A Moore method calculus II course

William S. Mahavier

Emory University
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Chapter 1

The first problems

**Definition 1.** If \(a\) and \(b\) are numbers and \(a < b\), then by the **interval** from \(a\) to \(b\) is meant the set of all numbers \(x\) such that \(a \leq x \leq b\).

**Notation 1.** If \(a\) and \(b\) are numbers and \(a < b\) then we will use \([a, b]\) to represent the interval from \(a\) to \(b\). Also if we say to let \([a, b]\) be an interval, we will mean that \(a\) is a number, \(b\) is a number, and \(a < b\).

You are probably used to writing the integral from \(a\) to \(b\) of the function \(h\) as \(\int_a^b h(x)\,dx\). We see no need for the \(x\) or the \(dx\) and will write this integral as \(\int_a^b h\).

You surely think of \(\int_a^b h\) as the area between the graph of \(h\) and the \(x\)−axis between the vertical lines through the points \((a, 0)\) and \((b, 0)\). We will also think of the integral as being this area.

We study Riemann sums because they give us a way of approximating an integral. Indeed, we will define an integral in terms of Riemann sums. Geometrically a Riemann sum is just the sum of the areas of a collection of non-overlapping rectangles. In the case of \(h\) to get an upper Riemann sum for \(\int_1^2 h\) we divide the interval \([1, 2]\) into non-overlapping intervals and for each of these intervals we form a rectangle whose base is that interval and whose height is the biggest value of \(h(x)\) for \(x\) in that interval. Since \(h\) is a decreasing graph, this would be at the left endpoint of the interval. The upper Riemann sum is the total area enclosed by these rectangles. A lower Riemann sum is gotten in the same way except we use the smallest value of \(h(x)\) for \(x\) in a given interval for the height. For \(h\) this would be at the left endpoint. Later we will define these sums analytically as you probably did in your AP calculus. Geometrically it should be clear that each lower Riemann sum for \(h\) on \([a, b]\) is less than \(\int_a^b h\) and each upper Riemann sum is larger than \(\int_a^b h\).

The simplest type of Riemann sums are those with only two partition points so that they determine only one rectangle. Explain why the values of the upper and lower Riemann sums for \(\int_1^2 h\), on a partition with only 2 points are 1 and 1/2. Later we will see that this integral gives us the natural logarithm function. At this point we do not have a definition for \(\int_1^2 h\), but thinking of the integral as an area explain why \(\frac{1}{2} < \int_1^2 h < 1\).

Explain why \(\int_1^2 h < \frac{1}{2}\). Do you see that this is because the graph of \(h\) is concave up?

**Problem 1.** Find a partition \(P = x_0, x_1, x_2\) of \([1, 2]\) such that the lower Riemann sum for \(\int_1^2 h\) on \(P\) is larger than any other lower Riemann sum for \(\int_1^2 h\) on a partition with only 3
The first problems

partition points.

**Problem 2.** Find a partition \( P = x_0, x_1, x_2 \) of \([1, 2]\) such that the upper Riemann sum for \( \int_1^2 h \) on \( P \) is smaller than any other upper Riemann sum for \( \int_1^2 h \) on a partition with only 3 partition points.

Note that these Riemann sums give the best upper and lower Riemann approximations for \( \int_1^2 h \) with only 3 partition points.

The next problem has two parts which are very much alike. In general for problems like this one I will have different students present each part. And if you decide to write up one of these problems, you do not have to do both parts.

**Problem 3.** Let \( f \) be the function such that \( f(x) = x \) for each number \( x \) such that \( 1 \leq x \leq 2 \). Find the best upper and lower approximations for \( \int_1^2 f \) with only 3 partition points.

**Problem 4.** Let \( f \) be the function such that \( f(x) = x^2 \) for each number \( x \) such that \( 1 \leq x \leq 2 \). Find the best upper and lower approximations for \( \int_1^2 f \) with only 3 partition points.
Chapter 2

The natural logarithm function

Definition 2. If $[a, b]$ is an interval then $P$ is called a partition of the interval $[a, b]$ if there is a positive integer $n$ such that $P$ is a finite increasing sequence of $n$ numbers with $a$ the first number and $b$ the last.

Notation 2. If $n$ is any positive integer, we will usually denote a partition $P$ having $n + 1$ points by $x_0, x_1, x_2, ..., x_n$ where $x_0 = a$, $x_n = b$, and if $j$ is an integer from 1 to $n$, then $x_{j-1} < x_j$. We often think of $P$ as a way of dividing the interval $[a, b]$ into the $n$ non-overlapping intervals $[x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$.

Review Previously we noted that the upper and lower Riemann sums for $\int_a^b h$ with only two partition points were $\frac{1}{2}$ and 1 so that we had $.5 < \int_a^b h < 1$. This was because the upper Riemann sum came from one rectangle which enclosed the area represented by $\int_a^b h$ and the lower Riemann sum came from one rectangle which was enclosed by this area.

Then we noted that there was a trapezoid of area $\frac{1}{4}$ which enclosed this area. You probably had a formula for the trapezoid rule which would give this value. So now we have that $.5 < \int_a^b h < .75$.

The word calculus comes from the same stem as calculate and that is what we will be doing most of this semester. Assuming for these calculations that $\ln 2 = \int_1^2 h$, then we could average our two approximations $.75$ and $.5$ to get $.625$ as an approximation for $\ln 2$ with an accuracy of $\pm .125$. Note that it is of little use to say that $\ln 2$ is approximately $.625$ unless you specify the accuracy of the approximation. It is sometimes very hard to determine the accuracy of a given approximation. Your calculator will indicate $\ln 2 = .69$ to two decimal places so this fits with our data.

Here is our definition of a Riemann sum. As you read this definition, let $n = 3$, draw a graph of $h$ and represent the lower Riemann sum for $\int_a^b h$ as the sum of the areas of 3 rectangles.

Definition 3. A lower Riemann sum for $\int_a^b h$ is a number obtained as follows. First select an integer $n$ and a partition $P = x_0, x_1, x_2, \cdots, x_n$ of an interval $[a, b]$ consisting of $n + 1$ points. Then for each positive integer $i$, $0 \leq i < n$, form a rectangle whose base is the interval $[x_i, x_{i+1}]$ and whose height is the smallest value of the function $h$ in the interval $[x_i, x_{i+1}]$. Then form the sum of all the rectangles. That number is a lower Riemann sum. To get an upper Riemann sum do exactly the same except that the height of each rectangle
The natural logarithm function should be the largest value of the function $h$ in the appropriate interval. For the function $h$ this would be at the left endpoint so you could use your formulas from your AP calculus to check that your answers are correct. But you should be able to explain at the board how to get these formulas from the sum of the rectangles described above.

**Problem 5.** Find the best lower Riemann approximation for $\int_{1}^{2} h$ with 4 partition points and 3 rectangles.

For many of the following problems, you are asked to do something for an integer $n$, and you may not see how to do this. In that case, work the problem for $n = 2$, then $n = 3$ and look for a pattern and see if you can guess an answer for $n = 4$ or for any choice of $n$. We can discuss in class how to do it for any choice of $n$.

**Problem 6.** Let $n$ be a positive integer. Find a formula for the lower Riemann sum for $\int_{1}^{2} h$ using $n$ rectangles of equal width and $n + 1$ partition points. If you do not know how to get a formula that works for any choice of $n$, then do it for $n = 2$, then $n = 3$ and see if you can guess a formula for $n = 4$. Then check it for $n = 4$, and if it is correct then try to guess one that works for any choice of $n$.

**Problem 7.** Do the same as in Problem 6 except do it for the upper Riemann sums.

**Problem 8.** Let $n$ be an arbitrary positive integer. Find a formula for the difference between the upper and lower Riemann sums for $\int_{1}^{2} h$ with $n + 1$ equally spaced partition points. Note that this will tell us how good (or bad) our approximation might be.

**Problem 9.** Let $n$ be any positive integer. Show that the lower Riemann sum for $\int_{1}^{2} h$ on a partition with $n$ equally spaced partition points is smaller than the lower Riemann sum for $\int_{1}^{2} h$ on a partition with $n + 1$ equally spaced partition points.

**Problem 10.** Find the best trapezoidal approximation for $\int_{1}^{2} h$ where the trapezoid lies on or under the graph of $h$, its base is the interval $[1, 2]$ and the top is tangent to the graph $h$.

**Problem 11.** Let $a$ and $b$ be numbers with $0 < a < b$. Find a formula for the lower Riemann sum for $\int_{a}^{b} h$ using $n$ equal width rectangles.

**Problem 12.** Let $a$ and $b$ be numbers with $0 < a < b$, and let $c$ be a positive number. Show that the lower Riemann sum for $\int_{ca}^{cb} h$ using $n$ equal width rectangles is the same as that for $\int_{ca}^{b} h$ using the same number of rectangles from the previous problem.

If you don’t see how to do the preceding problem, then find the lower Riemann approximation for $\int_{1}^{2} h$ using $n$ equal width rectangles and show that it is the same as that from Problem 6. Then do the same for $\int_{a}^{b} h$ and see if this helps on the original problem.

**Problem 13.** Let $a$ and $b$ be numbers with $0 < a < b$ and let $c$ be a positive number. Show that the upper Riemann approximation for $\int_{a}^{b} h$ with $n$ equal width rectangles is the same as the upper Riemann approximation for $\int_{ca}^{cb} h$ with $n$ equal width rectangles.

The following definition should have been given earlier when we started talking about Riemann approximations, but we did not need it. Now we do.
The natural logarithm function

Definition 4. If a and b are two numbers with $0 < a < b$ then the integral from a to b of $h$, written $\int_a^b h$, is the only number which is larger than each lower Riemann approximation for $\int_a^b h$ with equally spaced partition points and also smaller than each upper Riemann approximation for $\int_a^b h$ with equally spaced partition points.

I hope that you would agree that if $\int_a^b h$ is to represent the area between $h$ and the x-axis between $a$ and $b$, then it should be larger than each of our lower Riemann approximations and smaller than each of the upper Riemann approximations. We could actually prove that there is only one such number but I don’t see much point in doing that at this time. I am more interested in showing how to use these approximations to learn something about the integral.

Now for the problem I have been aiming for. There is really not anything to prove in Problem 14 as it simply follows from the Definition 4 and the preceding 3 problems.

Problem 14. If a, b, and c are positive numbers and $a < b$, then $\int_a^b h = \int_c^b h$.

Now we want to take care of the case where $a = b$ and $a > b$.

Definition 5. Assume that each of a and b is a positive number and $a > b$. Define $\int_a^a h = 0$, and $\int_b^a h = - \int_a^b h$.

We will need one more property of integrals which you surely were told about in high school and we could prove but we will just accept and use it. I hope it is clear that this should be true and I will not ask you to do the following problem. We call it a theorem because it is true and we could show it, but it would take too much time. So we will just accept it and use it as needed.

Theorem 1. If a, b, and c are numbers and $a < b < c$, then $\int_a^b h + \int_b^c h = \int_a^c h$.

In case $a < b < c$ it should be clear that Theorem 1 should be true if we think of the integrals as areas. But it works regardless of how $a$, $b$, and $c$ are related. You should be able to use Problem 1 and Definition 5 to do the next problem.

Problem 15. If a, b, and c are positive numbers and $a < c < b$, show that $\int_a^b h + \int_b^c h = \int_a^c h$.

There are lots of other cases to consider but they can all be done in much the same way as the preceding problem. So we will just accept and use the fact that Problem 1 works regardless of what number is represented by each of $a$, $b$, and $c$. They do not even have to be different numbers. The following problem states that fact for reference purposes. I will not ask anyone to present this one.

Problem 16. If each of $a$, $b$, and $c$ is a positive number then $\int_a^b h + \int_b^c h = \int_a^c h$.

Problem 17. Show that the 2 rectangles in Problem 1 have the same area and that the 3 rectangles from Problem 5 have the same area.

We have guessed in class that the best lower Riemann approximation for $\int_1^2 h$ with 4 rectangles is given by using the partition $(1, 2^{\frac{1}{4}}, 4^{\frac{1}{4}}, 8^{\frac{1}{4}}, 2)$. So if $n$ is a positive integer and $n > 1$, then we would probably guess that the best lower Riemann approximation with $n$ rectangles would be given by using the partition $(1, 2^{\frac{1}{n}}, 2^{\frac{2}{n}}, \cdots, 2^{\frac{n-1}{n}}, 2)$. 

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Problem 18. Show that if $n$ is a positive integer, then in the lower Riemann approximation for $\int_1^2 h$ using the partition $(1, 2^{1/n}, 2^{2/n}, \cdots, 2^{(n-1)/n}, 2)$, the rectangles all have the same size.

Problem 19. Show that if $n$ is a positive integer then in the upper Riemann approximation for $\int_1^2 h$ using the partition $(1, 2^{1/n}, 2^{2/n}, \cdots, 2^{(n-1)/n}, 2)$, the rectangles all have the same size.

Problem 20. Use the results of the preceding problem to show that if $n$ is a positive integer and $n > 1$ then the average of the upper and lower approximations for $\int_1^2 h$ using the partition $(1, 2^{1/n}, 2^{2/n}, \cdots, 2^{(n-1)/n}, 2)$ is $\frac{n}{2}(2^{1/n} - 2^{-1/n})$.

Problem 21. This is not one to be written up but use your calculator to compute the approximation for $\int_1^2 h$ given in the preceding problem for some large value of $n$. Say 1000 or 10000 and compare your answer with $\ln 2$ from your calculator. Your answer should be too large. Do you see why?

Notation 3. We are now ready to define the natural logarithm function as $\ln x = \int_1^x h$ for $x > 0$. We prefer to have a single letter to represent this function so we will define $L(x) = \ln x$.

As a reminder, for the following problems about the function $L$, you are not to use any properties of $\ln x$ from your high school calculus. Use only Definitions 4, 5 and 3, and Problems 14 and 16.

Problem 22. Show that $L(2) + L(3) = L(6)$. That is, show that $\int_1^2 h + \int_1^3 h = \int_1^6 h$.

Problem 23. Show that $\ln 4 + \ln 5 = \ln 20$.

Problem 24. Show that $\ln \frac{1}{3} = -\ln 3$.

Problem 25. Show that $\ln 2 = \frac{1}{2}\ln 4$.

Problem 26. Show that $\ln(\sqrt{2})^3 = \frac{3}{2}\ln 2$.

In the following problems, if you don’t see how to do them, then substitute values for $a$ or $b$ and see if you can work them then. If you can, see if the same method works for any choice of $a$ or $b$.

Problem 27. If $a$ is any positive number and $b$ is any positive number then $\ln a + \ln b = \ln ab$.

Problem 28. Show that if $a$ is any positive number and $b$ is any positive number, then $\ln \frac{a}{b} = \ln a - \ln b$.

Problem 29. Show that if $a$ is any positive number then $\ln \frac{1}{a} = -\ln a$.

For the problems which involve an integer $n$, do them for $n = 1$ and $n = 2$ and $n = 3$. Then decide if your method would work for any choice of $n$.

Problem 30. Show that if $a$ is any positive number and $n$ is any positive integer, then $\ln a^n = n \ln a$. 

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Definition 6. The statement that \( a \) is a rational number means that there is an integer \( n \neq 0 \) and an integer \( m \) such that \( a = \frac{m}{n} \).

You have been using some of the following facts about numbers. I list them here just as a review and to remind you that you do not know that they are valid unless the exponents are rational numbers. You surely have never seen a definition for \( 2^x \) in case \( x \) is an irrational number, for example: \( 2^{\sqrt{2}} \).

For the following problems you can use basic algebraic facts about exponents when the exponent is a rational number. So if \( x \) is a number, \( x \neq 0 \), and \( a \) and \( b \) are rational numbers, then

1) \( x^a x^b = x^{a+b} \), and \( x^a / x^b = x^{a-b} \),
2) \( x^{-a} = \frac{1}{x^a} \), and
3) \((x^a)^b = x^{ab}\)

In fact some of these rules are valid even if \( x = 0 \) depending on what \( a \) and \( b \) are.

Problem 31. Show that if \( a \) is any positive number and \( n \) is any positive integer, then \( \ln a^n = \frac{1}{n} \ln a \).

Problem 32. Show that if \( a \) is any positive number and \( n \) is any positive integer and \( m \) is any positive integer, then \( \ln a^m n = \frac{m}{n} \ln a \).

Problem 33. Show that if \( a \) is a positive number and \( r \) is any rational number, positive, negative or zero, then \( \ln a^r = r \ln a \).

Note that we can’t show that if \( a \) is a positive number then \( \ln a^{\sqrt{2}} = \sqrt{2} \ln a \) because we do not have a definition for \( a^x \) when \( x \) is an irrational number. First we need a definition for an irrational number. That much is easy.

Definition 7. The statement that the number \( x \) is irrational means that \( x \) is not a rational number.

You probably do not know that any number is irrational so let’s work on the following problem.

Problem 34. Show that \( \sqrt{2} \) is an irrational number.

If you can do the preceding problem, present it in class. If you want help, read on.

Again we need some definitions before proceeding.

Definition 8. The statement that \( n \) is an even integer means that there is an integer \( k \) such that \( n = 2k \). The statement that \( n \) is an odd integer means that there is an integer \( k \) such that \( n = 2k + 1 \).

To work Problem 34 you must show that there do not exist two integers \( m \) and \( n \) such that \( \frac{m}{n} = \sqrt{2} \). And you can assume that \( m \) and \( n \) are not both even integers, since, if they were then you would have \( m = 2m' \) and \( n = 2n' \) for some integers \( m' \) and \( n' \) and also \( \frac{m'}{n'} = \sqrt{2} \). And if \( m' \) and \( n' \) were both even you could repeat this process continuing until you get integers whose quotient is \( \sqrt{2} \) which are not both even.

So, to get started on Problem 34 try the following one as a start.
The natural logarithm function

**Problem 35.** Show that if \( m \) is an odd integer and \( n \) is an even integer, then \( \frac{m^n}{n^2} \neq 2 \).

**Definition 9.** The statement that \( f \) is an increasing function means that if \( a \) and \( b \) are in the domain of \( f \) and \( a < b \) then \( f(a) < f(b) \).

**Problem 36.** Show that \( L \) is an increasing function. To show this, you must use our definition for \( L(x) \) for \( x > 0 \). Use a Riemann sum to show that if \( a \) and \( b \) are positive numbers, and \( a < b \), then \( L(b) - L(a) > 0 \).

We now know that \( L \) is an increasing function and we know that the domain of \( L \) is the set of all numbers. Does this mean that \( L \) crosses every horizontal line?

**Problem 37.** Find a graph \( f \) whose domain is the set of all numbers which is an increasing graph and which lies entirely below the horizontal line with equation \( y = 1 \).

This completes most of our study of the function \( L \). Next we want to investigate properties of the inverse of \( L \).
Chapter 3

The exponential function

It seems to me to be surprising that the range of the graph $L$ is the set of all numbers. Use your graphing calculator to graph $L(x) = \ln x$ and see if you think the graph will cross the line with equation $y = 10$. You surely recall from your calculus that $L'(x) = 1/x$ for each positive number $x$. From this it follows that $L'(x)$ approaches zero as $x$ increases without bound.

**Problem 38.** Show that $\ln 4 > 1$.

You probably had the following Theorem in high school. It is called the intermediate value theorem and is one of those things that is “obvious” but not easy to prove. We will just assume that it is true without proof. We only need it right now for the function $L$ but it is valid for any continuous function.

**Theorem 2.** If each of $a$, $b$ and $y$ is a number, $a < b$ and $L(a) < y < L(b)$, then there is a number $x$ with $a < x < b$ such that $L(x) = y$.

We are now ready to define the base of our logarithm function $L$.

**Problem 39.** Recall that we showed earlier that $L(2) < 1$. Explain how you know from Theorem 2 and Problem 38 that there is a number which we will call $e$ such that $L(e) = 1$.

**Problem 40.** Show that if $n$ is a positive integer, there is a number $x$ such that $L(x) = n$. As usual, if you don’t see how to do this, try it for $n = 2$ or $n = 3$ and see if that helps.

For the next problem we will need the following property about numbers which I hope you will agree is “obvious”. We call it an Axiom to indicate that we think it would be difficult to prove it on the basis of anything simpler, so we just assume it to be true without proof.

**Axiom 1.** If $x$ is a number, then $x$ is an integer (positive, negative or 0) or else there is an integer $n$ such that $n < x < n + 1$.

**Problem 41.** Show that if $y$ is a positive number, then there is a number $x$ such that $L(x) = \ln x = y$. Hint: Use Theorem 2 and Axiom 1.

**Definition 10.** Recall that a function is simply a collection of ordered pairs $(x, y)$ of numbers such that no two of them have the same first term. Or, if you prefer, a function is a...
collection of points in a plane no two of which lie on the same vertical line. If $f$ is a function then by the **domain** of $f$ is meant the set of all numbers which are the first terms (the $x$-coordinates) of points on $f$. And the **range** is the set of all 2nd terms (y-coordinates) of points of $f$.

**Definition 11.** If $f$ is an increasing or a decreasing function, then the **inverse** of $f$ is meant the set of all points $(b,a)$ such that $(a,b)$ is a point of $f$.

**Problem 42.** Show that if $f$ is an increasing or decreasing function, then the inverse of $f$ is a function.

**Problem 43.** If $f$ is an increasing or decreasing function and $g$ is the inverse of $f$, use the definition and the usual notation of $(x,f(x))$ being a point on $f$, explain why $g(f(x)) = x$ if $x$ is in the domain of $f$ and $f(g(x)) = x$ if $x$ is in the domain of $g$.

**Definition 12.** $E$ will denote the inverse of $L$.

You know about exponents when the exponent is a rational number. So you know what is meant by $e^3$ or $\sqrt{3}$. And you know what $E(x)$ means for any number $x$, rational or irrational, from the definition of $E$ as the inverse of $L$. Our next set of problems will be to show that if $r$ is a rational number then $E(r) = e^r$. So at least for rational numbers the graph of $E$ is the same as the graph of the function $y = e^x$. We will begin as we did with the natural logarithm function by showing this for integers.

Using what we know about $L$ it is not hard to show what we want about its inverse $E$ since from the definition of inverse, if $(x,y)$ is on $L$ or $L(x) = y$, then $(y,x)$ is on $E$ so $E(y) = x$.

**Problem 44.** Explain why $E(0) = e^0$ and $E(1) = e^1$ just from the definition of $E$ and $e$ and what you know about exponents.

**Problem 45.** Show that if $n$ is a positive integer $E(n) = e^n$.

**Problem 46.** Show that if $n$ and $m$ are positive integers, then $E\left(\frac{n}{m}\right) = e^{\frac{n}{m}}$.

**Problem 47.** Show that if $r$ is a rational number then $E(r) = e^r$.

Now that we know that $E(r) = e^r$ if $r$ is a rational number and we know that the usual rules for exponents work for rational numbers then we know that if each of $a$ and $b$ is a rational number then we can express these rules in terms of $E$. So $E(a) \cdot E(b) = E(a+b)$ $E(-a) = 1/E(a)$, and $E(ab) = E(a)$. And we know that $E(x)$ is on $L$.

Actually it is easy to show that these rules work for any choice of $a$ and $b$, using what we know about the graph $L(x) = \ln x$. Here are a couple of examples.

**Problem 48.** Show that if each of $a$ and $b$ is a number, then $E(a + b) = E(a) \cdot E(b)$.

**Problem 49.** Show that if $a$ is a number, then $E(-a) = 1/E(a)$.

We are finally ready to define what we mean by $e^x$ if $x$ is an irrational number. We **define** $e^x$ to mean $E(x)$ which is the number $y$ such that $L(y) = x$. Noting that if $a > 0$, then $e^{\ln a} = E(L(a)) = a$ and we want $a^b = (e^{\ln a})^b = e^{\ln a}$, we make the following definition.

**Definition 13.** If $a > 0$ and $b$ is a number, $a^b = E(b \ln a)$.
Now we have the problem of showing that this definition agrees with our usual one. This is so much like what we have already done that we will just assume that the rules you are familiar with work not only for rational numbers, but irrational ones as well.

For the following problems I will assume that you know the formulas for the derivatives of polynomials and trigonometric functions. I don’t mean you have to memorize them. You can look them up as needed. We will also use the various rules for computing derivatives such as the product, sum, quotient and chain rules. The chain rule is often written in different ways. Here is the form we will use. If $f$ and $g$ are functions and the range of $g$ is the domain of $f$ and $h(x) = f(g(x))$ for each number $x$ in the domain of $g$, then under appropriate conditions, if $x$ is in the domain of $h$, $h'(x) = f'(g(x)) \cdot g'(x)$. We will just assume this works for all the functions we will consider.

You surely recall that $L'(x) = 1/x$ and $E'(x) = E(x)$. We could use Riemann sums again to show that if $x > 0$, $L'(x) = 1/x$. Instead we will use one of the fundamental theorems of calculus. The following is one of them.

**Theorem 3.** If $f$ is a continuous function whose domain includes the interval $[a, b]$, and $c$ is a number in $[a, b]$, and $A(x) = \int_c^x f$ for each number $x$ in $[a, b]$, then for each number $x$ in $[a, b]$, $A'(x) = f(x)$.

It follows from Theorem 3 that $L'(x) = h(x) = 1/x$ for each number $x > 0$. We will use this fact from now on.
Chapter 4

Taylor Polynomials

Problem 50. Use the definition of $E$, the chain rule and the fact that $L'(x) = \frac{1}{x}$ for each number $x > 0$ to show that $E'(x) = E(x)$ for each number $x$.

Problem 51. Find the tangent line to the graph of $E$ at the point $(0, 1)$.

Problem 52. Find a "tangent parabola" to the graph of $E$ at the point $(0, 1)$. That is, find a parabola $P$ such that $P$ and $E$ have the same value, derivative and 2nd derivative at the point $(0, 1)$.

Definition 14. If $n$ is a positive integer, the statement that $P$ is a polynomial of degree $n$ means that there are numbers $a_0, a_1, \ldots, a_n$ such that $a_n \neq 0$ and for each number $x$, $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. Or, if you prefer, $P(x) = \sum_{k=0}^{n} a_kx^k$.

Problem 53. Find a polynomial $P$ of degree 3 such that $P(0) = E(0)$, $P'(0) = E'(0)$, $P''(0) = E''(0)$ and $P'''(0) = E'''(0)$. Can you find a formula for a polynomial $P$ of degree $n$ so that $P(0) = E(0)$ and the first $n$ derivatives of $P$ and $E$ are the same at the point $(0, 1)$?

The polynomials in Problem 53 are called the Taylor polynomials for $E$ of degree 3 and $n$ based at 0.

Next we want to find Taylor polynomials for $E$ based at 1.

Problem 54. Find the tangent line to $E$ at the point $(1, e)$. Find a and b so that your tangent line has equation $y = a + b(x - 1)$.

Problem 55. As in Problem 52 find a tangent parabola to the graph of $E$ at the point $(1, e)$. That is, find a polynomial $P$ of degree 2 such that $P(1) = E(0)$, $P'(1) = E'(1)$ and $P''(1) = E''(1)$. As in the preceding problem, write your answer in the form $a + b(x - 1) + c(x - 1)^2$.

This is the Taylor polynomial of degree 2 based at 1 for $E$.

Notation 4. If $n$ is a positive integer and $f$ is a function and $x$ is a number in the domain of $f$ at which $f$ has $n$ derivatives, then $f^{(n)}(x)$ denotes the $n$th derivative of $f$ at $x$.

Question 1. Can you give an example of a function $f$ which you think is continuous at some number $x$ in its domain but does not have a derivative there? No proof required, nor could you give one without a definition for continuous and for derivative. Can you describe a function $f$ that has a derivative at some number $x$ but does not have a 2nd derivative at $x$?
Def. 15. If $f$ is a function and $n$ is a positive integer and $a$ is a number in the domain of $f$ and $f$ has a $j$th derivative for each positive integer $j \leq n$, then the **Taylor polynomial** for $f$ based at $a$ is the polynomial $P$ of degree $n$ such that if $0 \leq j \leq n$, then $f^{(j)}(a) = P^{(j)}(a)$.

Problem 56. Let $n$ be a positive integer. Find numbers $a_0, a_1, a_2, \cdots, a_n$ such that if $P_n(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots + a_n(x-1)^n$ for each number $x$, then $P_n$ is the Taylor polynomial of degree $n$ based at $1$ for $E$.

Problem 57. Show that there is no Taylor polynomial of degree 2 for $f$ based at $0$ if $f(x) = \sin x$ for each number $x$.

Problem 58. Let $n$ be an odd positive integer. Find the Taylor polynomial of degree $n$ based at $0$ for $f$ if $f(x) = \sin x$ for each number $x$. I don’t expect you to prove that your formula is correct. Work the problem for $n = 3$ and $n = 5$ and see if you can find a pattern.

Problem 59. Let $f(x) = \cos x$ for each number $x$. Let $n$ be an even positive integer. Find the Taylor polynomial of degree $n$ for $f$ based at $0$. Again, as in the previous problem, find a pattern.

Question 2. If $n$ is a positive integer, are you ready to guess a formula which gives the Taylor polynomial of degree $n$ based at the number $a$ for any function $f$ for which you can find the derivatives $f^{(j)}(a)$ for values of $j$ from $1$ to $n$?

Problem 60. Find the Taylor polynomial of degree 2 for $f$ based at $\frac{\pi}{2}$ if $f(x) = \cos x$ for each number $x$.

Problem 61. Find the Taylor polynomial of degree 4 for $f$ based at $\frac{\pi}{2}$ if $f(x) = \cos x$ for each number $x$. Use it and the polynomial of degree 4 from Problem 59 to compute an approximation for $\cos 1.5$. Which is the best approximation and why do you think it should be?

Problem 62. Let $f(x) = \tan x$ for each number $x$. Find the Taylor polynomial of degree 3 for $f$ based at $0$. I don’t know of any pattern.
Chapter 5

Series and sequences

By an increasing sequence we mean an unending sequence of numbers \(x_1, x_2, \ldots\) such that for each positive integer \(n, x_n < x_{n+1}\). A decreasing sequence is defined in a similar way. An increasing sequence is said to be bounded if there is a number larger than each number in the sequence. As an example, the sequence \(x_1, x_2, x_3, \ldots = 1, 2, 3, \ldots\) is not bounded since for each positive integer \(n, x_n = n\). and by Axiom 1 for any number \(x\), there is a positive integer \(n + 1\) which is larger than \(x\). On the other hand the sequence \(x_1, x_2, x_3, \ldots = 1/2, 2/3, 3/4, \ldots\) is bounded since for each positive integer \(n, x_n = 1 - 1/n + 1 < 2\). And it is increasing by the next problem.

Here are all the rules you need to be able to work with inequalities:

1. If \(a \neq b\), then \(a < b\) or \(b < a\).
2. If \(a < b\) and \(b < c\), then \(a < c\).
3. If \(a < b\) then \(a + c < b + c\).
4. If \(c > 0\) and \(a < b\), then \(ca < cb\).
5. If \(c < 0\) and \(a < b\), then \(ca > cb\).
6. \(0 < 1\).

I hope these are all familiar. You should be able to get other familiar facts from these. Use them on the next problem.

**Problem 63.** Use the preceding rules about inequalities to show that if \(n\) is a positive integer, \(1 - 1/n + 1 < 1 - 1/n^2\).

Consider the following sequence: \(1/2, 3/4, 7/8, 15/16, \ldots\). The number 2 is larger than every number in this sequence. But 2 is not the smallest such number. The number 1 is also larger than each number in the sequence. We will later see how to show that 1 is the smallest such number.

We now need another Axiom which I hope seems reasonable to you.

**Axiom 2.** If \(x_1, x_2, \ldots\) is an increasing (or decreasing) sequence of numbers and there is a number which is larger (smaller) than every number in the sequence, then there is a smallest (largest) such number. It is called the limit of the sequence.

**Problem 64.** Show that 0 is the limit of the sequence \(1, 1/2, 1/3, \ldots\). Note that clearly 0 is smaller than each number in the sequence so we must show that no positive number is
smaller than each number in the sequence. In other words, we want to know that if \( a \) is any positive number then \( a \) is not smaller than each number in the sequence. This means that you must show that if \( a \) is any positive number, then there is an integer \( N \) such that \( 1/N \) (which is a term in the sequence) is smaller than or equal to \( a \). Hint: Use Axiom 1.

**Definition 16.** A sequence is said to be **bounded above** if there is a number which is larger than every number in the sequence, and **bounded below** if there is a number smaller than every number in the sequence.

We often describe a sequence by starting with a number and then adding numbers over and over. For example the sequence: \( 1/2, 3/4, 7/8, 15/16, \ldots \) could be written as \( 1/2, 1/2 + 1/4, 1/2 + 1/4 + 1/8, \ldots \). We often abbreviate this sequence as \( 1/2 + 1/4 + 1/8 + \ldots \). When written like this we call it a **series** but it is just a different way of describing a sequence. Books would refer to the sequence as the sequence of partial sums: \( 1/2, 1/2 + 1/4, 1/2 + 1/4 + 1/8, \ldots \). By the limit of a series we mean the limit of the corresponding sequence.

**Definition 17.** The statement that a series or a sequence **converges** means that it has a limit.

So the problem of deciding if an increasing sequence converges is a matter of finding out if it is bounded above. This leads us to what is called the **integral test**.

For the rest of this course we will frequently use the fundamental theorems of calculus. One of them, Theorem 3 was stated and used earlier to show that \( E'(x) = \frac{1}{x} \) for \( x > 0 \). The other one is stated next. We will not prove this.

**Theorem 4.** If \( a \) and \( b \) are numbers with \( a < b \) and \( f \) is a function and \( g \) is a continuous function such that \( g(x) = f'(x) \) for each number \( x \) such that \( a \leq x \leq b \), then \( \int_a^b g = f(b) - f(a) \).

We will assume that all the functions we consider are continuous so we can use Theorem 4. You can also use what you know about derivatives and anti-derivatives or indefinite integrals for any of the rest of our problems.

**Problem 65.** Let \( f(x) = \frac{1}{x} \) for each number \( x > 0 \). Use Riemann sums to show that \( 1/2^2 < \int_1^2 f \) and \( \frac{1}{2} + \frac{1}{2} < \int_1^3 f \). Let \( n \) be a positive integer. Show that \( 1/2^2 + 1/3^2 + 1/4^2 + \ldots + 1/n^2 \) is a lower Riemann sum for \( \int_1^n f \).

**Problem 66.** Show that if \( f \) is the function so that \( f(x) = \frac{1}{x} \) for each number \( x > 0 \), and \( n \) is any positive integer, then \( \int_1^n f < 1 \).

From the preceding two problems we have that the sequence \( 1/2^2 + 1/3^2 + 1/4^2 + \ldots \) is bounded above by the number 1 and thus by Axiom 2 that the sequence has a limit.

**Problem 67.** Show that the limit of the series \( 1/2^2 + 1/3^2 + 1/4^2 + \ldots \) is less than 1.

The series \( 1/2^2 + 1/3^2 + 1/4^2 + \ldots \) is sometime written as \( \sum_{n=2}^{\infty} 1/n^2 \). But you must be careful when you see this as books often use the same symbol to mean the limit of the series. For example you might see, if \( -1 < x < 1 \), \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \) to mean that \( \frac{1}{1-x} \) is the
limit of the series \(1 + x + x^2 + \ldots\). Note that here and frequently when writing series using the \(\Sigma\) notation, \(x^0\) is used to represent 1. This is a bad use of notation, especially when \(x\) can be 0. I prefer to write \(1 + \sum_{i=1}^{\infty} x^n = \frac{1}{1-x}\).

Here is another example of the use of the integral test.

Problem 68. As was done for the series \(1/2^2 + 1/3^2 + 1/4^2 + \ldots\) in Problem 65 show that the series \(1/2^3 + 1/3^3 + 1/4^3 + \ldots\) has a limit.

Problem 69. For each positive integer \(n\), let \(a_n = \frac{1}{n^p}\). Use the integral test to show that \(a_1 + a_2 + a_3 + \ldots\) is a bounded series and thus converges.

Problem 70. Show that \(\sum_{n=2}^{\infty} \frac{1}{n^p}\) has a limit. Hint: use our rules for inequalities to show that if \(n\) is a positive integer and \(n > 1\), then \(\frac{1}{n} < \frac{1}{n^p}\)

Now we will use the integral test to get part of what is called the \(p\)-test.

We have shown that if \(p = 2\) or \(p = 3\), then the series \(1/2^p + 1/3^p + 1/4^p + \ldots = \sum_{n=2}^{\infty} 1/n^p\) has a limit. We now want to show that the same method shows that it has a limit for any number \(p > 1\).

Problem 71. Show that if \(p\) is a number, \(p > 1\), and \(f(x) = 1/x^p\) for each number \(x > 0\) and \(n\) is any positive integer then \(\int_0^n f < \frac{1}{p-1}\). As always if this seems too hard pick a value for \(p\) and work the problem. We have done this for \(p = 2\) and \(p = 3\). Try \(p = 1.1\)

Problem 72. Show that if \(p\) is a number and \(p > 1\), the series \(1/2^p + 1/3^p + 1/4^p + \ldots\) has a limit which is \(\frac{1}{p-1}\) or less. As before when we had \(p = 2\) or \(p = 3\) show that \(1/2^p + 1/3^p + 1/4^p + \ldots + 1/n^p\) is a lower Riemann sum for an integral.

This is part of the \(p\)-test. The other part is to show that if \(p \leq 1\), then the series \(1/2^p + 1/3^p + 1/4^p + \ldots\) does not have a limit.

Definition 18. The statement that a series or sequence diverges means it does not converge.

Thus we have that if an increasing sequence does not have an upper bound, then it diverges.

Problem 73. First consider the series \(1/2 + 1/3 + 1/4 + \ldots\) which is the \(p\)-series when \(p = 1\). Show that if \(n\) is a positive integer, then \(1/2 + 1/3 + 1/4 + \ldots + 1/n\) is a lower Riemann sum for \(\int_0^n f\). Here \(h(x) = 1/x\) for \(x > 0\) as before. Also show that \(1/2 + 1/3 + 1/4 + \ldots + 1/n\) is an upper Riemann sum for \(\int_{n/2}^{n+1} h\). Explain how this shows that \(\ln(n+1) - \ln 2 < 1/2 + 1/3 + 1/4 + \ldots + 1/n < \ln n\)

Problem 74. Use the proceeding problem and your calculator to find an integer \(n\) such that \(1/2 + 1/3 + 1/4 + \ldots + 1/n > 10\).

Problem 75. Show that \(1/2 + 1/3 + 1/4 + \ldots\) has no upper bound. That is show that if \(A\) is a number then \(A\) is not an upper bound by showing that there is an integer \(n\), depending on \(A\) so that \(1/2 + 1/3 + 1/4 + \ldots + 1/n > A\)

Next we consider the case where \(p < 1\). And again we will use the integral test. First we consider the case where \(p = 1/2\).
Problem 76. Show that \( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}} > \frac{n^2 + 1}{2\sqrt{n}} \).

Problem 77. Let \( A \) be a positive number. Find an integer \( n \) such that \( \sum_{k=1}^{k=n} \frac{1}{\sqrt{k}} > A \), to show that the series diverges.

Note that in the next problem I have used the \( dx \) that you are used to. I use it to describe what function I want to integrate because of the \( p \). I am assuming that \( p \) is a constant, and the function I want to integrate is \( f \) where \( f(x) = \frac{1}{x^p} \). It could have been written as \( \int_1^\infty f \) but that would have required us to define \( f \) as was done above. Most books do it as stated.

Problem 78. Assume that \( p < 1 \). Show that \( \int_1^\infty \frac{1}{x^p} dx = \frac{1}{1-p} \). Now find an integer \( n \), depending on \( p \) so that \( \int_1^n \frac{1}{x^p} dx > 10 \).

Problem 79. Assume that \( p < 1 \) and that \( A \) is a positive number. Find an integer \( n \), depending on \( A \) and \( p \), such that \( \int_1^n \frac{1}{x^p} dx > A \).

There is an easier way to do the preceding problem using what are called comparison tests. The next few problems are about these tests.

Problem 80. Assume that \( a_1, a_2, a_3, \ldots \) is a sequence of positive numbers and there is a number \( A \) such that for each positive integer \( n \), \( a_1 + a_2 + a_3 + \ldots + a_n < A \), so that the series \( a_1 + a_2 + a_3 + \ldots \) has a limit. Assume that \( b_1, b_2, b_3, \ldots \) is a sequence of positive numbers such that if \( n \) is a positive integer, \( b_n \leq a_n \). Show that for each positive integer \( n \), \( b_1 + b_2 + \ldots + b_n < A \) so that the series \( b_1 + b_2 + b_3 + \ldots \) has a limit also.

Problem 81. Assume that \( a_1, a_2, a_3, \ldots \) is a sequence of positive numbers and for each number \( A \) there is a positive integer \( n \) such that \( a_1 + a_2 + a_3 + \ldots + a_n > A \), so that the series \( a_1 + a_2 + a_3 + \ldots \) is unbounded and has no limit. Assume that \( b_1, b_2, b_3, \ldots \) is a sequence of positive numbers such that if \( n \) is a positive integer, \( b_n \geq a_n \). Show that for each positive integer \( n \), there is a positive integer \( n \) such that \( b_1 + b_2 + \ldots + b_n > A \) so that the series \( b_1 + b_2 + b_3 + \ldots \) has no limit also.

Problem 82. As an example of how one might use the comparison test, for each positive integer \( n \), let \( b_n = \frac{1}{n^{p+1}} \). We know that the series \( 1/2^2 + 1/3^2 + 1/4^2 + \ldots \) is bounded above by 1, so first show that the series \( 1/1^2 + 1/2^2 + 1/3^2 + \ldots \) is bounded above by 2. Then let \( a_n = \frac{1}{n^p} \) for each positive integer \( n \) and show that \( b_n \leq a_n \) so that \( b_1 + b_2 + b_3 + \ldots \) converges by Problem 80.

Problem 83. Use the comparison test to show that \( \sum_{i=1}^{i=n} \frac{1}{i^2} \) is bounded above and thus converges. Hint. Compare the terms with those of the series \( \sum_{i=1}^{i=n} \frac{1}{i^2} \).

Problem 84. Use Definition 13 and the fact that \( E \) is an increasing function to show that if \( a \) is a positive number, then \( a^p < a \) if \( p < 1 \) and \( a^p > a \) if \( p > 1 \).

Problem 85. Use the comparison test and Problem 75 to do Problem 79.

Integration by Parts

Suppose that \( f \) and \( g \) are functions with domain \( [a, b] \). You surely recall the product rule for the derivative of their product. It states that \( (fg)' = fg' + gf' \). That is if \( h(x) = f(x)g(x) \) for each number \( x \) in \( [a, b] \) then \( h'(x) = f(x)g'(x) + g(x)f'(x) \) for each number \( x \) in \( [a, b] \).
Series and sequences

Probably you are familiar with the use of what is basically the same rule in finding antiderivatives or indefinite integrals. Does this look familiar?

\[ \int u dv = uv - \int v du \]

Do you realize this is just the product rule in a different form?

We want to use a similar formula for definite Riemann integrals so, starting with \( h'(x) = f(x)g'(x) + g(x)f'(x) \) for each number \( x \) in the interval \([a, b]\), then we note that:

\[ f(x)g'(x) = h'(x) - g(x)f'(x) \]

So we can integrate to get:

\[ \int_a^b f(x)g'(x) dx = \int_a^b h'(x)dx - \int_a^b g(x)f'(x) dx \]

or, deleting the \( dx \)'s for simplicity:

\[ \int_a^b fg' = \int_a^b h' - \int_a^b gf' \]

and using Theorem 4, since \( h(x) = f(x)g(x) \) we have:

\[ \int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf' \]

**Problem 86.** Use Theorem 4 to show that \( e = 1 + \int_0^1 e \cdot g \).

**Problem 87.** Starting with the preceding problem, use the integration by parts formula and see what you get. Note that if you want to use the formula on the integral \( \int_0^1 E \), then you can use any two functions \( f \) and \( g \) as long as \( g(x) = x + c \) for each number \( x \) in \([a, b]\). I hope I don’t spoil the problem by suggesting that you let \( f(x) = E(x) \) and \( g'(x) = 1 \).

In the preceding problem, notice that if \( g'(x) = 1 \), then for any choice of the number \( c \) you could let \( g(x) = x + c \). If you let \( g(x) = x \) in the preceding problem, then do it again, this time letting \( g(x) = x - 1 \) and see what you get. This is the next problem.

**Problem 88.** Start with \( e = 1 + \int_0^1 E \), and integrate by parts to get \( e = 1 + \int_0^1 gE, \) where \( g(x) = x - 1 \). One would usually see this written as \( e = 1 + \int_0^1 e^x \cdot g \cdot (x - 1) dx \).

Our next goal is to get a formula that gives the same formula we got before for the Taylor polynomial of degree \( n \) based at 0, with the “error” expressed as an integral. That is, we want to find a simple function for \( g_n \) for each positive integer \( n \) so that:

\[ e = 1 + 1/1! + 1/3! + ... + 1/n! + \int_0^1 E \cdot g_n \]

Then we will make an estimate on the value of the integral to see how accurate our approximation is.

**Problem 89.** This is a bit tricky. Can you figure out how to repeat what we did in the preceding problem to find a different function for \( g \) and take another step and get

\[ e = 1 + 1 + 1/2! + 1/3! + ... + 1/n! + \int_0^1 E \cdot g ? \]
**Problem 90.** Do it again to find a function $g$ so that $e = 1 + 1 + 1/2 + + 1/3! + \int_0^1 Eg \ ?$. Can you see the pattern and guess the formula if we were to do this again?

Here is the result we have been working toward.

**Problem 91.** Show that if $n$ is a positive integer, then

$$e = 1 + 1 + 1/2! + \ldots + 1/n! + \int_0^1 \frac{(1-x)^n}{n!} e^x dx.$$ 

Now we want to use the result of the preceding problem to be able to compute the numerical value of $e$ to any desired degree of accuracy. Note that from the previous problem we have that, if $n$ is a positive integer, then

$$e - (1 + 1 + 1/2 + \ldots + 1/n!) = \int_0^1 \frac{(1-x)^n}{n!} e^x dx$$

Thus the integral in the previous equation is the error in using $1 + 1 + 1/2! + \ldots + 1/n!$ as an approximation for $e$. To estimate the error we will use the following fact without proving it. I hope it is easy to believe.

**Theorem 5.** If $f$ and $g$ are functions which have Riemann integrals on the interval $[a, b]$ and for each number $x$ in $[a, b]$, $0 \leq f(x) \leq g(x)$, then $0 \leq \int_a^b f \leq \int_a^b g$.

**Problem 92.** Recall that we have shown (see Problem 38) that $e < 4$. Use this and the preceding theorem to show that

$$\int_0^1 \frac{(1-x)^n}{n!} e^x dx < \frac{4}{(n+1)!}.$$ 

And that

$$e - (1 + 1 + 1/2 + \ldots + 1/n!) \leq 4/(n + 1)!$$

In the next problem I picked $10^{-13}$ in order to see how many terms of our series had to be added to get the 13 places of accuracy that your TI calculator claims to have. Note that if we wanted more that 13 places of accuracy there would be a problem using your calculator to add up the terms in the series.

**Problem 93.** Find an integer $n$ (use your calculator) such that

$$e - (1 + 1 + 1/2 + \ldots + 1/n!) \leq \frac{1}{10^{13}}$$

**Power series**

We began to study power series when we worked on Taylor Polynomials. Recall that we found that the Taylor polynomial approximation for $E$ of degree $n$ was $P_n(x) = 1 + x + x^2/2! + \ldots + x^n/n!$ for each number $x$. Let $x$ be a number. We got the sequence of approximations $1, 1+x, 1+x+x^2/2!, \ldots$ for $E(x)$. Written as a series we got the $1 + x + x^2/2! + x^3/3! + \ldots$ as our approximations for $E(x)$. And we saw how to get similar series approximations for other functions.

William S. Mahavier

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Definition 19. A power series is just a series of polynomial approximations. Taylor polynomials give one way of finding power series. But any series that can be written as \( a_0 + a_1 x + a_2 x^2 + \ldots \) where each of \( x, a_0, a_1, a_2, \ldots \) is a number is a power series.

You should be familiar from high school with the power series \( 1 + x + x^2 + x^3 + \ldots \). It is called a geometric series because each term is gotten from the preceding one by multiplying by the same number. For example, if \( x = 1/2 \), then the series is \( 1 + 1/2 + 1/4 + 1/8 + \ldots \). In high school you probably had the following formula: \( a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r} \).

Problem 94. Show that if \( n \) is a positive integer, then 
\[
\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n.
\]

Problem 95. Show that if \( n \) is a positive integer and \( 0 < x < 1 \), then 
\[
1 + x + x^2 + \ldots + x^n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} < \frac{1}{1-x}.
\]

Note that the preceding problem shows that if \( 0 < x < 1 \) then the series \( 1 + x + x^2 + \ldots \) is bounded above by \( \frac{1}{1-x} \). If time permits, we may show that the following is correct, but for now we will just assume it is true.

Theorem 6. If \( 0 < x < 1 \), then the increasing sequence \( 1, 1 + x, 1 + x + x^2, \ldots \) has \( \frac{1}{1-x} \) as its limit. That is, \( \frac{1}{1-x} \) is the smallest number larger than each number in the sequence.

Note 1. Actually the preceding theorem is true if \( -1 < x < 0 \) as well and if time permits we will show this also.

Problem 96. Let \( a_1, a_2, \ldots \) be a sequence of positive numbers, let \( r \) be a number such that \( 0 < r < 1 \). Assume that for each positive integer \( n \) \( a_{n+1} \leq r \cdot a_n \). Show that for each positive integer \( n \), 
\[
a_1 + a_2 + \ldots + a_n \leq \frac{a_1}{1-r}.
\]

Next we will just assume that \( e \) is the limit of the series \( 1 + \sum_{i=1}^{n} \frac{1}{n!} \). And that \( e-1 \) is the limit of \( \sum_{i=1}^{n} \frac{e^i}{i!} \) and \( e - 1 - 1/1! - 1/2! \) is the limit of the series \( \sum_{i=1}^{n} \frac{1}{n!} \), etc.

Problem 97. For each positive integer \( n \) let \( a_n = 1/n! \). Show that \( a_2 \leq \frac{e}{2} \), \( a_3 \leq \frac{e}{3} \) and for each positive integer \( n \), 
\[
a_{n+1} \leq \frac{e}{n+1}.
\]

Just as in the Problem 96, we have 
\[
a_1 + a_2 + \ldots + a_{n+1} < (1 + 1/2 + 1/2^2 + \ldots + 1/2^n) < \frac{1}{1-1/2} = 2.
\]

So \( 1 + \sum_{i=1}^{n} \frac{1}{n!} \) is bounded by 3. So we have \( e < 3 \).

Next we try to improve our approximation for \( e \). I will pick \( n = 9 \) arbitrarily and use the ratio test to estimate the error in using \( \sum_{i=1}^{9} \frac{1}{n!} \) as an approximation for \( e \).

Problem 98. First we want to show that the series \( 1/10! + 1/11! + \ldots \) is bounded above. To this end, let \( n > 10 \) be a positive integer and show that \( 1/10! + 1/11! + \cdots + 1/n! = \frac{1}{10!} (1 + \frac{1}{11} + \frac{1}{11!} + \cdots + \frac{1}{11 \cdot 12 \cdot \ldots \cdot n}) \).

Problem 99. Now show that if \( n > 10 \) is a positive integer, then 
\[
\frac{1}{10!} (1 + \frac{1}{11} + \frac{1}{11!} + \cdots + \frac{1}{11 \cdot 12 \cdot \ldots \cdot n}) < \frac{1}{10^9} (1 + 1/11 + 1/11^2 + \cdots + 11^{n-10}) < \frac{1}{10^9} (1 - \frac{1}{11^{n-10}}).
\]
Problem 100. Is it clear now that we have \( e - (1 + 1/2! + \ldots + 1/9!) < \frac{1}{10^4} \)? Use your calculator to show \( e - (1 + 1/2! + \ldots + 1/10!) < 0.0000031 \). Show that \( 1 + 1/2! + \ldots + 1/10! < e < 1 + 1/2! + \ldots + 1/10! + \frac{11}{10^10} \).

We have not discussed what it means to say that a series converges if it is neither increasing nor decreasing, as for example the series \( 1 - 1/2 + 1/3 - 1/4 + 1/5 \ldots \). Intuitively it means that we can compute an approximation to the limit to any desired accuracy, just by adding enough terms. We may not use this definition but for the sake of completeness in your notes, here is the definition.

Definition 20. If \( a_1, a_2, a_3, \ldots \) is a sequence of numbers, then the statement that \( L \) is the limit of the sequence \( a_1, a_2, a_3, \ldots \) means that if \( c \) is a positive number there is a positive integer \( n \) such that each of \( a_n, a_{n+1}, a_{n+2} + \ldots \) is in the segment \( (L - c, L + c) \).

Problem 101. We know from Problem 6 that the limit of the series \( 1 + 1/2 + 1/4 + 1/8 + \ldots \) is 2. Now consider the series \( 3 + 2 + 1/2 + 1/4 + 1/8 + \ldots \). Show that the limit of this series is 7.

And here is a similar problem which is a bit tricky to show.

Problem 102. If \( a_1 + a_2 + \ldots \) is a series of positive numbers which has a limit \( A \), and \( k \) is a positive integer, then the series \( a_k + a_{k+1} + a_{k+2} + \ldots \) has \( A - (a_1 + a_2 + \ldots + a_k) \) as a limit.

The next theorem we will not take the time to prove but it will help explain why I have spent so much time on series where all the terms are positive.

Theorem 7. If \( a_1 + a_2 + a_3 + \ldots \) is a series of numbers, not necessarily all positive then the series \( a_1 + a_2 + \ldots \) has a limit if the series \( |a_1| + |a_2| + \ldots \) has a limit.

Note 2. In the case of power series, the following is true. It will explain in part why I concentrated on only sequences of positive terms. We will not show this to be true, but you may well need this in later courses.

As an example of a series for which the preceding theorem does not apply, recall that \( 1 + 1/2 + 1/3 + \ldots \) is not a bounded series and thus does not have a limit. However the series \( 1 - 1/2 + 1/3 - 1/4 + 1/5 \ldots \) does have a limit. This is easy to believe from the next problem.

Problem 103. For each positive integer \( n \), let \( a_n = 1 - 1/2 + 1/3 - 1/4 + \ldots + \frac{(-1)^{n+1}}{n} \). Show that \( a_1, a_3, \ldots \) is a decreasing bounded sequence and \( a_2, a_4, \ldots \) is an increasing bounded sequence and they have the same limit.

Theorem 8. If \( a_0, a_1, a_2, \ldots \) is a sequence of numbers then
1) \( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \) converges only if \( x = 0 \), or
2) \( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \) converges for every number \( x \), or
3) there is a number \( R \) such that the \( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \) converges if \( -R < x < R \) and does not converge if \( x < -R \) or \( x > R \).

In the last case the series might or might not converge if \( x = R \) or \( x = -R \).

Here is the way the ratio test is usually stated for power series. Again we will not prove this.
Theorem 9. If $a_0 + a_1x + a_2x^2 + \ldots$ is a power series then the series converges if the sequence $\frac{|a_1|}{|a_0|}, \frac{|a_2|}{|a_1|}, \frac{|a_3|}{|a_2|}, \ldots$ has a limit which is less than 1. And the power series does not converge if the limit of this sequence is more than 1.

As an example of the use of Theorem 9 we return to our Taylor’s series for $f(x) = e^x$.

Problem 104. Consider the series $1 + x + x^2/2! + x^3/3! + \ldots$. Now form the sequence of ratios of consecutive terms of this series. To avoid the use of absolute values, assume that $x$ is positive. The ratios are $\frac{x^1}{1}, \frac{x^2/2!}{x}, \frac{x^3/3!}{x^2/2!}, \ldots$. Show that these have 0 as a limit. Thus the series converges by Theorem 9 for any choice of $x$.

Problem 105. Next consider the series $1 + x + x^2 + \ldots$. As before find the sequence which is made up of ratios of consecutive terms. What would you say is the limit of this sequence? One of the facts about sequences is that a sequence which repeats the same number forever has a limit which is that number. So the limit of our sequence is $x$. If $|x| < 1$, then the series converges and if $|x| > 1$ it does not converge by Theorem 9. So we know that the series converges for $-1 < x < 1$, and it does not converge if $x > 1$ or $x < -1$. The ratio test does not help in case $x = 1$ or $x = -1$. To see what happens in this case, return to the original series and substitute 1 for $x$ and then $-1$ for $x$ and look at the series of partial sums $s_1, s_2, \ldots$ where for each positive integer $n$, $s_n = 1 + x + x^2 + \ldots + x^n$. 

William S. Mahavier

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