2, x_3, \ldots\}$ is $[A, B]$?

Definition 29. The statement that the point sets $H$ and $K$ are mutually exclusive means no point belongs to both $H$ and $K$.

Definition 30. The statement that $H$ and $K$ are mutually separated means $H$ and $K$ are mutually exclusive and neither contains a limit point of the other.

Question 31. Is the interval $[A, B]$ the union of two mutually separated point sets?
Notation: If \( M \) is a point set, \( \overline{M} \) denotes the set to which the point \( P \) belongs if and only if \( P \) is a point of \( M \) or \( P \) is a limit point of \( M \).

**Definition 32.** The statement that the point set \( M \) is **closed** means if \( x \) is a limit point of \( M \) then \( x \) is a point of \( M \).

**Problem 33.** If the point set \( A \) is a subset of the point set \( B \) then \( \overline{A} \) is a subset of \( \overline{B} \).

**Problem 34.** If \( M \) is a point set, \( \overline{\overline{M}} = M \).

**Problem 35.** If \( M \) is a point set, \( M \) is closed.

**Problem 36.** If each of \( H \) and \( K \) is a point set, \( \overline{H \cup K} = \overline{H} \cup \overline{K} \).

**Question 37.** Is every finite point set closed?

**Question 38.** Can \( \cup \) in Problem 36 be replaced by \( \cap \)?

**Definition 39.** The statement that the collection \( \mathcal{G} \) of sets **covers** the set \( M \) means if \( x \) is in \( M \) then some element of \( \mathcal{G} \) contains \( x \).

**Question 40.** (Four Questions)

a. Does there exist a collection \( \mathcal{G} \) of segments covering the segment \([A, B]\) such that if \( \mathcal{H} \) is a finite subcollection of \( \mathcal{G} \) then \( \mathcal{H} \) does not cover \([A, B]\)?

b. Does there exist a collection \( \mathcal{G} \) of intervals covering the segment \([A, B]\) such that if \( \mathcal{H} \) is a finite subcollection of \( \mathcal{G} \) then \( \mathcal{H} \) does not cover \([A, B]\)?

c. Does there exist a collection \( \mathcal{G} \) of segments covering the interval \([A, B]\) such that if \( \mathcal{H} \) is a finite subcollection of \( \mathcal{G} \) then \( \mathcal{H} \) does not cover \([A, B]\)?

d. Does there exist a collection \( \mathcal{G} \) of intervals covering the interval \([A, B]\) such that if \( \mathcal{H} \) is a finite subcollection of \( \mathcal{G} \) then \( \mathcal{H} \) does not cover \([A, B]\)?

**Question 41.** Does there exist a closed point set \( M \) such that every point of \( M \) is a limit point of \( M \) and \( M \) contains no interval?

**Theorem 42.** The interval \([A, B]\) is not the union of two mutually separated point sets.

**Theorem 43.** If \( \mathcal{G} \) is a collection of segments covering the interval \([A, B]\), then there is a finite subcollection \( \mathcal{H} \) of \( \mathcal{G} \) that covers \([A, B]\).

**Question 44.** Does Theorem 43 remain true if the interval \([A, B]\) is replaced by a closed subset of \([A, B]\)?
Question 45. Let $M$ be a point set with the property that if $\mathcal{G}$ is a collection of segments covering $M$ then some finite subcollection of $\mathcal{G}$ covers $M$. Is it true that $M$ is closed?
Chapter 2

FUNCTIONS

Definition 46. A function is a set of ordered pairs such that no two pairs in the set have the same first term. If \( f \) is a function and \( M \) is the set of first terms of pairs in \( f \), then \( f \) is said to be a function defined on \( M \).

Notation: If \( f \) is a function and the pair \((x, y)\) is in \( f \), we sometimes write \( y = f(x) \).

Some Examples

1: \( f_1 \) is the set to which the pair \((x, y)\) of points belongs if and only if \( x \) is \( y \).

2: Suppose \( P \) is a point and \( Q_1 \) and \( Q_2 \) are two points. Denote by \( f_2 \) the set to which the pair \((x, y)\) of points belongs if and only if it is true that if \( x \) is to the left of \( P \) then \( y \) is \( Q_1 \) and \( y \) is \( Q_2 \) otherwise.

3: Suppose \( P \) is a point. Denote by \( f_3 \) the set to which the pair \((x, y)\) of points belongs if and only if \( y \) is \( P \).

Problem 47. Show that each of \( f_1, f_2, \) and \( f_3 \) is a function.

Definition 48. Suppose \( f \) is a function defined on the point set \( M \), \( P \) is a point of \( M \) and the set of second terms of pairs in \( f \) is a point set. The statement that \( f \) is continuous at \( P \) means if \( t \) is a segment containing \( f(P) \) then there is a segment \( s \) containing \( P \) such that if \( x \) is a point of \( M \) in \( s \) then \( f(x) \) is in \( t \).

Problem 49. Check the definition of \( f \) is continuous at \( P \) for each of the examples \( f_1, f_2, \) and \( f_3 \) at several points.

Problem 50. Write the bare denial of the definition of \( f \) is continuous at \( P \).
Theorem 51. Suppose $M$ is a point set, $P$ is a point of $M$ and $P$ is not a limit point of $M$. If $f$ is a function defined on $M$, then $f$ is continuous at $P$.

Theorem 52. Suppose $M$ is a point set, $P$ is a point of $M$ and $f$ is a function defined on $M$. If $f$ is continuous at $P$ and $Q$ is a point to the left of $f(P)$, then there is a segment $s$ containing $P$ such that if $x$ is a point of $M$ in $s$ then $Q$ is to the left of $f(x)$.

Problem 53. Suppose $f$ is a function defined on the interval $[A, B]$ such that if $P$ is a point of $[A, B]$ then $f$ is continuous at $P$. If $Q$ is a point, let $M_Q$ denote the set to which the point $x$ of $[A, B]$ belongs if and only if $f(x)$ is not to the left of $Q$. Show that for each point $Q$ such that $Q$ is to the left of $f(t)$ for some point $t$ of $[A, B]$, the set $M_Q$ is a closed point set.

Theorem 54. Suppose $f$ is a function defined on the interval $[A, B]$, $f(A)$ is not $f(B)$, and if $P$ is a point of $[A, B]$ then $f$ is continuous at $P$. If $Q$ is a point between $f(A)$ and $f(B)$ then there is a point $C$ between $A$ and $B$ such that $f(C)$ is $Q$.

Theorem 55. Suppose $f$ is a function defined on the interval $[A, B]$ such that if $P$ is a point of $[A, B]$ then $f$ is continuous at $P$. Then there exist points $C$ and $D$ such that if $x$ is a point of $[A, B]$, then $f(x)$ is in the segment $(C, D)$.

Definition 56. Suppose $f$ is a function defined on the point set $M$, $P$ is a limit point of $M$ and $L$ is a point. The statement that $f$ has limit $L$ at $P$ means if $t$ is a segment containing $L$ then there is a segment $s$ containing $P$ such that if $x$ is a point of $M$ in $s$ and $x$ is distinct from $P$ then $f(x)$ is in $t$.

Theorem 57. Suppose $M$ is a point set, $f$ is a function defined on $M$ and $P$ is a limit point of $M$. Then $f$ is continuous at $P$ if and only if $f$ has limit $f(P)$ at $P$.

Theorem 58. Suppose $M$ is a point set, $f$ is a function defined on $M$ and $P$ is a limit point of $M$ and $K$ and $L$ are points. If $f$ has limit $L$ at $P$ then $f$ does not have limit $K$ at $P$.

Definition 59. A collection $\mathcal{G}$ of point sets is called monotonic provided it is true that if $g$ and $h$ are in $\mathcal{G}$ then $h$ is a subset of $g$ or $g$ is a subset of $h$.

Problem 60. Show that if $\mathcal{G}$ is a monotonic collection of closed subsets of the interval $[A, B]$, then there is a point that belongs to every set in $\mathcal{G}$.

Definition 61. The statement that the point set $M$ has the finite covering property means if $\mathcal{G}$ is a collection of segments covering $M$ then there is a finite subcollection $\mathcal{H}$ of $\mathcal{G}$ that covers $M$.

Problem 62. Show that if the point set $M$ has the finite covering property then $M$ is closed.

Problem 63. Show that if $M$ is a closed subset of the interval $[A, B]$ then $M$ has the finite covering property.
Definition 64. If \( f \) is a function defined on the point set \( M \), denote by \( f[M] \) the set of all second terms of pairs in \( f \).

Definition 65. If \( f \) is a function defined on the point set \( M \) and \( f \) is continuous at each point of \( M \), we say \( f \) is continuous on \( M \).

Theorem 66. If \( f \) is continuous on the interval \( M = [A,B] \) and \( f[M] \) contains two points, then \( f[M] \) is an interval.

Problem 67. Show that if \( P \) and \( Q \) are points, there exist mutually exclusive segments \( s_P \) and \( s_Q \) containing \( P \) and \( Q \), respectively.

Definition 68. A set is said to be open if it is the union of a collection of segments.

Problem 69. Show that if \( P \) is a point and \( K \) is a closed set not containing \( P \), then there exist mutually exclusive open sets \( O_P \) and \( O_K \) containing \( P \) and \( K \), respectively.

Problem 70. Show that if \( H \) and \( K \) are mutually exclusive closed point sets, then there exist mutually exclusive open sets \( O_H \) and \( O_K \) containing \( H \) and \( K \), respectively.

Problem 71. Show that if \( H \) and \( K \) are mutually separated point sets, then there exist mutually exclusive open sets \( O_H \) and \( O_K \) containing \( H \) and \( K \), respectively.
Chapter 3

AXIOM 7 AND ITS CONSEQUENCES

Definition 72. A sequence is a function defined on the set of positive integers.

If $r$ is a sequence and $i$ is a positive integer, we often denote $r(i)$ by $r_i$ (as we have been doing).

AXIOM 7 There exists a sequence of points $r_1, r_2, r_3, \ldots$ such that if $A$ and $B$ are points then there is a positive integer $i$ such that $r_i$ is between $A$ and $B$.

Problem 73. Show that if $A$ and $B$ are points there is a point $x$ between $A$ and $B$ such that $x$ does not belong to \{r_1, r_2, r_3, \ldots\}.

Theorem 74. If $x$ is a point then $x$ is a limit point of \{r_1, r_2, r_3, \ldots\}.

Theorem 75. If $P$ is a point, there is a sequence $y_1, y_2, y_3, \ldots$ of points of \{r_1, r_2, r_3, \ldots\} such that $y_1$ is to the left of $y_2$, $y_2$ is to the left of $y_3$, and $P$ is a limit point of \{y_1, y_2, y_3, \ldots\}.

Theorem 76. If $P$ is a point, there is a sequence $s_1, s_2, s_3, \ldots$ of segments containing $P$ such that $s_2$ is a subset of $s_1$, $s_3$ is a subset of $s_2$, and $P$ is the only point common to $s_1, s_2, s_3, \ldots$.

Theorem 77. There exists a sequence $s_1, s_2, s_3, \ldots$ of segments such that if $s$ is a segment and $P$ is a point of $s$ then there is a segment $s_i$ in the sequence $s_1, s_2, s_3, \ldots$ that contains $P$ and is a subset of $s$.

Theorem 78. Suppose $P$ is a point, $M$ is a point set and $s_1, s_2, s_3, \ldots$ is a sequence of segments as in Theorem 77. Then, $P$ is a limit point of $M$ if and only if it is true that if $s_i$ is a segment in the sequence $s_1, s_2, s_3, \ldots$ and $s_i$ contains $P$ then $s_i$ contains a point of $M$ distinct from $P$. 
Definition 79. The statement that a set $X$ is countable means there is a sequence $x_1, x_2, x_3, \ldots$ such that $X = \{x_1, x_2, x_3, \ldots\}$. If a set is not countable we say that it is uncountable.

Theorem 80. If $M$ is an uncountable point set then some point of $M$ is a limit point of $M$.

Definition 81. Suppose $x_1, x_2, x_3, \ldots$ is a sequence of points and $x$ is a point. The statement that $x_1, x_2, x_3, \ldots$ converges to $x$ means if $s$ is a segment containing $x$ there is a positive integer $N$ such that if $n \geq N$ then $x_n$ is in $s$.

Problem 82. Guess what. Yes, write a bare denial for Definition 81.

Theorem 83. Suppose $x_1, x_2, x_3, \ldots$ is a sequence of points that converges to the point $x$ and $y$ is a point different from $x$. Then, $x_1, x_2, x_3, \ldots$ does not converge to $y$.

Theorem 84. Suppose the sequence $x_1, x_2, x_3, \ldots$ converges to the point $x$ and the sequence $y_1, y_2, y_3, \ldots$ converges to $x$. If $z_1, z_2, z_3, \ldots$ is a sequence of points such that, for each positive integer $i$, $z_i$ is between $x_i$ and $y_i$, then $z_1, z_2, z_3, \ldots$ converges to $x$.

Definition 85. If $x_1, x_2, x_3, \ldots$ is a sequence, the statement that $x_1, x_2, x_3, \ldots$ converges means there is a point $x$ such that $x_1, x_2, x_3, \ldots$ converges to $x$.

Question 86. If $x$ is a point and $x_1, x_2, x_3, \ldots$ is a sequence of points such that, for each positive integer $i$, $x_i = x$, does the sequence $x_1, x_2, x_3, \ldots$ converge?

Theorem 87. Suppose $x_1, x_2, x_3, \ldots$ is a sequence such that, for each positive integer $i$, $x_i$ is to the left of $x_{i+1}$. If there is a point $B$ such that, for each positive integer $i$, $x_i$ is to the left of $B$ then $x_1, x_2, x_3, \ldots$ converges.

Theorem 88. If the point $P$ is a limit point of the point set $M$ then there is a sequence of points of $M$ that converges to $P$.

Theorem 89. If $M$ is a point set, the point $P$ belongs to $\overline{M}$ if and only if there is a sequence of points of $M$ that converges to $P$.

Theorem 90. Suppose $M$ is a point set, $f$ is a function defined on $M$ and $P$ is a point of $M$. Then, $f$ is continuous at $P$ if and only if it is true that if $x_1, x_2, x_3, \ldots$ is a sequence of points of $M$ that converges to $P$ then $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(P)$.

Theorem 91. Suppose $[A, B]$ is an interval and $f$ is a function defined on $[A, B]$. Then, $f$ is continuous on $[A, B]$ if and only if for each subset $M$ of $[A, B]$, $f[M]$ is a subset of $f[M]$. 

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Chapter 4

SIMPLE GRAPHS

For the remainder of the course, the over-riding assumption is that \( S \) is the set of real numbers. We will use the word “point” in two different senses. Most of the time, point will mean “ordered number pair”. However, at times (and it should be clear from context) point and real number may be used synonymously.

**Definition 92.** A simple graph \( f \) is a set of points such that if \( h \) is a vertical line that intersects \( f \) then \( h \) contains only one point of \( f \).

**Definition 93.** The statement that the simple graph \( f \) has property \( U \) at the point \( P \) of \( f \) means if \( l \) is a horizontal line with \( P \) below it, there exist vertical lines \( h \) and \( k \) with \( P \) between them such that if \( Q \) is a point of \( f \) between \( h \) and \( k \) then \( Q \) is below \( l \).

**Definition 94.** If \( f \) is a simple graph, by the \textit{x-projection} of \( f \) is meant the set of first terms of pairs in \( f \).

**Theorem 95.** Suppose \( f \) is a simple graph with \textit{x-projection} an interval and \( f \) has property \( U \) at each point. If \( l \) is a horizontal line and \( M = \{ z \mid z \text{ is the } x\text{-projection of a point of } f \text{ that is on or above } l \} \) then \( M \) is closed.

**Definition 96.** If \( f \) is a simple graph, the statement that the point \( P \) of \( f \) is a \textit{high point} of \( f \) means if \( Q \) is a point of \( f \) then \( Q \) is not above the horizontal line passing through \( P \).

**Theorem 97.** If \( f \) is a simple graph with \textit{x-projection} an interval and \( f \) has property \( U \) at each point then \( f \) has a high point.

**Definition 98.** The statement that the simple graph \( f \) has property \( L \) at \( P \) means if \( l \) is a horizontal line with \( P \) above it, then there exist vertical lines \( h \) and \( k \) with \( P \) between them such that if \( Q \) is a point of \( f \) between \( h \) and \( k \) then \( Q \) is above \( l \).

**Problem 99.** State and prove theorems analogous to the two stated above for simple graphs with property \( L \).
**Theorem 100.** Suppose $f$ is a simple graph and let $P$ be a point of $f$. Then $f$ has both property $U$ and property $L$ at $P$ if and only if $f$ is continuous at the $x$-projection of $P$.

**Definition 101.** Suppose $f$ is a simple graph and $P$ is a point of $f$ such that the $x$-projection of $P$ is a limit point of the $x$-projection of $f$. The statement that $f$ has **slope** $m$ at $P$ means if $A$ is a line containing $P$ with slope greater than $m$ and $B$ is a line containing $P$ with slope less than $m$ then there exist vertical lines $h$ and $k$ with $P$ between them such that if $Q$ is a point of $f$ distinct from $P$ between $h$ and $k$ then $Q$ is between $A$ and $B$.

**Problem 102.** Show that if $f$ has slope $m_1$ at $P$ and $f$ has slope $m_2$ at $P$ then $m_1 = m_2$.

**Definition 103.** If $f$ has slope $m$ at $P$ and $l$ is the line through $P$ with slope $m$, then $l$ is called the **tangent line** to $f$ at $P$.

**Theorem 104.** Suppose $f$ is a simple graph with slope $m$ at $P$. If $l$ is a line containing $P$ such that no point of $f$ is above $l$ then $l$ is the tangent line to $f$ at $P$.

**Theorem 105.** If the simple graph $f$ has slope $m$ at $P$ then $f$ has property $U$ and property $L$ at $P$. 

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