JOURNAL OF INQUIRY-BASED LEARNING IN MATHEMATICS
No. **34**, (Nov. 2013)

Advanced Calculus with Generalizations: First Semester

Dr. Robert W. Vallin

Slippery Rock University

Contents

To the Instructor

0.1 Introduction - The Topic

In a way, Advance Calculus is an odd topic. There seems to not be anything new going on, as by the time a student gets to this class he/she has had at least three semesters of calculus. So everyone is familiar with limits, differentiation, integration, and sequences & series. It is because the material is familiar that the study of analysis can be misleading. It is the depth that makes the course what it is and oftentimes confusion over the depth can create issues.

I still recall sitting in this class as a student wondering how in the world I was going to *prove* that absolute value was non-negative when there was nothing to do. Everyone just knows it is so. Such is how I began to realize I was not in Kansas anymore. However, I was determined to conquer this class and understand the deeper parts of calculus. In fact for me it was life changing as I went on to earn my PhD in classical real analysis and continue to publish in this area.

In this anecdote lies important details. After so many years being told that something was true, it was difficult to think along the lines of why it was true. Especially since class at that time was all about listening to lecture, a passive act. However, there is something to get excited about as one sees that this is not a topic that has been wrung dry, but that there are generalizations and extensions to these topics which the interested student can follow to who knows where. There are ideas and proofs out there yet to be written. So this text was written with the idea of getting the students actively involved in thinking about mathematics and seeing some new(-ish) ideas out there.

0.2 Introduction - The Book

This book is for the first semester of a yearlong course in advanced calculus/real variables. It was used this way at Slippery Rock University where the sequence is a requirement for all math majors. Typically, such a class has between 8 and 20 students. The notes are parsed out to the students as a series of handouts. The pace of the course and the context of the handouts will vary from instructor to instructor - you should edit the problems based on the ability and needs of your students. During the refereeing process I was asked to place back the pagebreaks when I developed these notes. This is a request I respectfully decline. You may decide to devote more time/problems to a topic and skip other things entirely. The LaTeX code is available for anyone who wants to shift things around.

As I teach the course, each handout contains Definitions, Axioms, Problems, Exercises, and a few Theorems. Anything labelled Theorem (which by virtue of being called a theorem must be true) needs to be proven. Any Problem will start with the directive "Prove or Disprove." It is up to the student to determine the truthfulness of the statements and, if it is true, provide a proof; if it is false, give a counterexample with explanation. Both of these types of assignments can be presented at the board for points, and professionally written solutions are to be turned in. Exercises can be presented at the board for a grade, but are not turned in. The Exercises are to help illustrate definitions and topics. Similarly, there are Remarks throughout to explain concepts and introduce new discussions. Some Prove or Disprove items have quick and easy counter-examples. I have found that once the counter-examples are presented a good next step is to ask the student how to change the problem to make it into a theorem. Problem X then leads to Problem X, Part 2 which a student can then get credit for proving and presenting.

There is debate among practitioners of the Inquiry-Based Learning (IBL) about how to present the method to a class. Does one just plug away at the course, acting as if IBL is the usual thing or spend time explaining to the students how self-discovery is a better way to learn and own the material? Should the students be shown evidence of why IBL is best to help convince them that it is good? The decision of what to tell students is for most a personal matter. Each instructor must determine his/her own way. As for me, I steal an analogy from Ron Taylor of Berry College. At the beginning of the first day of class for the year, I talk about juggling. I define what juggling is and then give a short demonstration of juggling. Then I say to them, "Now you should be able to juggle. If you go home tonight and practice a little, you will able to juggle just fine." This is, in essence, the lecture-homework-move on format that most classes follow. Finally I pass out a review sheet of problems. These come from our introduction to proof course called Modern Concepts of Mathematics. There are two purposes to this: first, to review what I expect the students to know and be able to handle (induction, proof my contradiction, direct proof, etc.) and second, to get them up at the board on the second day, presenting something they should be more comfortable with as it is not new material. These problems, and these problems only, may be worked on in a group. The end of the second day is the first page of Axioms, Definitions, Exercises, and Conjectures. Of course, the first day of the second semester the students are already prepped on how the class runs. A group email before the first day can give them some problems to be thinking on for the new semester.

0.3 The Instructor in the Classroom

The most difficult part of being the instructor in an IBL setting is keeping quiet. This does not mean I remain absolutely quiet. Students should be encouraged to ask questions of the presenter and, especially at the start of the course, they sometimes need help formulating an exact question. So I will rephrase things or ask them to be more specific with their thoughts. In addition, if a presenter is having trouble, perhaps a fellow student has pointed out an issue that the presenter is struggling with resolving, I will suggest he/she takes a break and we revisit the problem the next class day.

The instructor is also the official keeper of the records. I have the official list of which problems have been presented and which are still open. These are two things the students do tend to keep track of, too. Additionally, I keep track of any tweaks that result in a new problem. The latter is something I really enjoy, though it took some getting used to. When new conjectures arise I do not always know the answer (as opposed to using a text where the problems are mostly the same regardless of the book). I learned to embrace the time of discovery via students. A typical day starts by asking if anyone has anything to present. Using my records, if I do see someone is falling behind in presentations I will say, “We haven’t heard from Z in awhile. Do you have something to present?” Some colleagues use software to keep track of such things. If you get involved in the IBL community you should be able to find them if you want.

At the beginning of the day I pass back write-ups, make a few general announcements (“Math Club meeting Thursday,” or “Make sure you are using the right notation in your limit proofs.”), and pass out any new pages. Then I take my seat *in the back of the room*. Students have a tendency to look at me to see if I react and use that to decide if they should or should not be looking for questions to ask. So the sides of the room are out. The presenter, the one person who is supposed to be looking in my direction, will still try and find confirmation in my face. It is important to remain impassive and let the presenter and his/her classmates work it all out.

After I sit down my role is that of moderator, helping to *further* the class discussion, but not to lead it. In the early days, I make sure the presenters are writing clearly and pausing between important statements so their classmates can keep up. I might spur students to ask questions by saying, “Does everyone agree with line 3?” or “Is there an assumption being made here when the theorem is applied?"

In summary, the instructor acts as bookkeeper with the problems and the grades and after that is a facilitator, making sure the class remains on task and that learning occurs.

0.4 Why these notes

One reason I decided to write my own notes is to introduce students to some of the generalizations and types of questions that someone like myself, who publishes in classical real analysis, faces in the scholarly life. Thus there are topics on these handouts that are not found in the typical analysis book/course. These include generalized continuity and convergence in metric spaces. None of these topics are covered too deeply (although metric spaces turn up again and again). It is amazing how well the students can do when they are not told that these topics are post-Bachelor's level or atypical.

0.5 Course Content

These notes are for the first semester of a two semester Advanced Calculus/Real Analysis sequence. There are several goals at work here. First is the in-depth study of a particular topic. Such depth is important for students of mathematics. Sometimes this is accomplished through taking an analysis sequence; sometimes a year-long algebra class; sometimes via two different, but related classes (a semester of algebra and one of number theory or a semester of real variables followed by one in complex variables). Secondly is honing the students' skills in writing proofs. All students, by the time they are ready for this class, have some learning in constructing proofs and proof techniques (direct, contradiction, contrapositive, induction, et al.), but one only gets better at this by writing more and more proofs, not by reading about proofs. Lastly, as with many "pure" math classes, this course should give students the opportunity to see how one extends the boundaries of what is known and not known via conjecture. This is where many textbook oriented classes fail as books do not rightly convey the stops and starts of what a mathematician studies, instead presenting a polished result that unfortunately gives students the impression that all that can be known is known.

The topics covered in the first semester are

- fundamentals of the real line,
- limits and convergence, and
- functions and continuity.

In the second semester, we continue onward by studying

- differentiation,
- integration,
- series of numbers, and
- series of functions.

Included in this text are several types of generalization of the main topics. For instance, symmetric derivatives and the Riemann-Stieltjes integral. The hope is to show students how things progress after what they learned in the Calculus sequence. Problems that are shown false by a counter-example (e.g. Let f have the property that $f(a) \cdot f(b) < 0$. Then there is a value c, with $a < c < b$ such that $f(c) = 0$.) are ripe for the question, "How can we change this to make it true?" during the problem's presentation. This shows how mathematicians hone an idea until it is both precise and correct. One of my favorite ideas is the so-called pathological example. Something so far afield from a certain property as to be considered a "mathematical monster." A continuous, nowhere differentiable function is an example of this. Many of the requests for examples in this text are leading the student toward some of these monsters.

0.6 Grading

My gradebook consists of three separate sections: homework, presentations, and exams. They are described below, however, for each instructor, there are many ways to determine grades. The ensuing paragraphs include the final percentages I use in this class. My technique is not a command, it's not even a suggestion. It is an example.

Section One is for written homework turned in. This assigned homework is worth 20% of the final grade. Someone once referred to this method as having perpetual homework. No one warned me about the perpetual grading. You are now forewarned. I insist that homework is written up neatly and professionally. Students do not need to use LaTeX (though I would not complain), but since there is an emphasis on learning to communicate mathematics, the students need to learn that doing sloppy work is *not* acceptable. Homework falls into two categories: before presentation and after presentation. Before presentation homework that is correct and well-written receives a grade of 10. Originally, my syllabus said homework that has issues will receive a grade between 1 and 9, but I found over time that unless something spectacular happened I only used the grades 9 (small error), 7 (essentially correct, but not a perfect presentation), 5 (on the way, but a lot wrong), and

3 (trying). The grade of 1 was very rare as students did not have a hard deadline and would wait until they had some confidence in their work before turning it in. So rather than 1 - 10, my grades are now 1, 3, 5, 7, 9, and 10. After the problem was presented, a student could still turn in a write-up for either a grade of 5 (correct) or 3 (mostly correct). At the date of an exam, I cut off the students from turning in any more problems from the sections covered. There is a bit of self-defense here. I do not want thirty problems from the beginning of the semester turned in at the end because students finally got around to writing their solutions to them.

Section Two is for student presentations. Percentage-wise this counts as 10% of the grade total. The grading scale is 2 points through 5 points. Two is the minimum since students do not really present unless they have some content and confidence in their work, so they have earned something. I do not recall anyone actually settling for a two, though, as this means there are some major flaws in the mathematics. So you may think of the 2 as a placeholder, which will be replaced with a grade for a subsequent presentation. Three is for an average attempt. The student has given correct ideas, with some flaws in terms of presentation and boardwork. A presentation at the level of 4 is correct and has a bit of polish to it. Something that can be followed by the students. A 5 is for an exceptional job. An absolutely correct solution with an excellent presentation where the concepts are broken down well and understood by the class earns this grade. On rare occasion, the presenter surprises me with a direction I never saw coming, I will give that a 5.5 out of 5 for creativity.

For a topic such as this, I tend to look at trends more than averages. I would not compare a student with 5 presentations worth 20 points against one with 8 presentations and 30 points. For the final grade I want to look at the *weighted* total. If the grade is borderline what I will look at is trends, to see if the presentation grades improved over time.

Students who are not actively presenting are still capable of earning points while a presentation is happening. This is Section Three. Those grades are for Questions, Insights, and Contributions (Q, I, C). Each of these types of grades is a single point, but students can earn multiple points per day. A "Q" is a good question. An example of this is, "Is your assumption the same as saying it's a Lipschitz function?" Questions such as, "Can you say why your second line works?" and "Should that be f' and not f?" are good questions, and students should be encouraged to ask them, but are not worth credit. The "I" is for some mathematical insight, a comment such as, "So a Riemann integral is a Riemann-Stieltjes integral where $g(x) = x$." The "C" for contribution is hard to nail down exactly, but as Supreme Court Justice said, "I know it when I see it." These points are part of the presentation grade. A good question or comment is as worthwhile as a nice presentation.

In addition to all of this, I give three exams and a comprehensive final. For each exam I will also give them a list of problems whose proof or counter-example I expect them to know. A majority of the exam, which is all proof and example, comes from that list with usually one problem thrown in that they have not seen to separate the A's from the B's in my class. These are two different parts of a student's final grade. The exam average is 50% of the course grade and the final is 20%.

Two things I wish to point out before finishing this section:

First, I have been criticized in the past for the percentages I use. The critics' point is mostly that students can pass this class without doing a lot of presentation and/or turning in their own version of solutions rather than what their peers show at the board. I, myself, don't see much of a problem with this. If a student knows enough of the material, then he/she knows enough of the material. I do believe my tests do have one or two problems that require enough creativity to separate the A's from the B's from the rest.

Second, the question comes up how all these sections and percents come together to make a final grade. I was given some very prescient advice before I started this class and it was absolutely spot on and something I wish to pass to you. The final totals tend to self-segregate. Each semester, when I computed the final spreadsheet there were obvious chunks where the A's, B's, C's, D's, and F's lay.

0.7 The Rules for Presentations

Presentations will be scored in the following way

- Accuracy of the problem you present, including the guidelines below.
- Defense of your work, including following the guidelines below.

Points: You will be awarded a point in the appropriate category every time you contribute in one of the following ways.

- P presentation points awarded
  - 5 points for a correct presentation
  - 2 - 4 points for a presentation with error(s)
- Q asking a good question of the presenter
- I contributing a demonstration of a mathematical insight
- C an oral contribution other than the two categories above

Remember: the presenter should be the person to answer a question.

Guidelines for Your Presentation

- Write the problem on the board.
- State what method/theorem/idea you will use.
- Clearly explain each step.
- Do not use the words stupid, trivial, obvious, etc.

Guidelines for Defending Your Work

- You must answer your classmates and professors questions in a respectful manner.
- Do not use the words stupid, trivial, obvious, etc.
- You must try to answer every question raised.
- It is okay to say, “I’m not sure that I understand your question.” It is not okay to say, “Your question doesn’t make any sense.”
- Talk to the class, not to the board.

Guidelines for Criticism of Classmates Work

- You are to ask questions about your classmates work. Do NOT suggest another technique. In some cases, there may be more than one way to solve a problem.
- Do not use the words stupid, trivial, obvious, etc.
- You must ask questions in a respectful manner.
- Its okay to say, “Can you explain how you got from line 3 to line 4?” It is not okay to say, “Line 4 is wrong,” or “Line 4 doesn’t make any sense.”

To the Student

0.8 Introduction - Advanced Calculus

Advanced Calculus (or Mathematical Analysis) is one of the most important classes one takes as a math student. Unlike a class in, say, abstract algebra, where the main concepts (rings, groups, fields) are new, in this class we will look at topics you should be familiar with from your earlier calculus sequence (limits, differentiation, integration, and sequences & series). That is the good news. Since you are familiar with all these topics, what this class then is about is an increase in rigor. This is not about determining the integral of a given function, but about proving what type of function is integrable. We will also, as is expected in higher-level mathematics, look at some generalizations of these ideas. This serves as an introduction to what many mathematicians do when researching a topic.

Now, a word about prerequisites. Most courses of this type require a C or better in a course on proof techniques. Let us be blunt here: A grade of C or better does *not* mean that you have some vague recollection a course that had a lot of proofs in it. It means that you learned the material and are capable of both creating your own proofs and following proofs of others. The types of proof techniques you should be familiar with include, but are not limited to, induction, direct proof, proof by contradiction, and proof by exhaustion.

This is an extremely important course as it develops the ability to apply logic, which is what makes the math major so desirable to employers (see www.maa.org/careers/ for more details). It takes hard work, but it is worth it.

Now, all that said, let me add that mathematics is one of the most creative disciplines out there. There is very little difference between the imagination, hard work, and focus needed to paint a still life and to determine a proof. Note that I did not say paint the *Mona Lisa* and prove *Fermat's Last Theorem*. Just as you don't have to paint masterpieces in order to enjoy painting, you do not have to solve the greatest open problems in mathematics to enjoy the thrill and pride associated with developing a proof. Learning to find proofs and write proofs is a skill that comes with practice; a lot of practice. If you start to feel discouraged and like you are flailing and lost, I suggest you think about the man who did solve Fermat's Last Theorem and his thoughts on doing math.

> I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of – and couldn't exist without – the many months of stumbling around in the dark that proceed them.

Andrew Wiles, Princeton University

Get ready and get excited to take a journey deeper into analysis than you have gone before. And enjoy the trip!

0.9 R.L. Moore and his Method

> That student is taught the best who is told the least.

R.L. Moore, 1882-1974

> [Dr. Moore] told us early on that he had no use for the university guidelines stating that we should expect three hours of outside class work for each hour in the classroom. He said he wanted us to think about his class all day, every day, to go to bed thinking about it, to wake up in the night thinking about it, to get up the next morning thinking about it, to think about it walking to class, to think about it while we were eating. If we weren't prepared to do that, he didn't want us in his class. It was also quickly evident that he meant what he said....

John Green, PhD, University of Texas, 1968, under R. L. Moore

The core of any course using the Moore Method (a type of Inquiry-Based Learning) is the understanding that people learn best by doing the work, not by being told the results. Moore developed his method in 1911 while teaching at the University of Pennsylvania and then took it with him to the University of Texas where he worked from 1920 until his retirement in 1969. The mathematics building at UT is named after him.

To begin with, a true Moore Method course has no book. Instead, the "book" for the course is written by the students as the semester/year goes on. That is, at the end of this course the collection of definitions, axioms, and results you have will be enough for a text. Each and every student is expected to do his/her own work. The use of outside sources (including, but not limited to, books, the internet, tutors, friends, classmates, and professors who are not me) is strictly forbidden.

You will be given handouts which contain Definitions, Axioms, Problems, Exercises, and a few Theorems. You are to provide proofs for anything labelled Theorem (which by virtue of being called a theorem must be true). Any problem will start with the directive "Prove or Disprove." It is up to you to determine the truthfulness of the statements and, if it is true, a proof; if it is false, a counterexample with explanation. Exercises can be presented at the board for a grade, but not turned in. The Exercises are to help illustrate definitions and topics. Similarly, there are Remarks throughout to explain concepts and introduce new discussions.

In addition, to written work, you will be presenting at the board. There will be little, if any, lecture in this class. On class days, you will be called on to present solutions. You may choose to show your work on any problem not yet presented in class. If you are not the first person chosen on a given day, you will not have your choice of all assigned problems. This means that you may not get to present your first choice problem, so you should be prepared with solutions to more than one problem.

During student presentations, you are encouraged (expected) to ask questions, and to think critically about the solution presented by the classmate. The class has the job of determining the validity of arguments presented, and the instructor will occasionally allow incorrect solutions to stand in class. These incorrectly presented problems may appear on tests. While "Should that two be a three?" is a question, you should also ask "Can you explain how you went from Step 2 to Step 3." Don't worry, if you have that question, so do others and the presenter should be able to answer that question and others. Every once in awhile there are questions someone doesn't know how to answer. It happens to everyone, so the presenter should not feel bad or embarrassed. The response of "I'm not sure, I'll have to get back to you," is absolutely fine, but *the speaker is then responsible for finding out the answer and showing it to the class*. It will not take you long to figure out how to be the speaker and how to be an active audience member. Then it's just a matter of enjoying the learning.

A quote from Paul Halmos[1]:

"Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?"

and another

"Mathematics is not a deductive science – that's a cliché. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork."

and a final one

"It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts."

[1] If you do not know who Paul Halmos was, you may find it worthwhile to look him up after you are done with analysis.

Chapter 1

Review of Proof Techniques

This is not graded work, but necessary work nonetheless. We will go over the proofs in the next class with you doing the presentation. However, unlike the rest of the problems, you may talk amongst yourselves about these.

1. Proofs by Induction
   An induction proof is used when the statement being proven is one that has truth values on a subset of the natural numbers $\mathbb{N}$ of the form $\{k, k+1, k+2, \ldots\}$. Usually, but not always, this is stated in the problem. For example, Prove that for each n in $\mathbb{N}$, $2^n \geq n+1$. *Go ahead and prove this.*

2. Direct Proofs
   This is for a statement of the form, "If P, then Q." The technique is to assume P is true and then deduce (this includes showing steps) that Q is true. For example, If n is an even integer, then n^2 is an even integer.

3. Indirect Proofs - Contrapositive
   Given the statement "P implies Q," its *contrapositive* is the logically equivalent "Not Q implies not P." Proving this second version is the same as proving the first. So we assume not Q is true and deduce that not P is true. Try this example, If xy is odd, then either x or y is odd.

4. Indirect Proofs - Proof by Contradiction
   Usually this is something like a proof by contrapositive. Given the statement "P implies Q," begin with the idea "P and not Q." Then deduce a statement of the form "R and not R," this is a contradiction. Prove with this method that there are infinitely many primes.

5. Proofs involving Quantifiers
   Quantifiers are statements of the form "for all" or "there exists." For the first, you must take an arbitrary member of this set and prove the statement is true always. For the latter, you can exhibit an example of one instance.

6. Counterexamples (Sometimes can be thought of as "Disproofs" involving Quantifiers)
   If you think a "for all" statement is false, the negation of "for all" is "there exists." Thus showing *ONE* instance where the statement is untrue is enough. On the other hand, one example will not work to disprove existential statements. One example will not contradict the existence of something, it just shows the something doesn't work in one case. It may still work under other circumstances. Determine the truthfulness of the statement, "For all rational numbers r there exists a rational number r^{-1} such that $r \cdot r^{-1} = 1$."

7. Finally, how proofs should be written up.

   Prove that for any real numbers a and b, $a^2 + b^2 \geq 2ab$.

   Scratchwork: If $a^2 + b^2 \geq 2ab$, then $a^2 + b^2 - 2ab \geq 0$. But the left side is $(a-b)^2$. Your scratchwork should NEVER be turned in, but it should be used (usually kind of backwards) in your proof.

   Proof:
   Let a and b be two real numbers. Since every square is non-negative, we know
   $$(a-b)^2 \geq 0.$$
   This is equivalent to
   $$a^2 - 2ab + b^2 \geq 0,$$
   which can be written as
   $$a^2 + b^2 \geq 2ab.$$
   Since a and b were arbitrary, this holds for all real numbers a and b.

   Notice that this is neither just a list of symbolic manipulations nor written out in longhand English with paragraph format. A combination of English and mathematical statements is what is called for.

8. More examples for you to prove/disprove:
   (a) Prove that if x and y are positive real numbers and $x \neq y$, then $x/y + y/x > 2$.
   (b) For all real numbers a, b, and c
   $$a^2 + b^2 + c^2 \geq ab + bc + ac.$$
   (c) Prove that $5^n - 2^n$ is divisible by 3 for any natural number n.

(d) Determine whether the statement, "There exists a real number x such that for all real numbers y, $x+y=x$." is true or false.

(e) If a non-negative x has the property that $x<\varepsilon$ for all $\varepsilon>0$, then $x=0$.

Chapter 2

Preliminaries

2.1 Notation

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ represent the natural numbers, integers, rational numbers, and real numbers, respectively
- $a \in A$ means a is an element (a member) of the set A
- $A \subseteq B$ means that A is a subset of B
- $A \subset B$ means A is a subset of B and not equal to B (we say A is a *proper* subset of B)
- $A \cup B$ means the union of A and B
- $A \cap B$ means the intersection of A and B
- $A \setminus B$ means all elements in the set A that are not elments of the set B.
- A^c means the complement of the set A (with respect to a known universal set)
- $\emptyset$ means the empty set

2.2 Fundamentals

Definition 1. *A* statement *is a sentence that is either true or false.*

Definition 2. *An* axiom *is a statement that we accept without proof.*

Definition 3. *A* theorem *is a proposition proved from other propositions, definitions, and axioms which are previously known.*

Remark 4. *We will usually, but not always, write xy for $x \cdot y$.*

Axiom 5. **(Closure Under Addition)** *If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$.*

Axiom 6. (Closure Under Multiplication) *If $x, y \in \mathbb{R}$, then $xy \in \mathbb{R}$.*

Axiom 7. (Commutativity) *If $x, y \in \mathbb{R}$, then $x + y = y + x$ and $xy = yx$.*

Axiom 8. (Associativity) *If $x, y, z \in \mathbb{R}$, then $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.*

Axiom 9. (Existence of Identities) *There exists an element $0 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x + 0 = x$ and there exists an element $1 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $1 \cdot x = x$.*

Remark 10. *For any $x \in \mathbb{R}$, $-x$ denotes $(-1)x$.*

Axiom 11. (Distributive Laws) *If $x, y, z \in \mathbb{R}$, then $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.*

Prove or Disprove 12. *Show that for any $x \in \mathbb{R}$ we have $x \cdot 0 = 0$.*

Axiom 13. (Existence of Inverses) *For all $x \in \mathbb{R}$ there exists $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and for all $x \in \mathbb{R} \setminus \{0\}$ there exists $x^{-1} \in \mathbb{R}$ such that $x \cdot x^{-1} = 1$.*

Prove or Disprove 14. *For real numbers x and y, if $xy = 0$, then either $x = 0$ or $y = 0$.*

Axiom 15. *If $x = y$ and $z \in \mathbb{R}$, then $x + z = y + z$ and $xz = yz$.*

Exercise 16. *Could either form of the axiom above be written as "if and only if"? Why or why not?*

Notation: We define subtraction and division in terms of the inverses of addition and multiplication, respectively. Thus if $x, y \in \mathbb{R}$, then $x - y = x + (-y)$ and if $y \neq 0$, then $x \div y = x \cdot y^{-1}$.

Definition 17. *The statement $x < y$ is left rigorously undefined, but we will use it to mean precisely as you have known. The statement $x \leq y$ means either $x = y$ or $x < y$; aslo $y \geq x$ means the same as $x \leq y$ and $y > x$ means the same as $x < y$.*

Axiom 18. *If $x, y \in \mathbb{R}$, then either $x \leq y$ or $y \leq x$.*

Axiom 19. *If $x, y \in \mathbb{R}$ with both $x \leq y$ and $y \leq x$, then $x = y$.*

Axiom 20. (Transitivity) *If $x \leq y$ and $y \leq z$, then $x \leq z$.*

Axiom 21. *If $x \leq y$ and $z \in \mathbb{R}$, then $x + z \leq y + z$.*

Axiom 22. *If $x \leq y$ and $z > 0$, then $xz \leq yz$.*

Prove or Disprove 23. *If $x \leq y$, then $-y \leq -x$.*

Prove or Disprove 24. *If $x \leq y$ and $z < 0$, then $yz \leq xz$.*

Prove or Disprove 25. *If $x, y \geq 0$, then $xy \geq 0$.*

Prove or Disprove 26. *For all $x \in \mathbb{R}$, $x^2 \geq 0$.*

Prove or Disprove 27. *If $x > 0$, then $x^{-1} > 0$.*

Prove or Disprove 28. *If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.*

Remark 29. *Each of the true properties we have looked at so far are true for the set $\mathbb{Q}$. This brings up the question of how the real numbers are different from the rational numbers and why that difference is important. That is the topic we shall now delve into.*

We begin with a set S which is a subset of $\mathbb{R}$.

Definition 30. (Upper Bound) *If there exists a real number M such that for all $x \in S$ we know $x \leq M$, then M is an upper bound for S.*

Definition 31. (Supremum, Least Upper Bound) *If there exists an $s \in \mathbb{R}$ such that s is an upper bound for S and, for any upper bounds M of S, we have $s \leq M$. then we call s the supremum of S (also called a least upper bound) and denote this by* $\sup(S)$.

Remark 32. *Define the analogous concepts of lower bound and infimum (greatest lower bound), written* $\inf(S)$.

Remark 33. *When a set S is bounded above* and *bounded below, we say that the set S is bounded.*

Axiom 34. (Completeness Axiom) *Let $S \subset \mathbb{R}$ be nonempty and bounded above, then* $\sup(S)$ *exists.*

Prove or Disprove 35. *Prove that for any bounded set S, there is only one supremum; that is, suprema are unique.*

Prove or Disprove 36. *Show that the Completeness Axiom does not hold when $\mathbb{R}$ is replaced by $\mathbb{Q}$.*

Prove or Disprove 37. *Let $S \subset \mathbb{R}$ be nonempty and bounded below. Then* $\inf(S)$ *exists.*

Prove or Disprove 38. *Let A and B be nonempty sets subsets of $\mathbb{R}$ such that $A \subseteq B$. If B is bounded above, then A is bounded above and* $\sup(A) \leq \sup(B)$.

Remark 39. *What is the analogous problem and answer for bounded below and infima?*

Prove or Disprove 40. *Let A and B be nonempty subsets of $\mathbb{R}$. If A is a proper subset of B, then* $\sup(A) < \sup(B)$.

Prove or Disprove 41. *If A and B are nonempty subsets of* $\mathbb{R}$ *such that for all* $x \in A$ *and for all* $y \in B$, $x \leq y$, *then* $\sup(A) \leq \inf(B)$.

Remark 42. *The set of real numbers (*$\mathbb{R}$*) naturally decomposes into two pieces: the rational numbers (*$\mathbb{Q}$*) and the irrational numbers (*$\mathbb{R} \setminus \mathbb{Q}$ *which is sometimes written as* $\mathbb{I}$*, but we will not be using that). So let us look at some of the properties of this decomposition. We begin with the Archimedean Property which is actually about the real numbers, not just rationals or irrationals. Some texts use the Archimedean Property as an axiom. This will be presented as a Theorem (for you to prove) here as it is a consequence of the Completeness Axiom despite the two looking very different.*

Theorem 43. (The Archimedean Property of $\mathbb{R}$**)** *For all positive* $x, y \in \mathbb{R}$ *there exists a natural number n such that* $nx > y$.

Prove or Disprove 44. (Denseness of the Rational Numbers) *Suppose* $x, y \in \mathbb{R}$ *with* $x < y$. *There is a* $z \in \mathbb{Q}$ *such that* $x < z < y$.

Prove or Disprove 45. (Denseness of the Irrational Numbers) *Suppose that* $x, y \in \mathbb{R}$ *with* $x < y$. *There is a* $z \in \mathbb{R} \setminus \mathbb{Q}$ *such that* $x < z < y$.

Prove or Disprove 46. *The sum (difference) of any two rational numbers is a rational number.*

Prove or Disprove 47. *If* $r \in \mathbb{Q}$ *and* $z \in \mathbb{R} \setminus \mathbb{Q}$, *then* $r + z \in \mathbb{R} \setminus \mathbb{Q}$.

Prove or Disprove 48. *If* $r \in \mathbb{Q}$ *and* $z \in \mathbb{R} \setminus \mathbb{Q}$, *then* $rz \in \mathbb{R} \setminus \mathbb{Q}$.

Prove or Disprove 49. *If* $x, y \in \mathbb{Q}$, *then* $(x + y)/2 \in \mathbb{Q}$.

Prove or Disprove 50. *If* $x, y \in \mathbb{R} \setminus \mathbb{Q}$, *then* $(x + y)/2 \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 51. *Consider* $s, t \in \mathbb{R}$ *with* $s < t$. *An interval I is a subset of* $\mathbb{R}$ *of any one of the following forms:*

$$
\begin{aligned}
[s,t] &= \{x \in \mathbb{R} | s \leq x \leq t\} \\
(s,t) &= \{x \in \mathbb{R} | s < x < t\} \\
(s,t] &= \{x \in \mathbb{R} | s < x \leq t\} \\
[s,t) &= \{x \in \mathbb{R} | s \leq x < t\} \\
[s,\infty) &= \{x \in \mathbb{R} | s \leq x\} \\
(s,\infty) &= \{x \in \mathbb{R} | s < x\} \\
(-\infty,t] &= \{x \in \mathbb{R} | x \leq t\} \\
(-\infty,t) &= \{x \in \mathbb{R} | x < t\}
\end{aligned}
$$

Also considered intervals are $(-\infty, \infty) = \mathbb{R}$ *and* $\emptyset$ *(which is called an empty interval). For some authors the singleton* $\{a\} = [a, a]$ *is referred to as a* degenerate *interval. We will not bother with a single point as an interval.*

Prove or Disprove 52. *The union of a collection of intervals will also be an interval.*

Prove or Disprove 53. *The intersection of a collection of intervals will also be an interval.*

Definition 54. *A nonempty collection of intervals J_i are called* nested *if* $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$.

Prove or Disprove 55. *The intersection or union of nested intervals is an interval.*

Remark 56. *To show the distance between two points on the number line we typically use the absolute value function. Although we have not yet actually defined a function explicitly, you do have an intuitive idea of what a function is and how functions work, and a large repertoire of examples at your disposal, so everything should work smoothly.*

Definition 57. *Let $x \in \mathbb{R}$. Then the absolute value of x, denoted by $|x|$, is given by*

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is sometimes referred to as the Euclidean distance between two points and can generalize to n-dimensions. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, then

$$d(x,y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Prove or Disprove 58. $|xy| = |x|\,|y|$.

Prove or Disprove 59. *For any two real numbers x and y, $|x-y| = |y-x|$.*

Prove or Disprove 60. (The Triangle Inequality) $|x+y| \leq |x| + |y|$

Prove or Disprove 61. $||x| - |y|| \leq |x-y|$

Prove or Disprove 62. *Let $x, y \in \mathbb{R}$. Then $x = y$ if and only if for all $\varepsilon > 0$, $|x-y| < \varepsilon$.*

Remark 63. *Using absolute value to find the distance between points on the number line is a specific example of a more general idea of a function which defines distance between points on a set S. Such a function is called a* metric *on S. We look at this more closely throughout this course. Before you read the definition below, you might want to take a minute and think about what you think defines a distance function.*

Definition 64. (Metric) *A metric (distance function) on a set S is a function d which satisfies the following three properties:*

1. *For all* $x,y \in S$, $d(x,y) \geq 0$ *(called nonnegativity) and* $d(x,y) = 0$ *if and only if* $x = y$.
2. *For all* $x,y \in S$, $d(x,y) = d(y,x)$ *(symmetry)*
3. *For all* $x,y,z \in S$, $d(x,y) \leq d(x,z) + d(z,y)$ *(The Triangle Inequality)*

(Convince yourself that these are all satisfied by $|\cdot|$ *on* $\mathbb{R}$.*)*

Prove or Disprove 65. *The function*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on any nonempty set S. This is called the discrete metric.

Prove or Disprove 66. *Let* $z = (z_1, z_2) \in \mathbb{R}^2$ *represent points in the plane. The function given by* $\overline{d}(x,y) = |x_1 - y_1| + |x_2 - y_2|$ *is a metric on* $\mathbb{R}^2$.

Prove or Disprove 67. *The function* $D(x,y) = 1 - 2^{-(x-y)}$ *is a metric on* $\mathbb{R}$.

Exercise 68. *To look at how different ideas of distance make for dissimilar results, look at the plane with two different metrics, the Euclidean metric*

$$d(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and the so-called taxicab metric, $\overline{d}$ *above. The unit circle is the set of all points whose distance from the origin is 1 unit. So draw the set of points where* $d(x,0) = 1$ *and the set of points where* $\overline{d}(x,0) = 1$ *and see how they are different circles.*

Chapter 3

Sequences and Limits

Definition 69. *A sequence is a function f with domain $\mathbb{N}$ and range $\mathbb{R}$.*

Remark 70. *The usual notation for a sequence is to let $x_n = f(n)$ and to write the sequence $(x_1, x_2, x_3, \ldots)$ as $\mathbf{x} = (x_n)_{n=1}^{\infty}$ or (x_n).*

Definition 71. **(Convergent Sequence)** *The sequence (x_n) converges to the real number L, if for every real number $\varepsilon > 0$ there exists a natural number N such that for all $n > N$ we have*

$$|x_n - L| < \varepsilon.$$

This is written as

$$\lim_{n\to\infty} x_n = L$$

or oftentimes $x_n \to L$ When such an L exists we say (x_n) is a convergent sequence.

Remark 72. *The intuitive, but incorrect, way many people think of a sequence converging to L is, "As n gets bigger and bigger, the x_n's get closer and closer to L." Our definition makes the concept rigorous. With it we see explicitly the relationship between n and "closer to L." Specifically, for each ε, we will have a formula to determine N so we are guaranteed that if $n > N$, then x_n will be within ε of L.*

Theorem 73. *If $x_n \to a$ and $x_n \to b$, then $a = b$.*

Remark 74. *Theorem 73 shows that if a sequence does converge to a limit, then the limit must be unique. Showing that an object is unique is a very typical problem in upper-level mathematics.*

Definition 75. **(Divergent Sequence)** *A sequence which does not converge is said to diverge. Note this can happen in several ways. Determine the negation of the definition of convergence for a formal definition we use when the limit does not exist.*

Remark 76. *There is also the possibility that the* $\lim x_n$ *exists, but it is not finite. We say* $\lim_{n\to\infty} x_n = \infty$ *if for every* $M \in \mathbb{R}$ *there exists a natural number* N *such that if* $n > N$*, then* $x_n > M$*. Formulate the analogous definition for* $\lim_{n\to\infty} x_n = -\infty$*.*

Example 77. *Prove that* $\lim_{n\to\infty} \frac{n}{n+1} = 1$*.*
Proof:
Let $\varepsilon > 0$ *be given. For this* ε *define* $N \in \mathbb{N}$ *so* $N > 1/\varepsilon$*. Then if* $n > N$ *we have*

$$|x_n - L| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \frac{1}{1/\varepsilon} = \varepsilon.$$

Since this ε *was arbitrary, we have shown that for all* $\varepsilon > 0$ *there is a way to find an* N*, depending only on* ε*, so that for* $n > N$ *we have* $|x_n - L| < \varepsilon$ *and thus* $\frac{n}{n+1}$ *converges to* 1*.*

Note: All of the scratch work was left out of the proof. The scratch work is where we find the relationship between ε *and* N*. This is done by starting with* $|x_n - L|$*, simplifying that expression down to one involving* N *and then setting* that *less than* ε *and solving for* N*. The scratch work then shows up as the offset line in the proof.*

Exercise 78. *Write a formal proof for each of the following:*

1. $x_n = \frac{1}{\sqrt{n}-1}$ *converges to* 0*.*
2. $y_n = \cos(n\pi)$ *diverges*
3. $z_n = (4/3)^n \to \infty$

Definition 79. *Let* x *and* y *be sequences and* $\alpha \in \mathbb{R}$*. Then we define the following sequences:*

$$\begin{aligned} \mathbf{x} \pm \mathbf{y} &= (x_1 \pm y_1, x_2 \pm y_2, \ldots) \\ \alpha\mathbf{x} &= (\alpha x_1, \alpha x_2, \ldots) \\ \mathbf{x} \cdot \mathbf{y} &= (x_1 y_1, x_2 y_2, \ldots) \end{aligned}$$

Prove or Disprove 80. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are convergent sequences, then* $\mathbf{x} + \mathbf{y}$ *is a convergent sequence.*

Prove or Disprove 81. *If* $\mathbf{x}$ *is a convergent sequence, then so is* $\alpha\mathbf{x}$ *where* $\alpha \in \mathbb{R}$*.*

Prove or Disprove 82. *If* $\mathbf{x}$ *is a convergent sequence, then so is* $\mathbf{1}/\mathbf{x} = (1/x_n)_{n=1}^{\infty}$*.*

Theorem 83. **(Squeeze Theorem)** *Let* $\mathbf{x}$*,* $\mathbf{y}$*, and* $\mathbf{z}$ *be sequences and* $N \in \mathbb{N}$ *so that* $x_n \to L$*,* $z_n \to L$*, and for* $n > N$

$$x_n \le y_n \le z_n.$$

Then $y_n \to L$*.*

Definition 84. *We say a sequence* $\mathbf{x}$ *is* bounded *if there exists numbers* m *and* M *such that for all* $n \in \mathbb{N}$

$$m \leq x_n \leq M.$$

Exercise 85. *Come up with corresponding definitions for* bounded above *and* bounded below. *Give an example of a sequence which is bounded above but is not bounded. Do the same for bounded below, but not bounded.*

Prove or Disprove 86. *If* (x_n) *converges then it is bounded.*

Definition 87. *Let* $\mathbf{x}$ *be a bounded sequence. Define sequences* (u_n) *and* (v_n) *by, for all* $N \in \mathbb{N}$

$$u_N = inf\{x_n : n > N\}$$

and

$$v_N = sup\{x_n : n > N\}.$$

Prove or Disprove 88. *If* $\mathbf{x}$ *is a bounded sequence, then* $\lim_{N\to\infty} u_N$ *and* $\lim_{N\to\infty} v_N$ *exists.*

Definition 89. (Limit Superior and Limit Inferior) *Let* $\mathbf{x}$ *be a bounded sequence and let* (u_N) *and* (v_N) *be defined as above. Define the* limit inferior *and* limit superior *of* $\mathbf{x}$, *denoted by* $\liminf x_n$ *and* $\limsup x_n$, *respectively, as*

$$\liminf x_n = \lim_{N\to\infty} u_N$$

and

$$\limsup x_n = \lim_{N\to\infty} v_N.$$

Exercise 90. *Find the* $\liminf x_n$ *and* $\limsup x_n$ *for each of the following sequences:*

1. $x_n = (-1)^n$
2. $x_n = e^{-n}$

Prove or Disprove 91. *If* $lim_{n\to\infty}x_n$ *exists and is finite, then* $limsup_{n\to\infty}x_n$ *and* $\liminf_{n\to\infty} x_n$ *exist and are finite.*

Prove or Disprove 92. *If both* $limsup_{n\to\infty}x_n$ *and* $\liminf_{n\to\infty} x_n$ *exist and are finite, then* $lim_{n\to\infty}x_n$ *exists and is finite.*

Definition 93. (Cauchy Sequence) *A sequence* (x_n) *of real numbers is called a* Cauchy *sequence if for every* $\varepsilon > 0$ *there exists a natural number* $\mathbb{N}$ *so that if* $n > m \geq N$, *then*

$$|x_n - x_m| < \varepsilon.$$

Exercise 94. *Show that* $(x_n) = (1/n)$ *is a Cauchy sequence.*

Prove or Disprove 95. *Every Cauchy sequence is a bounded sequence.*

Prove or Disprove 96. *If (x_n) is a convergent sequence of real numbers, then (x_n) is a Cauchy sequence of real numbers.*

Prove or Disprove 97. *Every convergent sequence is a bounded sequence.*

Remark 98. *In the real line, every Cauchy sequence converges to a real number which is why $\mathbb{R}$ is called a "complete metric space." This is again the difference between the spaces $\mathbb{Q}$ and $\mathbb{R}$. This is equivalent to the Completeness Axiom and from where the name comes. This is an extremely important property. One issue with the $\varepsilon - L$ definition of convergence is one must know the value of the limit before proving anything. One can show Cauchy without knowing the end result.*

Exercise 99. *Give an example of a sequence in $\mathbb{Q}$ with the Euclidean metric (recall Exercise 68) that does not converge to a rational number. (This shows that $\mathbb{Q}$ with the Euclidean metric is not complete, whereas $\mathbb{R}$ with the same metric is complete.)*

Remark 100. *Sometimes we are interested in not all of the terms of the sequence, but only some of them (infinitely many of them).*

Definition 101. *A* subsequence *is a sequence whose terms consist of some of the terms of (x_n) taken* in order. *The usual notation is to write our subsequence as*

$$(x_{n_k}) = x_{n_1}, x_{n_2}, x_{n_3}, \ldots$$

where n_1 denotes the subscript of the first term taken from (x_n), n_2 denotes the second term taken from (x_n), et cetera. Notice that for the value of n_k we must have $n_k > k$.

Exercise 102. *Give five different subsequences for the sequence $(1/n)$.*

Prove or Disprove 103. *If (x_n) converges then, for any subsequence (x_{n_k}) the subsequence converges and the limits are the same.*

Prove or Disprove 104. *If (x_n) diverges, then, for any subsequence (x_{n_k}) the subsequence also diverges.*

Definition 105. *Let $\mathbf{x} = (x_n)$ be a sequence. A tail of the sequence is a subsequence of the form $(x_n)_{n=k}^{\infty}$ for some $k \in \mathbb{N}$.*

Many of the results we have can be phrased in terms of tails of the sequence.

Theorem 106. *Let $\mathbf{x}$ be a sequence and $L \in \mathbb{R}$. Then the following are equivalent:*

1. $\mathbf{x} \to L$,

2. *for every* $k \in \mathbb{N}$*, the tail* (x_{n+k}) *converges to* L*,*

3. *there exists* $k \in \mathbb{N}$ *such that the tail* (x_{n+k}) *converges to* L*.*

Remark 107. *To prove a theorem which states multiple ideas are equivalent one does not need to prove "if an only if" for each pair of possible matchups. Instead, one can prove "circularly". For example, if there were four statements A, B, C, and D, a proof could be to show equivalence via*

$$A \Rightarrow B \Rightarrow D \Rightarrow C \Rightarrow A.$$

Definition 108. *Let* $\{x_n\}$ *be a bounded sequence of real numbers. A* subsequential limit *is any real number* x *such that there exists a subsequence* $\{x_{n_p}\}$ *that converges to* x*.*

Exercise 109. *Find the set of subsequential limits of* (x_n) *where* $x_n = \cos(\frac{n\pi}{4})$.

Prove or Disprove 110. *There is no sequence whose set of subsequential limits is* $\{1/n : n \in \mathbb{N}\}$.

Definition 111. *A sequence* $\mathbf{x}$ *is called* increasing *(*decreasing*) if for every natural number* n *we have* $x_n \le x_{n+1}$ *(*$x_n \ge x_{n+1}$*). If instead of an ordinary inequality it is a strict inequality, then we refer to the sequence as strictly increasing (strictly decreasing). In either case, we refer to* $\mathbf{x}$ *as monotonic.*

Remark 112. *If a tail of a sequence is increasing (decreasing) we say the sequence is* eventually increasing *(*eventually decreasing*).*

Prove or Disprove 113. *If a sequence is eventually increasing and bounded, then the sequence converges.*

Exercise 114. *Let* $x_1 = 1$ *and* $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$*. Prove that* $\mathbf{x}$ *converges and find its limit. (Hint: this problem uses a lot of induction.)*

Prove or Disprove 115. *Every sequence has a monotone subsequence.*

Theorem 116. *(Bolzano - Weierstrass Theorem) Every bounded sequence has a convergent subsequence.*

Definition 117. *The ideas of convergent sequence and Cauchy sequence generalize to any metric space. Given a space* X *with metric* d *we say the sequence of points* $x_n \in X$

1. *converges to* $L \in X$ *if for every* $\varepsilon > 0$ *there exists a natural number* N *so that if* $n > N$*, then* $d(x_n, L) < \varepsilon$*.*

2. *is Cauchy in* X *if for every* $\varepsilon > 0$ *there exists a natural number* N *so that if* $n, m > N$*, then* $d(x_n, x_m) < \varepsilon$*.*

Remark 118. *Look at these and the definitions you have for convergent and Cauchy in* $\mathbb{R}$*. Understand how they are saying the same thing when* $X = \mathbb{R}$ *and* d *is Euclidean distance.*

Prove or Disprove 119. *Let the space X be the real numbers with the "discrete metric"*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

There is no such thing as a convergent sequence here.

Chapter 4

Functions and Continuity

We begin with the most general definition of a function.

Definition 120. *A function f from $X \subseteq \mathbb{R}$ to $Y \subseteq \mathbb{R}$ is a set of ordered pairs (x,y) in the Cartesian product $X \times Y$ for which each $x \in X$ is the first coordinate in exactly one ordered pair. We write this as $f : X \to Y$ and say that f maps X to Y.*

Remark 121. *We will adopt the standard functional notation $f(x)$ to represent the value of y in $(x,y) \in f$. The domain of f, denoted dom(f), is X and the range of f, denoted ran(f), is the set $\{y \in Y|$ there exists an $x \in X$ with $f(x) = y\}$. We say f maps X into itself if $Y \subseteq X$.*

Remark 122. *Although domains are important, we will mostly write f rather than $f : X \to Y$ and assume that unless stated otherwise f has as large a domain as possible, usually $\mathbb{R}$.*

Remark 123. *Let f and g be functions and $c \in \mathbb{R}$. We assume the reader is familiar with the definitions and properties of the functions named $f \pm g$, fg, f/g, $f \circ g$ and cf.*

Remark 124. *The elementary trigonometric, exponential, and logarithmic functions, and their respective properties, are assumed.*

Some additional terminology for functions include

- f is called *one-to-one*, sometimes abbreviated $1-1$, if for every $x, x' \in X$, $x \neq x'$ implies $f(x) \neq f(x')$.
- f is called *onto* if for every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.
- f is called a bijection if it is both one-to-one and onto.

For now we shall restrict ourselves to functions on $\mathbb{R}$.

Exercise 125. *Give an example of a function which is*

1. *one-to-one, but not onto*
2. *onto, but not one-to-one*
3. *neither one-to-one, nor onto*
4. *a bijection*

Definition 126. (Limit of a Function) *Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. We say* the limit of f as x approaches x_0 is L *if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that if $x \in X$ and $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. Notationally, we write this as*

$$\lim_{x \to x_0} f(x) = L.$$

Example 127. *Prove that* $\lim_{x \to 2} x^3 = 8$.

Proof: Let $\varepsilon > 0$. For this ε, let $\delta = \min\{1, \varepsilon/19\}$. Then if $|x-2| < \delta$ we have

$$|x^3 - 8| = |x-2| \cdot |x^2 + 2x + 4| < 19 \cdot |x-2| < 19\delta < 19 \cdot \varepsilon/19 = \varepsilon.$$

Since ε was arbitrary, this works for any positive epsilon and thus $\lim_{x \to 2} x^3 = 8$.

Exercise 128. *Show that* $\lim_{x \to 3} x^2 = 9$.

Exercise 129. *The part which reads $0 < |x - x_0|$ is important. Why do we not let $|x - x_0|$ be zero?*

Prove or Disprove 130. *If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = K$, then*

$$\lim_{x \to x_0} (f \pm g)(x) = L \pm K.$$

Definition 131. (Continuity of a Function) *Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous at a point x_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ with $|x - x_0| < \delta$ we have*

$$|f(x) - f(x_0)| < \varepsilon.$$

Note: this is continuity at a point.
We say f is a continuous function if it is continuous at every point $x_0 \in X$.

Remark 132. *The definitions of continuous at x_0 and the limit at x_0 look almost identical, but are not. It's important to see the difference and understand why they are not the same.*

Exercise 133. *Show that $f(x) = 3x + 2$ is continuous at $x = 1$.*

Exercise 134. *Show that $f(x) = 3x + 2$ is a continuous function.*

Exercise 135. *Show that* $f(x) = x^2$ *is a continuous function.*

Prove or Disprove 136. *Let* $c \in \mathbb{R}$. *If* $f : X \to Y$ *is continuous at* x_0, *then* cf *is continuous at* x_0.

Prove or Disprove 137. *If* $f : X \to Y$ *is continuous at* x_0, *then* $|f|$ *is continuous at* x_0.

Remark 138. *A function is bounded if its range is a bounded set in* $\mathbb{R}$. *In other words,* f *is bounded if there exist* $m, M \in \mathbb{R}$ *such that*

$$m \leq f(x) \leq M$$

for all x *in the domain of* f.

Theorem 139. *If* $f : X \to Y$ *is continuous at* x_0, *then there is an interval* I *containing* x_0 *so that* $f(x)$ *is bounded on* I.

Prove or Disprove 140. *If* $f : X \to Y$ *and* $g : X \to Y$ *are continuous at* x_0, *then*

1. $f + g$
2. fg
3. f/g

are continuous at x_0.

Remark 141. *There is another definition of continuous which is equivalent to the ordinary definition for real functions. This one utilizes sequences and while it is sometimes difficult to use for positive results (that is proving continuity) it works very well for negative results.*

Definition 142. *Let* $D \subseteq \mathbb{R}$. *A function* $F : D \to \mathbb{R}$ *is continuous at a point* x *if for every sequence* $(x_n) \subseteq D$ *such that* x_n *converges to* x *we have*

$$f(x_n) \text{ converges to } f(x).$$

Exercise 143. *Write the negation of Definition 142 to see how to use it to prove the function* f *is not continuous at the point* x.

Prove or Disprove 144. *If* $g : U \to V$ *is continuous at* x_0, *and* $f : V \to \mathbb{R}$ *is continuous at* $y_0 = g(x_0)$, *then* $f \circ g : U \to \mathbb{R}$ *is continuous at* x_0.

Exercise 145. *Let*

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f *is* not *continuous at* 0.

Definition 146. *Let I be an open interval, let $x_0 \in I$, and let f be a function defined on I (except possibly at x_0). Then*

1. *f has a* jump discontinuity *at x_0 if the one-sided limits of f exist at x_0, but are not equal.*
2. *f has a* removable discontinuity *at x_0 if $\lim_{x \to x_0} f(x)$ exists, but $f(x)$ either does not exist or has a value different from the limit.*

Exercise 147. *Let*

$$F(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ \\ a & \text{if } x = 1 \end{cases}$$

Determine the value of a so that F is continuous at $x = 1$.

Remark 148. *The next two theorems and the subsequent problems lead to many applications of continuity.*

Theorem 149. (Intermediate Value Theorem) *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If v is a number between $f(a)$ and $f(b)$, then there is a point $c \in (a,b)$ such that $f(c) = v$. (Hint: Assume $f(a) < v < f(b)$ and look at $S = \{x \in [a,b] \,|\, f(x) < v\}$.)*

Remark 150. *For awhile it was thought that a function having this Intermediate Value Property was the same as a continuous function. This was disproved by (among others) Gaston Darboux. Nowadays, functions with the Intermediate Value Property are called Darboux Functions.*

Theorem 151. (Extreme Value Theorem) *If $f : [a.b] \to \mathbb{R}$ is continuous on $[a,b]$, then there exist points c and d in $[a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.*

Exercise 152. *If $g : \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$, with $g(a) \cdot g(b) < 0$, then there exists a value $r \in [a,b]$ such that $g(r) = 0$.*

Definition 153. *Let $f : D \to \mathbb{R}$ be a function and let I be an interval for which $I \subseteq D$. Then the image of I under f is given by*

$$f(I) = \{\, f(x) \,|\, x \in I\}.$$

Prove or Disprove 154. *If f is continuous and nonconstant on the interval I, then $f(I)$ is an interval.*

Definition 155. *A function f is increasing (strictly increasing) on the interval (a,b) if for all $x,y \in (a,b)$ with $x < y$ we have $f(x) \leq f(y)$ ($f(x) < f(y)$). Decreasing and strictly decreasing are similarly defined. A function is called* monotone *on (a,b) if it is either increasing on (a,b) or decreasing on (a,b).*

Definition 156. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called* symmetric *at* x_0 *if*

$$\lim_{h\to 0^+}[f(x_0+h)+f(x_0-h)-2f(x_0)] = 0.$$

Prove or Disprove 157. *If* f *is continuous at* x_0*, then* f *is symmetric at* x_0*.*

Prove or Disprove 158. *If* f *is symmetric at* x_0*, then* f *is continuous at* x_0*.*

Definition 159. *Let* $D \subseteq \mathbb{R}$ *and* $f : D \to \mathbb{R}$*. We say* f *is* uniformly continuous *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $x, y \in D$ *with* $|x-y| < \delta$ *we have*

$$|f(x) - f(y)| < \varepsilon.$$

Remark 160. *Note how this differs from ordinary continuity. Then we fix x and* ε *and then find* δ*; that is, concentrating on one point we see how close the x's have to be in order for the* $f(x)$*'s to be a prescribed distance away. For uniform continuity we have when the x's are close, the* $f(x)$*'s are the given distance apart for all points.*

Prove or Disprove 161. $f(x) = 3x+2$ *is a uniformly continuous function.*

Prove or Disprove 162. $f(x) = x^2$ *is a uniformly continuous function.*

Theorem 163. *If* f *is continuous on the closed interval* $[a,b]$*, then* f *is uniformly continuous on* $[a,b]$*.*

Exercise 164. *Show the theorem above is not true if we replace* $[a,b]$ *with* (a,b)*.*

Exercise 165. *Show that* $f : [1,\infty) \to \mathbb{R}$ *given by* $f(x) = \sqrt{x}$ *is uniformly continuous.*

Theorem 166. *If* f *is uniformly continuous on the interval* $[a,c]$ *and* f *is uniformly continuous on the interval* $[c,b]$*, then* f *is uniformly continuous on* $[a,b]$*.*

Prove or Disprove 167. *If* f *is uniformly continuous on the interval* $[a,c)$ *and* f *is uniformly continuous on the interval* $[c,b]$*, then* f *is uniformly continuous on* $[a,b]$*.*

Theorem 168. *If* $f : [0,\infty) \to \mathbb{R}$ *is continuous on* $[0,\infty)$ *and uniformly continuous on* $[k,\infty)$ *for some* $k > 0$*, then* f *is uniformly continuous on* $[0,\infty)$*.*

Definition 169. *Let* $X \subseteq \mathbb{R}$ *and let* $\{f_n\}$ *and* f *be real functions with domain* X*. We say* f_n *converges pointwise to* f *if for each* $x_0 \in X$ *we have*

$$\lim_{n\to\infty} f_n(x_0) = f(x_0);$$

that is, for every $x_0 \in X$ *and every* $\varepsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that for* $n > N$

$$|f_n(x_0) - f(x_0)| < \varepsilon.$$

We write this as $f_n \xrightarrow{p} f$*.*

Exercise 170. *Define* $f_n : [-1,1] \to \mathbb{R}$ *by*

$$f_n(x) = \cos(nx)/\sqrt{n}$$

Determine the pointwise limit of f_n*.*

Exercise 171. *Define* $f_n : [0,\infty) \to \mathbb{R}$ *by* $f_n(x) = \frac{x^n}{1+x^n}$*. Find the pointwise limit of* f_n*.*

Exercise 172. *Let* $\{r_n\}$ *be an ordering of the rational numbers. Define* $f_n : \mathbb{R} \to \mathbb{R}$ *by*

$$f_n(x) = \begin{cases} 1 & x = r_k, k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Determine the pointwise limit of f_n*.*

Prove or Disprove 173. *The pointwise limit of a sequence of continuous functions is a continuous function.*

Definition 174. *Let* $X \subseteq \mathbb{R}$ *and let* $\{f_n\}$ *and* f *be real functions with domain* X*. We say* f_n *converges uniformly to* f *if for every* $\varepsilon > 0$ *there exists and* $N \in \mathbb{N}$ *such that if* $n > N$ *we have*

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in X$*. We denote this by*

$$f_n \stackrel{u}{\to} f.$$

Prove or Disprove 175. *The uniform limit of continuous functions is a continuous function.*

Remark 176. *As usual, our concentration is in the real line. These definitions we have, though, do not depend on the domain and codomain being* $\mathbb{R}$*. Here is a generalization of the* $\delta - \varepsilon$ *definition to any metric space along with an example.*

Definition 177. *Start with metric spaces* (S,d) *and* (S^*,d^*)*. A function* $f : S \to S^*$ *is continuous at* $s_0 \in S$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$d(s_0,s) < \delta \text{ implies } d^*(f(s_0), f(s)) < \varepsilon.$$

Remark 178. *Look back at the definition of continuity and see how this metric space definition is the same, but different.*

Let $S = \{(x_n) : (x_n) \text{ is a bounded sequence}\}$ and define distance between points in this space by

$$d((x_n),(y_n)) = \sup\{|x_n - y_n|\}.$$

The function $F : S \to S$, called the left shift operator, is given by $F(x_n) = (x_2, x_3, \ldots)$.

Prove or Disprove 179. *The left shift operator is continuous.*

Remark 180. *The sequence definition also exists in general metric spaces and is again helpful in showing something is discontinuous at a point.*

Definition 181. *Start with metric spaces* (S,d) *and* (S^*,d^*). *A function* $f : S \to S^*$ *is continuous at* $s_0 \in S$ *if for every sequence* (s_n) *in* S *converging to* s_0 *we have*

$$f(s_n) \to f(s_0)$$

in (S^*,d^*).

Prove or Disprove 182. *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \textit{otherwise} \end{cases}$$

is discontinuous at the $(0,0)$.

Remark 183. *This function is from the study of* separate versus joint continuity. *One of many types of generalized continuity currently being researched.*