Place-Value Model for the Numbers

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To the Instructor

These notes are intended for a course in which students may be proving theorems on their own for the first time. The mathematical content addresses the question “Can we give meanings for ‘number’, ‘<’, ‘+’, and ‘∗’ so that the axioms for the numbers are consequences of the definitions?”

In these notes, a place-value model is pursued. The students are given that the natural numbers exist, and functions from the natural numbers into a two-element set become the objects of study.

The problem set is designed so that even a class that plods will be exposed to ideas of comparing sets and imposing an order on a set. A class that experiences success from the beginning can be expected to get at least to the point of recognizing that the model is Dedekind complete, and experience the difficulties of imposing an algebra driven by algorithms that are not universally defined. I teach these notes as a one-semester course. I routinely have classes that prove the order complete, and get some of the set comparisons, but I have never had a class get entirely through the imposition of the algebra on the model. These notes stop short of an entire model for the numbers. Students that can get through most of these problems will have the maturity to do mathematics of more importance (perhaps computer science majors could derive major benefit) than what would follow. Nevertheless, for a teacher determined to pursue this inquiry to its “completion”, I have given a brief description at the end of Section 3 of what might come next.

The storyline for the course goes like this:

• Defining a place-value number consists of specifying the digit in each place value.
• With the comparison principle “find the first place-value in which the objects are different and use it for comparison”, an order is defined, but it contains pairs of elements so that there is no object between them.
• Having corrected this flaw, the set admits an order with no minimum and no maximum, and with the Dedekind cut property.
• Meanwhile, we are also counting sets. The natural numbers are shown to be infinite.
• The natural numbers are shown to be as large as the set of elements whose “partners” were purged to eliminate the “holes”, and this set is shown to be dense in the order.
• Segments are shown to be as large as the entire set.
• The natural numbers are shown to be not as large as this set, or for that matter, any Dedekind-complete set.
• Using the addition algorithm from grade school arithmetic, and being careful, an algebra is imposed.

• With this algebra, the set admits an order-preserving local semigroup which lacks an identity, but in which one can solve all of the “tractable” equations.

Since progress through these notes depends on the students finding the conjectures which are not theorems and then addressing the issues raised in trying to produce structures about which the conclusions are true, the instructor must exercise some care in when access to subsequent problems is granted to the students. Also, students often uncover theorems while working on problems, and theorems can often be sifted out of students’ arguments; these make nice addenda to the notes. I have included some examples from a past class of mine in Section 2. Section 3 contains remarks about particular problems or definitions, and possible timing schemes for presenting the problems. The order in which I have listed the problems need not be the optimal sequence for a particular class.

When I first taught a course of this type, I began by giving some background in logic. I no longer do this. The approach I use now is to give out a sheet on quantification when the course begins, and deal with points of logic as they arise. When students argue correctly, they give lectures as good as yours. Particular points of logic may be emphasized after students finish arguments, by going back and focusing on a part of the argument that used logic in a particular way. Correcting logically impaired arguments affords great opportunities for teaching the logic; the mistakes that students make often reflect the misunderstandings of other students in the class.

The instructor will also have available the opportunities that occur as students deal with the set of natural numbers, which is assumed to exist along with its arithmetic, and in terms of which the objects of the model are defined. In this course, the students typically address structures of the natural numbers as they work on the counting problems. The notions of odd and even, the fact that \( \mathbb{N} \) is well-ordered by \(<\), the infinitude of the primes, and the “uniqueness” of a prime factorization of a natural number, routinely appear in the students’ work. I usually demand a definition for odd and even, since the experience of formalizing “can be written as \( \ldots \)” affords an opportunity for the student to consciously use quantification. Discussion of \(<\) naturally occurs the first time finite induction is used (or is appropriate). I let the class have unique prime factorization; a student clever enough to construct a counting argument based on unique factorization deserves to be rewarded. Once the class has shown that a proper subset may command its superset, I am willing to grant the infinitude of the primes, and even to share an argument for it.

I give only one test in this course: the final examination. I offer one credit each time a student presents an argument for a problem that the class judges as being correct. If a student has a problem that someone else presents, that student is allowed to turn in her/his write-up at the end of the class period in which the problem was finished. If the write-up is correct, the student gets one-half of a credit. The final examination is given as a take-home, “use your notes but nobody else’s” test. It consists of a section of problems solved during the semester, and a section of problems that the students have not yet solved and may not even have seen. Successful proofs for the problems proven during the course allow students to earn a grade of “C”, or to keep whatever grade their work during the semester warranted. Successful work on the second section allows a student to increase her/his grade, or to atone for slip-ups on the first section.
To the Student

Introduction to students - An Introduction to Doing Mathematics

In this course, not only will you be responsible for understanding why the mathematics we cover is correct, but the responsibility for discovery will also be assigned to the class. One of the immediate results of this responsibility for doing mathematics yourself rather than just learning how someone else did it, will likely be an acute awareness of the difference between the challenge associated with understanding why something is correct, and discovering for yourself whether or not a conjecture is a theorem.

Doing mathematics can be extremely exhilarating when one succeeds in the discovery process; failing to do mathematics when one is putting in the time trying to do mathematics can be extremely frustrating. This introduction is intended to alert you to some tips that are designed to optimize the chances for success.

First, you must put in the time necessary to give your creative intelligence a chance to work. Flashes of insight typically occur after information is organized and mulled over. Commitment to solving problems often leads to help from the subconscious. Students often tell me that they got “the big idea” while walking across campus or after turning in for the night.

Second, solutions to problems need not come all at once. You may need to solve many small problems on the way to proving a theorem or disproving an incorrect conjecture. Some of the most important work in mathematics is the creation of technique. Take pride in progress toward a goal, as well as reaching the goal. Any information you uncover is more than you knew before, and solving a problem is usually just a matter of putting together enough small solutions to allow you to see why the big problem is correct.

Students often tell me that they would be glad to put in the time if they just knew where to start. The following scheme is offered toward that end.

The awareness stage:

1. Identify all the words in the problem, and make sure that you know the definition of each of them. Try to recall examples that have dealt with these notions before. If a definition is new, make some examples for the definition.

2. Identify any theorems that may have already dealt with ideas present in the problem.
Put techniques that gave rise to proofs in those contexts firmly in mind.

The direct approach:

3. Make an example that models the hypothesis to the problem, and try to show that the example exhibits the properties of the conclusion. (If you can prove that your example fails to have the properties of the conclusion, you will have shown that the problem is not a theorem!)

4. See if what allowed you to establish the conclusion in the example is a property of all examples covered by the hypothesis. If it is, write a proof. If not . . .

5. . . . make an example which models the hypothesis, but fails to have whatever special properties you used to get the conclusion in the previous example. Go to 3.

The indirect, or contrapositive approach:

6. Suppose that the conclusion is false, and try to show that the hypothesis must be false as well. If the problem is not a theorem, any conclusions you get must be qualities which an example that disproves the conjecture must have.

7. Try to be aware of properties that if they were added to the hypothesis, would guarantee the conclusion. Alternatively, you might also try to find conclusions that follow from the hypothesis, even if they do not include the one you seek. Even if you are not able to solve the problem as stated, you may be able to create a substitute theorem.

The main mindset is to be aware that even when arguments do not come quickly or easily, the hunt itself may be an important learning experience. Working on problems yourself is the central ingredient. Not only will it provide you with theorems that are “your own”, but even when someone beats you to a solution, it will put you in a much stronger position to analyze the argument given.

A Theory of Sets and Ordered Pairs

We will not create an axiomatic set theory. However, following is an idiomatic presentation of some conventions that axiomatic set theory implies. We presuppose the existence of formal English as a language for expressing properties.

The primitive words are “set”, “element”, “ordered pair”, “first coordinate”, and “second coordinate”.

(i.) A set consists of an element or elements.

(ii.) An element of a set, and the set consisting of that element, are different objects.

(iii.) A set is defined by stating the properties its elements have. (The plural has been chosen here, but the definition of a set may be made by stating a single property, and a set may have a single element.)
(iv.) Given a definition for a set, any object having the properties specified is an element of the set, and any element of the set has the properties specified in the definition.

(v.) An ordered pair consists of a first coordinate and a second coordinate.

(vi.) An ordered pair’s first coordinate may be the same set-theoretic object as the second coordinate, but as a part of the ordered pair, being the first coordinate is distinguishable from being the second coordinate.

We reserve a notation for the creation of definitions of sets, and for defining ordered pairs.

Reserved symbols for definitions of sets are \{ : \}. A symbol is created to follow the open brace and precede the colon, and then properties that an element must have are stated in terms of that symbol after the colon and before the closed brace. Thus

\{ x : x \text{ is a number and } x > 5 \}

stands for “the set to which an element belongs provided that it is a number and it is greater than 5”.

Reserved symbols for definitions of ordered pairs are \( ( , ) \). The first coordinate of the ordered pair is written after the open parenthesis and before the comma; the second coordinate of the ordered pair is written after the comma and before the closed parenthesis. Thus, \( (p, 5) \) stands for the ordered pair whose first coordinate is \( p \) and whose second coordinate is 5.

The purpose of this course is to build a model for the numbers. Our ultimate goal is to prove that the statements which are typically taken as axioms for the numbers are theorems in our model. In an axiomatic treatment, “number”, “<”, “+”, and “∗” are taken as primitive words; thus we provide definitions within the model so that if they are interpreted as the primitive words, the statements made by replacing the primitive words in the axioms with their analogues in the model become the topics of consideration.

You may assume that the natural numbers exist, and have whatever properties number theory says they do. If there is doubt about a property of the natural numbers, we will either prove the property or indicate what property we are assuming.
Chapter 1

Problem Sequence

**Definition 1**: Suppose that each of $X$ and $Y$ is a set. The statement that $f$ is a *function* from $X$ into $Y$ means that $f$ is a set so that

(i.) each element of $f$ is an ordered pair whose first coordinate is an element of $X$ and whose second coordinate is an element of $Y$; and

(ii.) if $p$ is an element of $X$, then there is an element of $f$ whose first coordinate is $p$; and

(iii.) if $p$ and $q$ are elements of $f$, then the first coordinate of $p$ is not the first coordinate of $q$.

**Notation**: If $f$ is a function from $X$ into $Y$, and $(p, q)$ is an element of $f$, then we may write $f(p) = q$.

**Definition 2**: Suppose that each of $X$ and $Y$ is a set, and that $f$ is a function from $X$ into $Y$. The statement that $M$ is the *range* of $f$ means that $M$ is the set to which an element belongs provided that there is an element of $f$ of which it is the second coordinate.

**Problem 1**: Suppose that $X$ is a set with more than one element. (That $X$ has more than one element means that if $p$ is an element of $X$, then there is an element of $X$ different from $p$.) Show that the set to which an element belongs provided that it is an ordered pair whose first coordinate is an element of $X$ and whose second coordinate is an element of $X$, is not a function from $X$ into $X$.

**Definition 3**: Suppose that $X$ is a set, and that $L$ is a set each element of which is an ordered pair whose first coordinate is an element of $X$ and whose second coordinate is an element of $X$. The statement that $L$ is an *order* on $X$ means that

(i.) if $p$ is an element of $X$, then $(p, p)$ is not an element of $L$; and

(ii.) if $p$ and $q$ are elements of $X$, then $(p, q)$ is an element of $L$ or $(q, p)$ is an element of $L$; and

(iii.) if $(p, q)$ and $(q, r)$ are elements of $L$, then $(p, r)$ is an element of $L$.

**Problem 2**: Suppose that $X$ is a set with exactly one element. Show that there is no order on $X$. 


Problem 3: Suppose that $X$ is a set with more than one element, and that $L$ is an order on $X$. Show that $L$ is not a function from $X$ into $X$.

Definition 4: $U$ is the set to which an element belongs provided that it is a function from the natural numbers into $\{0, 1\}$ so that its range is $\{0, 1\}$.

Definition 5: Suppose that $f$ and $g$ are elements of $U$. The statement that $f$ precedes $g$ means that if $n$ is the smallest natural number in $\{k : f(k) \neq g(k)\}$, then $f(n) = 0$ and $g(n) = 1$.

Definition 6: $G = \{(p, q) : p$ is an element of $U; q$ is an element of $U; and p precedes $q\}$.

Problem 4: Suppose that $x$ is an element of $U$. Show that there is an element of $U$ (call such an element $y$) so that $(x, y)$ is an element of $G$.

Problem 5: Suppose that $x$ is an element of $U$. Show that there is an element of $U$ (call such an element $y$) so that $(y, x)$ is an element of $G$.

Problem 6: Suppose that $x$ and $y$ are elements of $U$, and $(x, y)$ is an element of $G$. Show that there is an element of $U$ (call such an element $w$) so that $(x, w)$ and $(w, y)$ are elements of $G$.

Problem 7: Show that $G$ is an order on $U$.

Definition 7: Suppose that each of $X$ and $Y$ is a set. The statement that $X$ commands $Y$ means that there is a function from $X$ into $Y$ whose range is $Y$.

Problem 8: Show that $U$ commands the natural numbers.

Definition 8: Suppose that each of $X$ and $Y$ is a set. The statement that $X$ is a subset of $Y$ means that if $p$ is an element of $X$, then $p$ is an element of $Y$.

Problem 9: Suppose that each of $X$ and $Y$ is a set, and that $X$ is a subset of $Y$. Show that $Y$ commands $X$.

Problem 10: Suppose that $X$ and $Y$ are sets, and that $X$ is a subset of $Y$. Show that it is not the case that $X$ commands $Y$.

Definition 9: Suppose that each of $X$ and $Y$ is a set, and that there is an element of $X$ which is an element of $Y$. The intersection of $X$ with $Y$ is

$$\{x : x \text{ is an element of } X \text{ and } x \text{ is an element of } Y\}.$$  

Notation: $X \cap Y$ stands for “the intersection of $X$ with $Y$”.

Definition 10: Suppose that $X$ is a set, $L$ is an order on $X$, and $a$ and $b$ are elements of $X$ so that $(a, b)$ is an element of $L$, and there is an element of $X$, call such an element $c$, so that $(a, c)$ is an element of $L$ and $(c, b)$ is an element of $L$. The segment from $a$ to $b$ by $L$ is $\{x : (a, x)$ is an element of $L$ and $(x, b)$ is an element of $L\}$.

Notation: If $(a, b)$ is an element of the order $L$, $(a, b)$ stands for “the segment from $a$ to $b$ by $L$”.

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Problem 11: Suppose that \( x \) is an element of \( U \). Show that there is a segment by \( G \) so that \( x \) is an element of it.

Problem 12: Suppose that \( X \) is a set, \( L \) is an order on \( X \), \( (p,q) \) and \( (r,s) \) are segments by \( L \), and \( x \) is an element of \( (p,q) \cap (r,s) \). Show that \( (p,q) \cap (r,s) \) is a segment by \( L \).

Problem 13: Suppose that \( x \) and \( y \) are elements of \( U \) and that \( G \) is an order on \( U \). Show that there are segments by \( G \), call them \( P \) and \( Q \), so that

(i.) \( x \) is an element of \( P \),
(ii.) \( y \) is an element of \( Q \), and
(iii.) if \( w \) is an element of \( P \), then \( w \) is not an element of \( Q \).

Definition 11: Suppose that each of \( X \) and \( Y \) is a set. The union of \( X \) with \( Y \) is \( \{ p : p \) is an element of \( X \) or \( p \) is an element of \( Y \} \).

Notation: \( X \cup Y \) stands for “the union of \( X \) with \( Y \)”.

Definition 12: Suppose that \( X \) is a set, \( L \) is an order on \( X \), and \( T \) and \( V \) are subsets of \( X \). The statement that \( (T,V) \) is a cut of \( X \) by \( L \) means that

(i.) \( T \cup V = X \); and
(ii.) if \( x \) is an element of \( T \), and \( y \) is an element of \( V \), then \( (x,y) \) is an element of \( L \).

Problem 14: Suppose that \( X \) is a set, \( L \) is an order on \( X \), and \( (A,B) \) is a cut of \( X \) by \( L \). Show that if \( p \) is an element of \( A \), then \( p \) is not an element of \( B \).

Problem 15: Suppose that \( X \) is a set, and \( L \) is an order on \( X \). Show that there is a cut of \( X \) by \( L \).

Problem 16: Suppose that \( X \) is a set, \( L \) is an order on \( X \), and \( (p,q) \) is an element of \( L \). Show that there is a cut of \( X \) by \( L \) (call it \( (A,B) \)) so that \( p \) is an element of \( A \) and \( q \) is an element of \( B \).

Definitions 13: Suppose that \( X \) is a set, \( L \) is an order on \( X \), \( p \) is an element of \( X \), and \( M \) is a subset of \( X \). The statement that \( p \) is the max of \( M \) by \( L \) means that \( p \) is an element of \( M \), and if \( q \) is an element of \( M \) different than \( p \), then \( (q,p) \) is an element of \( L \). The statement that \( p \) is the min of \( M \) by \( L \) means that \( p \) is an element of \( M \), and if \( q \) is an element of \( M \) different than \( p \), then \( (p,q) \) is an element of \( L \).

Definition 14: Suppose that \( X \) is a set, and \( L \) is an order on \( X \). The statement that \( L \) has the Dedekind cut property means that if \( (A,B) \) is a cut of \( X \) by \( L \), then

(i.) \( A \) has a max by \( L \), or \( B \) has a min by \( L \); and
(ii.) it is not the case that both \( A \) has a max by \( L \), and \( B \) has a min by \( L \).

Problem 17: Suppose that \( L = \{(x,y) : x \) is a natural number, \( y \) is a natural number, and \( x < y \} \). Show that \( L \) does not have the Dedekind cut property.

Definition 4′: \( U' = \{ x : x \) is an element of \( U \}; \) and if \( n \) is a natural number, then there is a natural number greater than \( n \), call it \( m \), so that \( x(m) = 1 \).
Problem 25: Show that the natural numbers do not command $G'$. 

Problem 23: Suppose that $x$ is an element of $U'$. Show that there is an element of $U'$, call such an element $y$, so that $(x,y)$ is an element of $G'$. 

Problem 24: Suppose that $x$ is an element of $U'$. Show that there is an element of $U'$, call such an element $y$, so that $(x,y)$ is an element of $G'$. 

Problem 26: Suppose that $x$ and $y$ are elements of $U'$, and $(x,y)$ is an element of $G'$. Show that there is an element of $U'$, call such an element $w$, so that $(x,w)$ and $(w,y)$ are elements of $G'$. 

Problem 7: Show that $G'$ is an order on $U'$. 

Problem 8: Show that $U'$ commands the natural numbers. 

Problem 11: Suppose that $G'$ is an order on $U'$, and that $x$ is an element of $U'$. Show that there is a segment by $G'$ so that $x$ is an element of it. 

Definition 15: $D = \{x : x \text{ is an element of } U', \text{ and there is a natural number, call such a natural number } n, \text{ so that if } k \text{ is a natural number greater than } n, \text{ then } x(k) = 1\}$. 

Problem 18: Suppose that $x$ and $y$ are elements of $U'$, and $(x,y)$ is an element of $G'$. Show that there is an element of $D$, call such an element $w$, so that $(x,w)$ and $(w,y)$ are elements of $G'$. 

Problem 19: Show that the natural numbers command $D$. 

Problem 20: Show that $U'$ commands $U$. 

Problem 21: Show that $G'$ has the Dedekind cut property. 

Problem 22: Suppose that $C$ is a function from the natural numbers into $U'$, and that $x = \{(p,q) : p \text{ is a natural number, } q \text{ is an element of } \{0,1\}, \text{ and } q \text{ is not } C(p)(p)\}$. Show that $x$ is not an element of the range of $C$. 

Problem 23: Show that the natural numbers do not command $U'$. 

Problem 24: Suppose that $x$ is an element of $U'$, and $n$ is a natural number. Show that $\{(p,q) : p \text{ is a natural number; and if } p < n, \text{ then } q = x(p); \text{ or if } p = n, \text{ then } q = 0; \text{ or if } p > n, \text{ then } q = 1\}$ is an element of $U'$. 

Problem 25: Suppose that $(x,y)$ is an element of $G'$. Show that $(x,y)$ commands $U'$. 

Definition 16: Suppose that $X$ is a set, $L$ is an order on $X$, and $(p,q)$ is an element of $L$. The interval from $p$ to $q$ by $L$ is $\{x : x \text{ is an element of } [p,q], \text{ or } x = p, \text{ or } x = q\}$. 

Notation: If $(p,q)$ is an element of the order $L$, $[p,q]$ stands for “the interval from $p$ to $q$ by $L$”. 

Problem 26: Suppose that $M = \{x : \text{ there is an element of } G', \text{ call it } (p,q), \text{ so that } x = [p,q]\}$. Show that there is a function from the natural numbers into $M$, call such a function $f$, so that if $n$ is a natural number, then $f(n+1)$ is a subset of $f(n)$.
Problem 27: Suppose that \((A,B)\) is an element of \(G'\), and \(x\) is an element of \(U'\), so that \(x\) is an element of \((A,B)\). Show that there is an element of \(G'\), call it \((p,q)\), so that \([p,q]\) is a subset of \(\overline{A,B}\), and \(x\) is not an element of \([p,q]\).

Problem 28: Suppose that \(M = \{x: \text{there is an element of } G', \text{call it } (p,q), \text{so that } x = [p,q]\}\); \(s\) is a function from the natural numbers into \(M\) so that if \(k\) is a natural number, then \(s(k+1)\) is a subset of \(s(k)\); and \(A = \{x: \text{there is a natural number, call it } k, \text{so that if } p \text{ is an element of } s(k), \text{then } (x,p) \text{ is an element of } G'\}\). Show that \((A,\{x:x \text{ is an element of } U' \text{ and } x \text{ is not an element of } A\})\) is a cut of \(U'\) by \(G'\).

Problem 29: Suppose that \(M = \{x: \text{there is an element of } G', \text{call it } (p,q), \text{so that } x = [p,q]\}\); and \(s\) is a function from the natural numbers into \(M\) such that if \(k\) is a natural number, then \(s(k+1)\) is a subset of \(s(k)\). Show that there is an element of \(U'\), call it \(w\), so that if \(k\) is a natural number, then \(w\) is an element of \(s(k)\).

Problem 30: Suppose that \(X\) is a set, and that \(L\) is an order on \(X\) so that \(L\) has the Dedekind cut property. Show that the natural numbers do not command \(X\).

Definition 17: \(pc = \{(0,0), (0,0)\}, \{(0,1), (0,1)\}, \{(1,0), (1,0)\}, \{(0,0), (0,1)\}, \{(1,0), (1,0)\}, \{(0,1), (1,1)\}, \{(1,0), (1,1)\}, \{(1,1), (1,1)\}\)

Problem 31: Show that \(pc\) is a function from \(\{(x,y):x \text{ is an element of } \{p,q\}, p \text{ is an element of } \{0,1\}\}\), and \(y\) is an element of \(\{0,1\}\), into \(\{(x,y):x \text{ is an element of } \{0,1\}\}\), and \(y\) is an element of \(\{0,1\}\).

Definition 18: Suppose that \((x,y)\) is an ordered pair. The projection of \((x,y)\) into its first coordinate is \(x\), and the projection of \((x,y)\) into its second coordinate is \(y\).

Notation: Suppose that \(p\) is an ordered pair. \(\Pi_1 p\) stands for “the projection of \(p\) into its first coordinate”, and \(\Pi_2 p\) stands for “the projection of \(p\) into its second coordinate”.

Definition 19: Suppose that \(m\) is a natural number, and each of \(x\) and \(y\) is a function from \(\{k: k \text{ is a natural number and } k \leq m\}\), into \(\{0,1\}\). \(\rho ((x,y)) (m) = \Pi_2 pc((x(m),y(m)),0))\), and if \(t\) is a natural number so that \(t < m\), and \(\rho ((x,y)) (t+1) = \Pi_2 pc((x(y(t)+1),y(y(t+1))),\Pi_1 pc((x(t+1),y(t+1)),w))\))

Problem 31': Suppose that \(m\) is a natural number, and each of \(x\) and \(y\) is a function from \(\{k: k \text{ is a natural number, and } k \leq m\}\), into \(\{0,1\}\). Show that \(\rho ((x,y))\) is a function from \(\{k: k \text{ is a natural number, and } k \leq m\}\), into \(\{0,1\}\).

Definition 20: \& = \{((x,y),z): \text{each of } x \text{ and } y \text{ is an element of } U', \text{and } z \text{ is an element of } U' \text{ so that if } m \text{ is a natural number, then there is a natural number, call it } n, \text{so that if } n' > n, \text{then } \{(s,t): s \leq m \text{ and } t = z(s)\} \subset \rho (((s,t): s \leq n' \text{ and } t = x(s)),\{(s,t): s \leq n' \text{ and } t = y(s)\}};\text{and if } \rho (((s,t): s \leq n' \text{ and } t = x(s)),\{(s,t): s \leq n' \text{ and } t = y(s)\})\} = pc((x(1),y(1)),w))\), then \(\Pi_1 pc((x(1),y(1)),w)) = 0\}.

Problem 32: Show that “\&” is a function from \(\{(x,y):x \text{ is an element of } U', \text{and } y \text{ is an element of } U'\}\), into \(U'\).

Problem 33: Suppose that \(((x,y),z)\) is an element of “\&”. Show that \(((y,x),z)\) is an element of “\&”.

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Problem 34: Suppose that “&” is a function from \( \{(x,y) : x \text{ is an element of } U', \text{ and } y \text{ is an element of } U'\} \) into \( U' \); and that each of \( x, y, \) and \( z \) is an element of \( U' \). Show that \( &((x,&((y,z)))) = &(&(x,y),z)) \).

Problem 35: Show that there is an element of \( U' \), call it \( z \), so that if \( x \) is an element of \( U' \), then \( ((x,z),x) \) is an element of “&”.

Problem 36: Suppose that \( x \) and \( y \) are elements of \( U' \) so that \( ((x,y),z) \) is an element of “&”. Show that \( (x,z) \) is an element of \( G' \).

Problem 37: Suppose that \( x \) and \( y \) are elements of \( U' \) so that \( (x,y) \) is an element of \( G' \). Show that there is exactly one element of \( U' \), call it \( w \), so that \( ((x,w),y) \) is an element of “&”.

Definition 21: \( E = \{(x,y) : \text{there is an element of } U', \text{ call it } z, \text{ so that } ((x,y),z) \text{ is an element of "&"}\} \).

Problem 32: Show that “&” is a function from \( E \) into \( U' \).

Problem 34': Suppose that \( (x,&((y,z))) \) is an element of \( E \), and \( &(&(x,y),z) \) is an element of \( E \). Show that \( &((x,&((y,z)))) = &(&(x,y),z)) \).

Problem 38: Suppose that \( (x,y) \) is an element of \( E \), and \( (w,x) \) is an element of \( G' \). Show that \( (w,y) \) is an element of \( E \).
Chapter 2

Examples

Following are some theorems that students proved during a past offering of the course, for which a selection from the problems above formed the corpus from which they worked. The problem from the notes which each addresses is noted in parentheses.

**Theorem (6′):** Suppose that \(x\) and \(y\) are elements of \(U′\), \(x\) precedes \(y\), \(n\) is the least natural number so that \(x(n) \neq y(n)\), and \(p\) is a natural number greater than \(n\) so that \(y(p) = 1\). Then \(\{(s, t) : s\) is a natural number; and if \(s \neq p\), then \(t = y(s)\), or if \(s = n\), then \(t = 0\}\) is an element of \(U′\) so that \(x\) precedes it, and it precedes \(y\).

**Theorem (21):** Suppose that \((P, Q)\) is a cut of \(U′\) by \(G′\). Then if \(P\) has a maximum by \(G′\), then \(Q\) does not have a minimum by \(G\); or if \(Q\) has a minimum by \(G′\), then \(P\) does not have a maximum by \(G′\).

**Theorem (21):** Suppose that \((P, Q)\) is a cut of \(U′\) by \(G′\), \(n\) is a natural number, and \(s\) is a function from \(\{k : k\) is a natural number no greater than \(n\}\) into \(\{0, 1\}\) so that

(i.) \(s(n) = 0\);

(ii.) if \(a\) is an element of \(P\), then \(\{(j, k) : j\) is a natural number no greater than \(n,\) and \(k = a(j)\}\) = \(s\); or there is an element of \(P\), call such an element \(a′\), so that \(\{(j, k) : j\) is a natural number no greater than \(n, and k = a′(j)\}\) = \(s\), and \(a\) precedes \(a′\); and

(iii.) if \(b\) is an element of \(Q\), then \(b(n) = 1\), or there is an element of \(Q\), call such an element \(c\), so that \(c(n) = 1\) and \(c\) precedes \(b\).

Then \(\{(j, k) : j\) is a natural number; and if \(j \leq n\), then \(k = s(j)\); or if \(j > n, k = 1\}\) is an element of \(U′\) so that if \(a\) is an element of \(P\) different than it, then \(a\) precedes it; and if \(b\) is an element of \(Q\) different than it, then it precedes \(b\).

**Theorem (25):** (The class called this the “bead-chain” theorem.) Suppose that \(x\) and \(y\) are elements of \(U′\), \(x\) precedes \(y\), \(a\) is the least natural number so that \(x(a)\) is not \(y(a)\), \(b\) is a natural number so that \(b > a\) and \(y(b) = 1\), and \(ES = \{p : there is an element of \(U′, call it \(w\), so that \(p = \{(c, d) : c\) is a natural number; and if \(c < b, then \(d = y(c)\); or if \(c = b, then \(d = 0\); or if there is a natural number, call it \(e\), so that \(c = b + e, then \(d = w(e)\}\}\}. Then \(ES\) is a subset of \(\{(x, y)\}\), and \(ES\) commands \(U′\).
Theorem (25): Suppose that each of $X$, $Y$, and $Z$ is a set; $X$ commands $Y$; and $Y$ commands $Z$. Then $X$ commands $Z$. 
Chapter 3

Remarks

The following comments contain information about my intent for many of the problems, and experiences that my students have had with them.

Definition 1: Notice that the idea of “domain” is included in the definition of “function” by proviso ii.

Definition 2, Problem 1, and Definition 4: I have written the words for definitions of the sets in question here. Students often translate these to the notation for these words suggested back on the section about sets. If they don’t, I usually suggest that they see if they can.

Problems 1 and 3: The students I teach in this course are typically very naïve, and seldom have been forced to deal with the need to say things carefully. I put Problems 1 and 2 in the notes to address the fact that early in every course, a student would define a “function” as \{ (x, y) : x is an element of X and y is an element of Y \} and then claim whatever additional properties he or she needed as he or she needed them. When it happens now, I can point to Theorem 1 and start the discussion there. Problem 3 reinforces “not every set of ordered pairs whose coordinates are in the right sets is a function”.

Problem 2: This problem is here to emphasize that in this set theory, sets have at least one element each. Discussion of the “empty set” will naturally occur here.

Problems 4, 5, 6, and 7: These are all properties assumed about < on \mathbb{R}. Although U will be modified when it fails to yield a density property for “precedes”, the proofs made in U for 4, 5, and 7 usually go over to the subsequent modification.

Problem 6: This is the first false conjecture since, for instance,

\{(x, y) : x is a natural number; and if x = 1, then y = 1, or if x > 1, then y = 0\}

and,

\{(x, y) : x is a natural number; and if x = 1, then y = 0, or if x > 1, then y = 1\}

have nothing between them.
Historically, the resolution to this problem was to identify each such pair as representing a single element. I prefer to emphasize that in a model, different objects must be different, and to continue with a subset of the objects with which we are working.

**Definition 7:** “Commands” is the concept central to counting. The Schroeder-Bernstein Theorem, which is not addressed in this course, guarantees that it is sufficient to admit the classical results.

**Problems 9 and 10:** Although all students in your class will know that squaring maps $[1, 2]$ onto $[1, 4]$, they will not likely realize that this precludes Problem 10 from being a theorem. Indeed, most of the time classes try to prove Problem 10 and the argument includes as its punch line something equivalent to “because the containing set contains more elements than the subset”. This affords a marvelous opportunity to clarify the difference between ordinary language and formal language, since “more” in the subset sense turns out to be different than “more” in the counting sense (sometimes!). The questions that show incorrect arguments are incorrect often lead to the example $n + 1 \rightarrow n$.

**Problem 11:** If Problem 7 has been done at this time, use Problem 11′ here instead.

**Problem 12:** This problem virtually guarantees a case argument will be forthcoming.

**Problem 13:** Since Problem 6 is not a theorem, then if this problem is solved, it will be done using the order structure. Thus, together with Problem 12, we get that segments defined by an order with neither max nor min create a basis for a Hausdorff topology on the set on which the order is defined.

**Definitions 12, 13, and 14:** In an axiom system for the numbers, one has (at least) the choice of the greatest lower bound property, the Bolzano-Weierstrass property, the Heine-Borel property, or the Dedekind cut property, as a completeness axiom. I choose the Dedekind cut property, since it can be articulated without reference to any structure other than the order itself. This is the first idea in the course that the students are likely not to have encountered in another context.

**Problems 15 and 16:** These problems are usually solved using cuts exhibiting the Dedekind cut property, thus establishing a pretext for asking “must all cuts be like these?”

**Problem 17:** The order “<”, as we find it in counting, does not have the Dedekind cut property.

**Definitions 4′ and 6′, and Problems 4′ - 8′:** Problem 6 will be solved at some point. The example that shows Problem 6 is not a theorem will display two elements of $U$ that have nothing between them. Often students have shown that elements of a particular type do have the property before finding counterexamples, and sometimes the student finding a counterexample will show that any “such pair of elements” fails to have the property. Since the only pairs of elements which fail the property have the “all the rest 0s” and the “all the rest 1s” property, the instructor will have in hand at least an example that makes the structure in Definition 4′ plausible. I have placed Problems 4′ - 8′ after Problem 17 only because it has been typical in my experience that Problem 6 gets solved before Problems 15 - 17 get solved. Whenever Problem 6 gets solved, it is time for Definitions 4′ and 6′ and Problems 4′ - 8′. Until Problem 6 gets solved,
it is not time for Definitions 4′ and 6′ or Problems 4′ - 8′. An interesting sidelight is to see which constructions from the solutions to 4 - 8 give elements of $U'$. These problems offer an outstanding context to point out the power of “some arguments”, and always afford at least an opportunity to show how to modify an argument to meet new conditions.

Problems 18 and 19: $U'$ has a countable dense subset.

Problem 4′ - 7′, 18, 19, and 21: Collectively, these problems will demonstrate that $U'$ and $G'$ model the geometry axioms for the numbers and “<”. A class that gets this far has a decent foundation for studying the topology of ordered sets.

Problems 22 and 23: This is the scheme for Cantor’s proof that the (place-value model for the) numbers are (is) not denumerable. Some technical care is necessary to ensure that the construction of “x” from 22 is modified to ensure that the object that is made is in $U'$, in order to make it work for 23.

Problem 24: This problem has usually been done in the context of solving 4′ - 8′ (sometimes even earlier), so it may not need to be stated. It is offered here since its construction technique is viable in 25, thus making it a nice lemma for 25.

Problems 26 - 30: These problems establish that by mimicking something that you can do in $U'$, any set which admits an order with the Dedekind cut property must be non-denumerable. Their inclusion is dependent on whether you wish to concentrate on the model itself (leave them out), or seize the opportunity to illustrate the power of the type of thinking that the students have been doing (include them).

Definitions 17 - 20: These definitions formalize the place-value addition algorithm. 17 is the digit arithmetic and the carry, 18 creates a notation that will distinguish digit from carry, and 19 manages truncation. 20 matches addition of “terminating decimals” (the ones purged from the system after 6 turned out not to be a theorem) to elements of $U'$. 19 involves finite induction, so if the issue has not come up before now, here is a chance to teach it.

Problem 32: Although “&” is “single-valued”, it is not an operation on all of $U'$, since any pair of elements which pair 1 with 1 violate the “carry” property (if $\rho((x,y))(1) = pc((x(1),y(1),w))$, then $\Pi_1 pc(((x(1),y(1)),w)) = 0$). Students may also find ways to exclude 0 or to exclude 1 from the range of a “prospective answer”. This is the first evidence that $U'$ might not be the numbers, since the geometry for the numbers is in place. If the former example is found, the problem on which equations have solutions, 37, is motivated. If the other type of example is found, the plausibility of this model only being the segment from 0 to 1 is established.

Problems 33 - 35: Wherever it is defined, “&” has the commutative and associative properties, but does not have an identity. In showing no identity, the solver will likely show that if the function that pairs every natural number with 0, even though it isn’t in $U'$, works as an identity when the rule from the algorithm for “&” is applied.

Problem 36: If 36 is solved before 35, it precludes 35 from being a theorem.

Problems 32′: The function “&”, even though it is not defined on all of $U' \times U'$, is at least single-valued wherever it makes sense.
Problem 38: If 32 is solved by showing the “carry in the first place” is violated, 38 provides an opportunity to show that not having range \{0, 1\} (alternatively, having range \{1\}) is a possible consequence of applying the algorithm.

What would come next? Using \(\rho\), the algorithm for multiplication can be defined, and the resulting multiplication is an order-reversing, commutative, and associative quasigroup on \(U'\) which distributes over “&” wherever “&” makes sense. Search for an identity would show that the function that pairs every natural number with 1 would do the job if it were an element of the set on which the algebra acts.

Another tack to take is to extend \(U'\) by making element of \(U''\) mean element of \(U'\), a natural number, or ordered pair whose first coordinate is a natural number and whose second coordinate is an element of \(U'\). The order is extended lexicographically using “<” on the natural numbers in the first coordinate, and \(G\) in the second. The function “&” is extended by having elements of \((N \times N) \times (U' \times U')\), paired with pairs whose first coordinates are the natural number sums or the natural sums plus 1, depending on whether or not the element of \(U' \times U'\) is in \(E\).
Chapter 4

Some Axioms for the Numbers

The primitive words are “number”, “<”, “+”, and “∗”.

Axiom G1: “<” is an order on the set of numbers.

Axiom G2: It is not the case that the set of numbers has a min by “<”, and it is not the case that the set of numbers has a max by “<”.

Axiom G3: There is a sequence in the set of numbers, call it $Q$, so that if $x$ and $y$ are numbers, then there is a natural number, call it $k$, such that $x < Q(k)$ and $Q(k) < y$.

Axiom G4: “<” has the Dedekind cut property.

Axiom A1: If each of $x$ and $y$ is a number, then $x + y$ is exactly one number, and $x ∗ y$ is exactly one number.

Axiom A2: If each of $x$ and $y$ is a number, then $x + y = y + x$, and $x ∗ y = y ∗ x$.

Axiom A3: If each of $x$, $y$, and $z$ is a number, then $x + (y + z) = (x + y) + z$, and $x ∗ (y ∗ z) = (x ∗ y) ∗ z$.

Axiom A4: 0 is a number so that if $x$ is a number, then $0 + x = x$; and 1 is a number so that if $x$ is a number, then $1 ∗ x = x$.

Axiom A5: If $x$ is a number, then there is exactly one number, call it $y$, so that $x + y = 0$; and if $x$ is a number different than 0, then there is exactly one number, call it $w$, so that $x ∗ w = 1$.

Axiom A6: If each of $x$, $y$, and $z$ is a number, then $x ∗ (y + z) = (x ∗ y) + (x ∗ z)$.

The Combining Axiom: If $x$ and $y$ are numbers so that $x < y$, and $w$ is a number, then $x + w < x + z$. 

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