Trigonometry

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To the Instructor

This course came to life because, during ten years of teaching trigonometry, I constantly found myself veering away from the recommended textbooks due to a lack of rigor. The emphasis of the texts was on the quantity and diversity of problems rather than on the precision of the material and the development of problem-solving skills in the students. This modified Moore-method sequence is my attempt to produce in the students an understanding of the trigonometric functions, their properties, and their applications, while honing the students’ communication, presentation, and problem-solving skills. The typical class size I taught was about ten students, although this is smaller than an optimal class size of perhaps twenty to twenty-five students. For class sizes between thirty and fifty students, I would expect to have the students work in groups where one person represents the group for the presentation of a given problem. This technique is known as small group discovery and I have used it successfully in calculus courses of a similar nature.

This sequence was developed in the late 1990s for use in a one-semester trigonometry course at Nicholls State University in Thibodaux, Louisiana, where I taught it three times, refining the notes during each iteration. Most of the students were Honors students who either were not prepared for the calculus sequence or simply preferred the three-hour trigonometry course to the six-hour calculus course. The notes were later used by Dr. B. Dale Daniel at Lamar University in Beaumont, Texas, and by Dr. Pamela Roberson at Stephen F. Austin University in Nacogdoches, Texas. Each instructor modified the notes to meet specific needs. For example, one added problem sets and another added a section on complex numbers to mesh with the departmental syllabus.

While too long to include here, Dr. Roberson wrote a final report regarding her course which could be quite helpful in preparing to teach the course and is available upon request. While teaching the course, she wrote:

_I am having a wonderful time with the trigonometry class in which we are using your problem sequence. The students were very skeptical at the beginning, but not having to pay for a high priced book was enough to keep most of them from dropping out after the first day. No matter what happens the rest of the semester, I have already had enough rewarding experiences to make it worthwhile. One young lady spoke with me after class last Friday and said that she had taken intelligence tests in the past and was told that she was extremely talented in mathematics, but she never made good grades in math classes in school. She said that no one had ever bothered to tell her “why” things worked like they did, but just asked her to memorize lists of formulas. She says now she has more confidence than she ever had and very much enjoys working on these problems. Another student piped up in class and said he had finally figured me out. I asked him what he meant and he said that_
he realized that every time they ask me a question, I direct it to the class to deal with. He said, “We’re going to end up practically writing the book ourselves!” The rest of the class got a good laugh out of it and told him they were surprised it took him that long to figure it out. I think one of the most fun things to watch as the students work through these problems is that some of the problems are almost trivial and some are quite sophisticated, yet the students never seem to notice the difference. They just dive in and tackle each problem as a new challenge.

The structure of the class is simple. I pass the notes to the students a few pages at a time and have them work on and present the problems. It is understood that they are to look only to themselves and to me for guidance; no books or outside help of any other kind are allowed. As a problem is presented, I turn to the class and ask if it is correct or if there are questions. While I might lead with questions to the audience, I rarely point out mistakes at the board and, early in the semester, I place the burden of determining the correctness of each problem completely on the class. If I am asked if a problem is correct, I merely take a vote from the class or ask what they are worried about. To help clarify a potential problem, I may rephrase a student’s question by saying, “Are you asking...?” and then accurately paraphrasing the question so the class can clearly understand the question. But, I won’t answer it.

Generally, I offer minimal guidance until the students have nothing to present. Then I chat informally about what we have covered, using examples to solidify understanding of previous work or discuss the upcoming definitions and problems so that they have a better intuition. Direction of this type can easily double the speed of a class and I definitely use this technique to assure that we cover what I consider “sufficient material.” These are not lectures. Rather, they are discussions where my questions and students’ questions are offered to the class for debate. The most important aspect of my successful classes has been constant open discourse between the students so that they feel comfortable presenting material at the board, asking questions of myself or students who are presenting, and defending their arguments at the board.

At the beginning of the semester, I always fear that we are making minimal progress, since students may spend an entire class struggling to put up a simple problem. However, by the end of the course, they are putting up many correct problems per class period. It is common for me to attend a conference and leave the students in charge of the class, merely providing me with individual reports before the next class so that I know where we are and what was covered. The learning curve is exponential and the patience required at the beginning is rewarded as they learn to read carefully and do the mathematics on their own. The approach and demeanor of the instructor is the critical element for success.

Perhaps four to five times during the semester, I create one-page, hand-written practice sheets. My experience has been that without these simple drill sheets, the students become very proficient at discovering mathematical truths during the course, but lack the ability to quickly knock out simple problems that calculus instructors expect. These sheets are typically a page with ten to twenty problems on it, three to five from each major concept that the students have discovered. These sheets offer the additional benefit that a student who has not successfully discovered the mathematics, but is one step behind the rest of the class, can quickly catch up, rather than fall further behind. These sheets are not included here because they are written specifically depending on the progress students have made and at what I deem to be the right time in the course.
While I describe a grading scheme below, I have actually used two grading schemes for this type of class, although not in this particular class. The scheme not described is much simpler. I give credit for presentations and nothing else – no written work, no tests, no homework, nothing. Students may turn in written work for feedback, which I grade, but the grades are not recorded and do not count toward a final grade. All is determined by the work presented. The class atmosphere in such a setting is truly unique, as the course is 100% about the mathematics – discovering it, presenting it, and explaining it. There is no pressure to do anything but enjoy the mathematics for its sheer beauty. To this remark, one reader once commented, “Come on – the students are under tremendous pressure to present!” But I have not found this to be the case. On the first day, I ask a few questions and the moment someone can present a solution, I encourage him or her to do so. By the end of the period, several students have put up several problems (or attempts at solutions) and they realize how relaxed the class is going to be. As they leave, I give out more problems and explain that this is the way I will conduct the class every day and that I look forward to more solutions tomorrow. I don’t tell them right away that this is their entire grade – that might frighten them, but I do tell them within the week. In my experience, the low pressure allows the students to relax, and relaxing allows the students to excel.

I do not assume that these brief introductory remarks are adequate to assure a successful implementation of a modified Moore-method course. For extensive writings on my thoughts and implementation of modified Moore-method courses, please refer to “The Moore Method: A Pathway to Learner-Centered Instruction,” by Coppin, Mahavier, May, and Parker. For mentoring using these notes, contact me directly or refer to the mentoring link at www.discovery.utexas.edu/rlm.

Two technical notes are in order:

- We do not necessarily proceed linearly through the notes. On occasion, a solution is not available for a problem, but a solution to a subsequent problem is available that uses this first problem. In this case, we assume the former to solve the latter, leaving the former problem for later. One must be careful to avoid circular reasoning. Hence, the first problem must now be solved independently of the latter.

- There are three theorems listed among the problems that we do not necessarily prove during class, although I encourage students to work on them and we will take a look if a student finds a proof.
To the Student

The structure of this course will quite likely be different from previous courses you have taken. There will be no book and all the notes that you will need will be provided. These notes and my office hours are to be your only resources. The notes that you will develop as you work through the problems in this sequence will be your book, a collection of problems and solutions that you and your peers, rather than me or the author of a textbook, have worked out. The purpose of this format is to actively involve you in the process of doing mathematics as opposed to passively viewing a lecture and then mimicking problems slightly modified from those you have been shown. The bad news is that this approach is very different from your previous classes. The good news is that it is a lot more fun to participate than to watch.

We will derive the subject of trigonometry essentially from scratch. Therefore, when you work a problem, you are not allowed to use anything that we have not already discussed in the course with the exception of a few concepts listed in the next paragraph or any concepts that you may create or define on your own. You will not be allowed to use information that you may recall from previous courses such as the fact that $\sin(\theta) = o/h$ until we have defined these terms and proved this fact based on material developed in this course.

On the other hand, there are concepts that we will assume are familiar and you should feel free to use as needed. We might refer to these as our “undefined terms” since we will use them without stating a formal definition. Here are a few examples.

- circle, coordinate axes, origin, plane, point
- line, line segment, ray
- arc, area, center, circumference, diameter, radius, and tangent of a circle
- triangle, similar triangles, square

All work presented or turned in is to be yours or that of your group. You are not to discuss any problems with anyone other than your group members or me, and you are not to look into any source for further guidance. Grading for the course will be the average of three grades: (i) the average of your board work grades (group grades), (ii) the average of your written assignments, and (iii) the average of midterm and final exam grades. Anyone who is regularly presenting material at the board will certainly have adequate work for good grades on the written assignments and thus will do well on the midterm and final. I emphasize that the goal of the course for each student should be clear presentations at the board of well-prepared problems.
If a problem is about to be presented at the board and you do not wish to see the presentation because you are still working on it, then you may choose to leave the room. In this case, you may turn in this problem for credit as original work. You must write original at the top of the page. There is no limit on the number of original problems you can submit.

You must turn in exactly one new problem each week. A new problem means one that you have not turned in before. If this problem has been presented in class, label it write-up. If it has not been presented, label it original. If you receive a grade of less than “B,” you may resubmit this problem the following week and I will record the higher of the two grades. You only get one chance to resubmit. Please write resubmit at the top. Be sure that everything you turn in is double-spaced with your name, problem number, and problem statement on it. Be sure and write either write-up, original, or resubmit, at the top of each problem turned in.

Grading for turn-in assignments and board work will be based on the following scale.

- A: This is a correct problem.
- B: I believe you know how to do the problem but some of what you have written is not correct.
- C: I cannot tell if you understand the problem based on what you have written, but I think you have an intuitive idea of the solution.
- D: There is at least one major flaw in your argument.

The purpose of these exercises is to teach you to solve problems and write the solutions correctly. It is not expected that you started the class with this knowledge; hence, some low grades are to be expected. Do not be upset – just come see me and resubmit the problem.

Finally, there are three theorems listed among the problems. You don’t need to prove these. If you do prove one of them or if you already know a proof of one of them, then let me know and we will look at them in class just like we look at the problems.
Problem Sequence

Definition 1. The unit circle is the circle centered at the origin and of radius one.

Problem 1. Graph the unit circle, and subdivide it into eight arcs of equal length such that one division lies at the point (1,0). Working in a counter-clockwise direction, determine the distance along the unit circle from the point (1,0) to each of the divisions, and label each division with this distance.

Problem 2. Repeat the previous exercise using a total of twelve divisions.

Problem 3. Repeat the previous two exercises, using a circle centered at the origin and of radius two.

Definition 2. An angle is a subset of the plane consisting of two distinct rays (or two distinct line segments) with a common endpoint called the vertex.

Definition 3. An angle in standard position is an angle where one of the two rays is the positive x-axis. This ray is referred to as the initial side of the angle. The other ray is referred to as the terminal side of the angle.

Definition 4. Given a circle centered at the origin and an angle in standard position, let $P_1$ be the intersection of the circle with the initial side of the angle, and let $P_2$ be the intersection of the circle with the terminal side of the angle. The arc associated with the angle is the portion of the circle traced by a point traversing the circle in a counter-clockwise direction from the point $P_1$ to the point $P_2$.

Definition 5. Given a circle centered at the origin and an angle in standard position, the radian measure of the angle is the ratio of the length of the arc associated with the angle to the radius of the circle.

Notation: If the radian measure of an angle is $\theta$, then we will say that such an angle has measure $\theta$ radians.

Problem 4. Determine the radian measure of each angle illustrated below.
Problem 5. Given an angle, not necessarily in standard position, make up a definition for the radian measure of the angle.

Definition 6. Given a circle centered at the origin and an angle in standard position, the degree measure of the angle is \( \frac{360}{\pi} \) times the radian measure of the angle.

Historically, the circle was evenly divided into 360 arcs, and an angle was said to have degree measure \( \theta \) if the terminal side of the angle intersected the circle at the \( \theta \)th division. I have read two explanations of this – the choice of 360 was because it was believed there were 360 days in a year, or because much mathematics was done based on multiples of 60. If pressed, I could probably come up with a source supporting these statements.

Definition 7. A right triangle is a triangle that has one angle with degree measure 90.

Theorem 1. The sum of the degree measures of the interior angles of a triangle is 180.

Definition 8. The hypotenuse of a right triangle is the side that is not adjacent to the angle of degree measure 90.

Theorem 2. Pythagorean Theorem: Given a right triangle with sides of length \( a \), \( b \), and \( c \), where \( c \) is the length of the hypotenuse, we have \( a^2 + b^2 = c^2 \).

Notation: If the degree measure of an angle is \( \theta \), then we will say that such an angle has measure \( \theta \) degrees.

Problem 6. Determine the degree measure of each angle in the previous illustration.

Problem 7. Graph the unit circle, and the angle in standard position whose measure is 150 degrees.

Problem 8. Graph the unit circle, and the angle in standard position that has measure \( \frac{\pi}{4} \) radians. What are the xy-coordinates of the point that is the intersection of the terminal side of this angle with the unit circle?

Problem 9. Graph the unit circle, and the angle in standard position that has measure \( \frac{\pi}{3} \) radians. What are the xy-coordinates of the point that is the intersection of the terminal side of this angle with the unit circle? What is the length of the arc associated with this angle?
Problem 10. Graph a circle of radius 2 centered at the origin, and the angle in standard position so that the length of the arc associated with the angle is $\frac{3\pi}{2}$. What is the radian measure of this angle? What is the degree measure of this angle?

Problem 11. Given a circle centered at the origin with radius 5 centimeters, and an angle in standard position with radian measure $\frac{3\pi}{4}$, determine the length of the arc associated with this angle.

Problem 12. Suppose that a unicycle with a wheel of radius 9 inches is rolled 4 feet. Through what radian measure has one spoke on this wheel traveled? How many revolutions has the wheel made?

Problem 13. Suppose that it took 2 seconds to roll the unicycle (described in the previous problem) 2 feet. What is the speed of the unicycle as measured in inches per second? As measured in radians per second? As measured in revolutions per second?

Problem 14. Given an angle of degree measure 270, determine the radian measure of the angle.

Problem 15. Given an angle of radian measure $\frac{5\pi}{3}$, determine the degree measure of this angle.

The development above hinges on the choice of the counter-clockwise direction in the definition of the arc associated with the angle. An analogous series of definitions can be made using the clockwise direction. From this point forward, we will use the convention that an angle measured in the clockwise direction from the positive $x$-axis will have negative measure and be referred to as a negative angle, while an angle measured in the counterclockwise direction from the positive $x$-axis will have a positive measure and be referred to as a positive angle.

Problem 16. Locate the point on the unit circle so that the angle formed by the radius of the circle containing this point and the positive $x$-axis is $-\frac{3\pi}{4}$ radians. What is the measure of this angle in degrees?

Problem 17. Given an angle of degree measure $-405$, determine the radian measure of this angle.

Problem 18. Prepare an essay addressing the question: Does mathematics have value to society? Defend your answer and state your sources. If you are unsure as to what an essay is, please request my “essay resource kit” that describes an essay and indicates grading guidelines.

Problem 19. Determine the coordinates in the xy-plane of each point on the unit circle whose distance from the point $(1,0)$ along the circle in a counter-clockwise direction is an integer multiple of $\frac{\pi}{4}$.

Problem 20. Determine the coordinates in the xy-plane of each point on the unit circle such that the angle that is in standard position with terminal side containing the point has radian measure of an integer multiple of $\frac{\pi}{3}$.

Definition 9. A function is a collection of points in the plane with the property that no two of these points lie in the same vertical line.
You have probably seen the concept of a function denoted by such algebraic expressions as perhaps $f(x) = x^2$ or $y = x^2$. These are not functions; rather, they are expressions that represent a function. The function itself is the collection of points (or ordered pairs) that you might graph to form a graphical representation of the function. Thus, I would say that we define a function $f$ by the equation $f(x) = x^2$, and it is understood that $f$ is the function consisting of the ordered pairs generated by the equation. Thus $f = \{(1, 1), (2, 4), (-3, 9), \cdots \}$.

**Definition 10.** The first coordinates (x-coordinates) of all ordered pairs of a function $f$ are referred to as the **domain** of $f$, while the second coordinates (y-coordinates) are referred to as the **range** of $f$.

In the next definitions, we will use the same type of notation to define a function $C$ as a collection of ordered pairs which are determined by a rule.

**Definition 11.** We define the function $C$ such that if $P = (x, y)$ is the point on the unit circle such that the radius of the circle that contains $P$ forms an angle of radian measure $\theta$ with the positive x-axis, then $C(\theta) = x$. Hence, $(\theta, x) = (\theta, C(\theta))$ is a point of the function $C$.

**Problem 21.** Determine values for $C(0)$, $C\left(\frac{\pi}{6}\right)$, $C\left(\frac{\pi}{4}\right)$, $C\left(\frac{\pi}{3}\right)$, $C\left(\frac{\pi}{2}\right)$, $C\left(\frac{2\pi}{3}\right)$, $C\left(\frac{3\pi}{4}\right)$, $C\left(\frac{\pi}{2}\right)$, and $C(2\pi)$.

**Problem 22.** Graph the function $C$, using the ordered pairs $(\theta, C(\theta))$ computed in the previous problem.

**Problem 23.** Find a value for $\theta$ where $C(\theta) = 0$. Are there others?

**Problem 24.** Determine an approximate value for $C\left(\frac{\pi}{4}\right)$.

**Problem 25.** List all values for $\theta$ where $C(\theta) = \frac{1}{2}$.

**Problem 26.** List all values for $\theta$ where $C(\theta) = \frac{\sqrt{2}}{2}$.

**Problem 27.** Graph the function $S$.

**Problem 28.** Solve $S(\theta) = 1$ for $\theta$.

**Problem 29.** Solve $S(\theta) = \frac{\sqrt{2}}{2}$ for $\theta$.

**Problem 30.** Solve $S(\theta) = -\frac{1}{2}$ for $\theta$.

**Problem 31.** Let $f$ be the function defined by $f(\theta) = -S(\theta)$ for every real number $\theta$. Graph $f$.

**Problem 32.** Let $g$ be the function defined by $g(\theta) = C\left(\theta + \frac{\pi}{2}\right)$ for every real number $\theta$. Graph $g$.

**Definition 13.** Let $T$ be the function defined by $T(\theta) = \frac{S(\theta)}{C(\theta)}$ for every real number $\theta$.
Problem 33. For what values of $\theta$ will $T$ be undefined?

Problem 34. Write down a set that is the domain of $T$.

Problem 35. Graph $T$.

Problem 36. Write an essay on any mathematician’s contribution to society. Explain the contribution in words that your classmates can understand. Make enough copies (with your name omitted if you desire) for your classmates.

Notation: Given two points $A$ and $B$ in the plane, we denote the line segment between $A$ and $B$ by $AB$, and the length of the line segment by $l(AB)$.

Problem 37. Let $\theta$ be a number such that $0 < \theta < \frac{\pi}{2}$. Draw the unit circle, the line that is tangent to the unit circle at $(1,0)$, and the line that forms an angle of radian measure $\theta$ with the positive x-axis.

Problem 38. Refer to the picture from the previous problem. Determine the length of the line segment between $(1,0)$ and the intersection of the two lines.

Note: Refer to the figure above for the next three problems. Assume that the circle is a unit circle, $\theta$ is the radian measure of the angle $EAB$, and $0 < \theta < \frac{\pi}{2}$.

Problem 39. Show that $S(\theta) = \frac{l(\text{CD})}{l(\text{AD})}$.

Problem 40. Show that $C(\theta) = \frac{l(\text{AC})}{l(\text{AD})}$.

Problem 41. Show that $T(\theta) = \frac{l(\text{CD})}{l(\text{MC})}$.

We have now defined three functions, referred to as $S$, $C$, and $T$. We have also shown that if we have a right triangle with one angle of radian measure $\theta$, and we assume that the side adjacent to this angle has length $a$, the side opposite from this angle has length $o$, and the remaining side (the hypotenuse) has length $h$, then these functions satisfy $S(\theta) = \frac{o}{h}$, $C(\theta) = \frac{a}{h}$, and $T(\theta) = \frac{o}{a}$. Of course, these are the three trigonometric functions sine, cosine, and tangent, that are commonly abbreviated as sin, cos, and tan, respectively. Notice that we also showed that the tangent function gets its name from the fact that it represents the length of a line segment associated with a certain line tangent to the unit circle. We now define three more functions in terms of the sine, cosine, and tangent functions.
Definition 14. For any number $\theta$ for which cosine is non-zero, let secant be the function defined by $\sec(\theta) = \frac{1}{\cos(\theta)}$.

Definition 15. For any number $\theta$ for which sine is non-zero, let cosecant be the function defined by $\csc(\theta) = \frac{1}{\sin(\theta)}$.

Definition 16. For any number $\theta$ for which cosine is non-zero, let cotangent be the function defined by $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.

Problem 42. Graph the secant function, and list the domain and range.

Problem 43. Find all numbers $u$ that satisfy $2 \sin(u) = 1$.

Problem 44. Graph the function defined by $t(x) = 3 \sin(x - \pi)$.

Problem 45. Graph the function defined by $f(x) = 5 - \cos(2x)$.

Problem 46. Graph the function defined by $r(x) = \sin(x) + \cos(x)$.

Problem 47. Graph the function defined by $z(u) = u \sin(u)$.

Problem 48. At what minimum height above ground level must I place a satellite dish so that at a 30-degree angle, it will be able to “see” the sky over the top of a building that is 40 feet tall and 50 feet away from the dish?

Problem 49. Solve $2 \cos(\theta) = -\sqrt{3}$ for $\theta$.

Problem 50. Solve $\tan(\theta) = 1$ for $\theta$.

Note: The previous problem could be written “Solve $2 \sin(u) = 1$ for $u$”.

Problem 54. Solve $2 \sin^2(x) - 3 \sin(x) + 1 = 0$ for $x$.

Problem 55. Krista is standing at the edge of a long straight beach when she sees Jared drowning. Assume that Jared is at a distance of 76 meters straight out from a point on the beach that is 380 meters from where Krista is standing. Assume that Krista can run at 6.5 meters per second and swim at 1.4 meters per second. Krista runs down the beach toward Jared to a point $P$ on the beach, and then dives into the water and swims to Jared. The angle at the point $P$ between the beach and the line from $P$ to Jared has a measure of 77 degrees. How long does it take Krista to save Jared? Could she have saved him faster by taking a different path?

Problem 56. Graph cotangent, and list its domain and range.

Problem 57. Solve $\cot(\theta) > 0$ for $\theta$. 

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Problem 58. Graph the function defined by \( f(x) = 3 - 2 \sin(2x + \pi) \).

Problem 59. Write an essay addressing how you expect to use mathematics after leaving the university.

Problem 60. Let \( S \) be a square, let \( M \) and \( N \) be the midpoints of two adjacent sides, and let \( V \) be the corner of the square that is opposite both \( M \) and \( N \). Let \( \theta \) be the measure of the angle between the two lines connecting \( M \) and \( N \) with \( V \). Compute \( \sin(\theta) \).

Problem 61. Graph the function defined by \( f(x) = -2 + 4 \cos(\pi x + \frac{\pi}{4}) \).

Problem 62. Solve \( 2 \sin(4x) - \sqrt{3} = 0 \) for \( x \).

Problem 63. A plane passes directly over your head at an altitude of 500 feet. Two seconds later, you observe that the angle of elevation of the plane is 42 degrees. What is the plane’s average speed over those 2 seconds?

Problem 64. Solve \( 2 \sin(3x) - 1 = 0 \) for \( x \).

Problem 65. In aerial and nautical navigation, 0 degrees represents due north, and directions are measured in degrees clockwise from north, so 90 degrees is due east. If a plane travels 200 miles at a bearing of 300 degrees, how far west of the airport is the plane? How far north? What are the coordinates of the plane?

Definition 17. Let \( f \) be a function and \( A \) be a positive number. We say that \( f \) has period \( A \) if \( A \) is the smallest positive number satisfying \( f(x + A) = f(x) \) for all \( x \) in the domain of \( f \).

Problem 66. Solve \( 4 \cos^2(2x) - 4 \cos(2x) + 1 = 0 \) for \( x \).

Problem 67. Determine the periods of the six trigonometric functions.

Problem 68. Assume that \( \theta \) is a number such that neither \( \sin(\theta) \) nor \( \cos(\theta) \) is zero. Simplify \( \tan(\theta) \cot(\theta) \). Why did we make this assumption about sine and cosine not being zero?

Problem 69. Show that \( \sin^2(t) + \cos^2(t) = 1 \) for any number \( t \).

Problem 70. Prove that \( \tan^2(\theta) + 1 = \sec^2(\theta) \) is valid for all \( \theta \) where \( \cos(\theta) \neq 0 \).

Problem 71. Show that \( 1 + \cot^2(\theta) = \csc^2(\theta) \) is valid for all \( \theta \) where \( \sin(\theta) \neq 0 \).

The three identities you have just derived are referred to as the Pythagorean identities.

Problem 72. Solve \( 1 + \cos(x) = \sin(x) \) for \( x \).

Problem 73. Solve \( \csc(x) + \cot(x) = 1 \) for \( x \).

Problem 74. Solve \( \cos^2(x) = \cos(x) + \sin^2(x) \) for \( x \).

Problem 75. Simplify \( \frac{\tan(\alpha)}{\sec(\alpha) \csc(\alpha)} \). For what values of \( \alpha \) is this expression defined? For what values of \( \alpha \) is your simplified expression defined? Are the two expressions equal for all values of \( \alpha \)?

Problem 76. Simplify \( (\cos^2(\theta) - 1)(\tan^2(\theta) + 1) \). For what values of \( \theta \) is this expression meaningful?
Problem 77. Simplify \( \tan(x) \cos(x) \).

Problem 78. Simplify \( \frac{\sec(x)}{\csc(x)} \).

Problem 79. Let \( f(x) = \frac{\sec(x) + \csc(x)}{1 + \tan(x)} \). Simplify \( f(x) \), and state the domain of \( f(x) \).

Problem 80. Simplify \( (\csc(x) - 1)(\csc(x) + 1) \).

Problem 81. Simplify \( \frac{1 + \sec(x)}{\tan(x) + \sin(x)} \).

Problem 82. Write an expression in terms of \( \sin(\alpha), \cos(\alpha), \sin(\beta) \), and \( \cos(\beta) \) for the distance between \( P \) and \( Q \) in the first illustration.

Problem 83. Write an expression for the distance between \( P' \) and \( Q' \) in the second illustration.

Problem 84. Set the results from the two previous problems equal, and simplify this expression.

The expression that you computed in the last problem is referred to as the subtraction identity for the cosine function. This single identity gives rise to a multitude of identities, which we will now develop.

Definition 18. If \( f \) is a function, and we have \( f(x) = f(-x) \) for every number \( x \) in the domain of \( f \), then we say that \( f \) is even.

Definition 19. If \( f \) is a function, and we have \( f(x) = -f(-x) \) for every number \( x \) in the domain of \( f \), then we say that \( f \) is odd.

Problem 85. Show that every function can be written as the sum of an even and an odd function.

From a graphical point of view, these definitions correspond to symmetry about the \( y \)-axis and symmetry about the origin, respectively.

Problem 86. Let \( f, g, \) and \( h \) be the functions defined by \( f(x) = x^2 \), \( g(x) = x^3 \), and \( h(x) = f(x) + g(x) \), for all numbers \( x \). Prove that \( f \) is even, \( g \) is odd, and \( h \) is neither even nor odd.
Problem 87. Determine which of the six trigonometric functions are even and which are odd.

Problem 88. Prove that if f is an even function and g is an odd function, then the function defined by \( h(x) = f(x)g(x) \) is an odd function.

Problem 89. Substitute \( \alpha = a \) and \( \beta = -b \) into the subtraction identity for cosine, to develop the addition identity for cosine.

Problem 90. Substitute \( \alpha = \frac{\pi}{2} \) and \( \beta = b \) into the subtraction identity for cosine, to develop one of the cofunction (or translation) identities. What does this identity say about the graphs of sine and cosine? List at least three additional cofunction identities.

Problem 91. Substitute \( \alpha = a \) and \( \beta = b - \frac{\pi}{2} \) into the addition identity for cosine, to develop yet another identity. What would you call this identity?

Problem 92. There are a total of four addition and subtraction identities for sine and cosine, and we have developed three. Develop the fourth.

Problem 93. Compute a subtraction identity for tangent by simplifying the quotient of the subtraction identity for sine and the subtraction identity for cosine.

Problem 94. Compute an addition identity for tangent in a similar manner.

Problem 95. Prove or disprove: \( \cot(x) + \cot(y) = \frac{\cos(x-y)}{\cos(x)\sin(y)} \).

Problem 96. Prove that \( \cot\left(\frac{\pi}{2} - x\right) = \tan(x) \).

Problem 97. Compute an exact value for \( \cos\left(\frac{\pi}{12}\right) \).

Problem 98. Compute an exact value for \( \sin\left(\frac{7\pi}{12}\right) \).

Now we can generate exact values for the sine and cosine of a new set of numbers, specifically those that are integer multiples of \( \frac{\pi}{12} \). How many angles on the unit circle can we find exact values of sine and cosine for now? What percentage of the total number of numbers can you get an exact value for?

Problem 99. Prove that \( \sin(2x) = 2\sin(x)\cos(x) \) is valid for all values of x.

Problem 100. Prove that \( \cos(2x) = \cos^2(x) - \sin^2(x) \) is valid for all values of x.

Problem 101. Prove that \( \tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} \) is valid for all values of x for which both sides of the equality are defined.

Problem 102. Solve \( \sin(2x) = \sin(x) \) for x.

These last three identities are referred to as the double angle identities, and show up quite a bit in calculus courses. The next two are referred to as the half angle identities, and can be derived easily from the double angle identities if you can figure out what substitution to make.

Problem 103. Prove that \( \sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos(\alpha)}{2} \) is valid for all values of \( \alpha \).

Problem 104. Prove that \( \cos^2\left(\frac{b}{2}\right) = \frac{1 + \cos(b)}{2} \) is valid for all values of \( b \).
Recall that a function is a collection of ordered pairs satisfying a certain property. Not every function has an inverse, but if one does, then the inverse of the function is the collection of ordered pairs obtained by reversing the 1st and 2nd coordinate of each ordered pair of \( f \). Thus if \( f \) has an inverse and we denote it by \( f^{-1} \), then \( f(x) = y \) if and only if \( f^{-1}(y) = x \). The dilemma is that given a simple function such as \( f(x) = x^2 \), if we reverse the coordinates, then the new set of coordinates do not satisfy the necessary property to be a function. We solve this by restricting the domain. Thus, \( f(x) = x^2 \) has as its inverse \( f^{-1}(x) = \sqrt{x} \) over the domain \( x \geq 0 \). The six inverse trigonometric functions are all defined in this manner.

**Definition 20.** We define the **arcsine** (or **inverse sine**) to be the function satisfying \( \arcsin(x) = y \) if and only if \( \sin(y) = x \), and having domain \(-1 \leq x \leq 1\) and range \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\).

**Problem 105.** Graph the inverse sine function.

**Definition 21.** We define the **arccosine** (or **inverse cosine**) to be the function satisfying \( \arccos(x) = y \) if and only if \( \cos(y) = x \), and having domain \(-1 \leq x \leq 1\) and range \(0 \leq y \leq \pi\).

**Problem 106.** Graph the inverse cosine function.

**Definition 22.** We define the **arctangent** (or **inverse tangent**) to be the function satisfying \( \arctan(x) = y \) if and only if \( \tan(y) = x \), having domain of all real numbers and range \(-\frac{\pi}{2} < y < \frac{\pi}{2}\).

**Problem 107.** Graph the inverse tangent function.

**Notation:** The arcsine, arccosine, and arctangent functions are often denoted by \( \sin^{-1} \), \( \cos^{-1} \), and \( \tan^{-1} \). This is an abuse of notation, since \( \sin^{-1}(x) \neq \frac{1}{\sin(x)} \). Hence, \( \sin^n(\theta) \) means \( (\sin(\theta))^n \) for all values of \( n \) except \( n = -1 \) when it denotes the inverse sine function.

**Problem 108.** What is the value for \( \arcsin\left(-\frac{\sqrt{3}}{2}\right) \)?

**Problem 109.** Compute \( \arctan(1) \).

**Problem 110.** What is the difference between the results of the next two problems?

1. Solve \( \cos(\theta) = \frac{1}{2} \).
2. Compute \( \cos^{-1}\left(\frac{1}{2}\right) \).

**Problem 111.** Compute \( \arccos\left(\frac{1}{\sqrt{2}}\right) \).

**Problem 112.** Compute \( \sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) \).

**Problem 113.** Compute \( \cot\left(\arccos\left(-\frac{1}{2}\right)\right) \).

**Problem 114.** Graph \( f(x) = \arcsin(x + 3) \) for \(-4 \leq x \leq 2\).

**Problem 115.** Graph \( g(x) = -2\cos^{-1}\left(\frac{1}{2}\right) \) for \(-3 \leq x \leq 3\).

**Problem 116.** Graph \( r(x) = \arctan(1 - 2x) \) for \(-2 \leq x \leq 3\).
Problem 117. Go to the library. Choose any book on trigonometry, and copy the pages that list all the identities. Check to be sure they are correct! These can often be found on the inside cover of trigonometry or calculus books.

Theorem 3. The solutions to \( ax^2 + bx + c = 0 \) are \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

Problem 118. Solve \( \tan^2(x) = 2 \tan(x) + 1 \) for \( x \), listing all solutions. Approximate these solutions using your calculator.

Problem 119. Solve \( \sin^2(x) + 2 = 4 \sin(x) \) for \( x \).

Problem 120. Viewing the triangle above, show that \( \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} \).

Problem 121. Viewing the triangle above, show that \( a^2 = b^2 + c^2 - 2bc \cos(\alpha) \).

For the next few problems, we will adhere to the convention that triangles will be labeled with angles \( \alpha, \beta, \) and \( \gamma \), and the sides opposite these angles will have lengths \( a, b, \) and \( c \), respectively. The expression “solve the triangle” means to provide the lengths of any sides, and the measure of any angles, that are not supplied in the problem. For each of the following problems, both solve and draw the triangle to a reasonable degree of accuracy.

Problem 122. Draw and solve any triangles satisfying: \( \alpha = 32^\circ, a = 2.5, \beta = 41^\circ \).

Problem 123. Draw and solve any triangles satisfying: \( \alpha = 29^\circ, a = 7, c = 14 \).

Problem 124. Draw and solve any triangles satisfying: \( \alpha = 43^\circ, b = 6, c = 10 \).

Problem 125. Draw and solve any triangles satisfying: \( \beta = 72.2^\circ, b = 78.3, c = 145 \).

Problem 126. Draw and solve any triangles satisfying: \( a = 6, b = 8, c = 9 \).

Problem 127. Draw and solve any triangles satisfying: \( \alpha = 26^\circ, a = 11, b = 18 \).

Problem 128. Draw and solve any triangles satisfying: \( a = 8, \beta = 60^\circ, c = 11 \).

Problem 129. Draw and solve any triangles satisfying: \( \alpha = 20^\circ, b = 10, c = 16 \).

Problem 130. Write an essay describing what, if anything, you have learned from this course that will have a lasting impression on you. Sign, date, and seal this essay. Give me this essay (anonymously, if you wish) after the semester is over.

Problem 131. Find the length of one side of a nine-sided regular polygon inscribed in a circle of radius 8.32 centimeters.

Problem 132. Two bird watchers, located at points A and B, are twelve and one-half miles apart. A Yellow-Bellied Sap-Sucker is located at point C by both birders. Careful measurements indicate that \( \angle BAC = 14^\circ \), while \( \angle ABC = 82^\circ \). Which birder is closer to our Yellow-Bellied Sap-Sucker, and how far is he from the Sap-Sucker?
Problem 133. An uptight observer stands at ground level, some unknown distance away from the base of a building at point A, and measures the angle between ground level and the top of the building (called the angle of elevation) to be 63°. After taking this measurement, she walks 140 feet directly away from the building to point B (also at ground level), where she measures the angle of elevation to be 55°. Being weak in trigonometry, she gives you this data. Find the height of the building.

The following problem was brought to my attention by a very seasoned sailor. He had a Captain's license, and regularly commanded vessels with length in excess of 100 feet. He gave the example of a lighthouse that was there to alert sailors to a reef that was 200 yards off the point where the lighthouse was located. However, whenever he sailed there, we could not discern how far off-shore he was. He was familiar with angles and navigation utilizing trigonometry, but he was not familiar with the law of sines. To solve his problem, we must learn a bit more about nautical and aerial navigation. A direction such as N30°E is read as the direction 30° East of due North. Thus in terms of the unit circle, this is the direction determined by \( \frac{\pi}{6} \). S40°W is 40° West of South, and corresponds to 230° on the unit circle, or 220° if we view 0° as due North.

Problem 134. A sailor spots a lighthouse at N28°E, and then proceeds east 7.5 nautical miles where he sites the lighthouse at N16°E. Find the distance from the boat to the lighthouse. If the boat continues along the same path, determine the minimum distance between the boat and the lighthouse.

Problem 135. In order to seal an oil pipeline, we must make a plate of 1/4″ steel to place on a flange at the open end of an open pipe. Our plate must be circular with radius 6″, and must have 7 holes drilled in it. These holes must be 3/16″ in diameter, they must be equally spaced, and their centers must be 1″ from the perimeter of the plate. What should be the distance between the centers of two adjacent holes? Note: Industry standards require an answer accurate to one ten-thousandth of an inch.

Problem 136. Jack and Jill take off from the same airport at the same time in their new Cesna and Beechcraft planes. Jack flies N35°W at 160 miles per hour (mph), while Jill flies S70°W at 170 mph. How far apart are the planes after two hours? Determine a function that gives the distance between the planes as a function of time \( t \), of hours of flight.

Problem 137. Two joggers in Central Park are resting on park benches at points A and B, where point A is 1.2 miles north of point B. At midnight, both spot a walker. The jogger at point A observes the walker at a heading of N20.0°E while the jogger at point B observes the walker at a heading of S70°E. At 12:10 A.M., the jogger at point A observes the walker at N34.0°E, and the jogger at point B observes her at S55°E. Find the walker's average speed.

Problem 138. Write an essay on trigonometry.

Congratulations! You have completed a one-semester course in trigonometry! More importantly, you have done it on your own, without the usual crutch of lectures and examples. Sometimes it is easy to lose sight of the forest when you are working within the trees, so I want to take this opportunity to review some of the hard work you have done, placing it in context and reminding you of the key accomplishments.

We began with the study of the unit circle and used it to define the functions, sine and cosine. From these we defined the remaining four trigonometric functions. After
graphing and observing the properties of these functions (intercepts, asymptotes, amplitude, frequency, etc.), we learned to solve equations involving the trigonometric functions. Then came a key result that most remembered from high school, the relationship between the trigonometric functions and right triangles, for example that $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$. It is worth noting that these are what we as mathematicians call, equivalent definitions. That is, we could have defined the trigonometric functions from the right triangles and all that we did still holds. Our original definition based on the unit circle would then be a consequence of this new definition. From here, we began deriving the trigonometric identities – such as $\sin^2(\theta) + \cos^2(\theta) = 1$ for all angles $\theta$. We derived the Pythagorean identities, the double angle identities, the half-angle identities, the sum and difference identities and others. We then developed the inverse trigonometric functions, the law of sines and the law of cosines and considered applications requiring each of these tools.

If you leave this course with more than the mathematics your learned, then I hope that you leave realizing that you were an active mathematician during this course, doing exactly what mathematicians do every day – you made conjectures, you read mathematics, and most importantly, you created mathematics. The fact that this mathematics was already known is all that separates you from a doctorate in mathematics. When you create mathematics that is not already known, then you will earn your Ph.D. in mathematics. Again, congratulations, and good luck in your future courses.