JOURNAL OF INQUIRY-BASED LEARNING IN MATHEMATICS
No. **12**, (Mar. 2009)

Analysis[1]

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[1]If each of X and Y is a person, then the relation $X < Y$ indicates that the notes originated with Y and were subsequently modified by X, who takes full responsibility for the current version. While Y is credited with the genesis of the notes, s/he makes no claim to the accuracy of the current version which may or may not reflect her/his original version.

Contents

To the Student

Analysis is an area of mathematics, just as Algebra, Geometry, and Topology are areas of mathematics, and is usually defined heuristically or not at all. In your calculus sequence you learned about the topics of limits, continuity, differentiability, and integration at an introductory level. In this course, we will follow the same order that you followed in calculus, but we will spend more time on the mathematical structures than on the application of the concepts. We will define each concept rigorously and present material that you will recognize from calculus such as the Extreme Value Theorem, Mean Value Theorem, Rolle's Theorem, and the Fundamental Theorem of Calculus. From here, we will explore the notions of uniform continuity, uniform convergence, the existence and uniqueness of solutions to differential equations, and a bit of measure theory.

These notes are intended to be virtually self-contained, only assuming basic understanding of the real numbers. If you work through the first fifty problems independently, then you have mastered the topic that most universities would cover in a course titled "Real Analysis" or "Advanced Calculus." Universities typically offer such a course after a "transitional" course intended to move students away from working problems and toward a theoretical viewpoint of the subject.

The goal is to solve all the statements labeled as "Problems," "Theorems," or "Lemmas." The titles don't necessarily represent the level of difficulty because the real work in proving a theorem may have been done in a lemma or problem. A handful of the problems or theorems are labeled with (CA) which implies they require the use of the Completeness Axiom.

The remainder of this introduction is important only to those taking the course from me for credit.

All work presented or submitted is to be your own. You are *not* to discuss any problem with any one other than me, nor are you to look to any other reference such as a book or the internet for further guidance.

Grading for the course will be no less than the average of three grades: your presentation grade, your submission grade, and the average of your midterm and final exam grades. Anyone who is regularly presenting material at the board will certainly have adequate work for good submission grades and thus will likely do well on the midterm and final. The midterm and final grades are opportunities for those who do not regularly make it to the board. However, it is my experience that those who do not work toward successful presentations rarely do well on the midterm and final. Thus, I emphasize that the *goal* of the course for each student should be well prepared, well presented problems at the board. You will know your grades on submissions and tests. My policy for your presentation grade is:

D = the student made it to class every day, was attentive and alert, and his or her cell phone never rang

C = requirements for D plus made a few successful presentations

B = requirements for C plus made numerous successful presentations

A = requirements for B plus presented some truly impressive problems

Turn-ins. You must turn in exactly one "new" problem each week. A "new" problem means one that you have *not* turned in before. This problem should be neatly written and double spaced. You should label this problem with TURN IN at the top of the page along with your name, the problem number and the problem statement.

Grading for TURN IN assignments will be based on the following scale.

A = This is a correct proof.

B = You know how to prove the theorem but some of what you have written is not correct.

C = You have a mistake in your work or I do not understand what you have written, but I believe you have a good idea.

D = There is at least one major flaw in your argument.

Please understand that the purpose of the TURN IN assignments is to *teach* you to prove theorems. It is not expected that you started the class with this skill; hence, some low grades are to be expected. Do not be upset - just come see me.

Resubmissions. If you receive a grade of less than B, you may resubmit a TURN IN problem on the following week and I will average the two grades. You still must turn in a new problem as well and you only get one chance to resubmit. Please write RESUBMIT at the top. Feel free to come see me anytime if you do not understand my comments. It is expected that a certain amount of time in my office will be required to help students. I prefer to give guidance in my office rather than in class because this allows me to tailor the hints to the person who needs help, but I will always answer questions in class as well.

Boardwork. If you have solved a problem that is about to be presented at the board *or* you feel you have made significant progress on a problem that is about to be presented then you may opt to leave the room for the presentation. In this case, you may turn in a write up of this problem for credit as BOARD WORK. You must write BOARD WORK at the top of the page. There is no limit on the number of BOARD WORK problems you can submit.

Last Comments. Be sure that everything you turn in has your name, problem number, and problem statement on it. Be sure to double space and write either TURN IN, RESUBMIT, or BOARD WORK at the top.

I reserve the right to vary from these policies and probably will.

Chapter 1

Limit Points and Sequences

You will find undefined words in these notes such as *collection*, *element*, *in*, *member* and *number*. We assume that you have an intuitive understanding of these words and an intuitive understanding of the algebra associated with the real numbers.

Definition 1. *By a* **point** *is meant an element of the real numbers,* $\mathbb{R}$.

Definition 2. *By a* **point set** *is meant a collection of one or more points.*

Definition 3. *The statement that the point set M is* **linearly ordered** *means that there is a meaning for the words "less than" and "greater than" so that if each of a, b, and c is in M, then*

1. *if $a < b$ and $b < c$ then $a < c$ and*
2. *one and only one of the following is true:*
   - i. $a < b$,
   - ii. $b < a$, *or*
   - iii. $a = b$.

Axiom 1. $\mathbb{R}$ *is linearly ordered.*

Axiom 2. *If p is a point, then there is a point less than p and a point greater than p.*

When we refer to "two points," we adhere to standard usage of the English language and thus imply that they are not the same point. For example, if you went to "two stores" we would not assume that you visited the same store twice. On the other hand if we were to write, "let each of a and b be a point" then it would be possible that a is the same point as b.

Axiom 3. *If p and q are two points then there is a point between them, for example,* $(p+q)/2$.

Axiom 4. *If $a < b$ and c is any point, then $a + c < b + c$.*

Axiom 5. *If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$. If $c < 0$, then $a \cdot c > b \cdot c$.*

Axiom 6. *If x is a point, then x is an integer or there is an integer n such that $n < x < n + 1$.*

Definition 4. *The statement that the point set O is an* **open interval** *means that there are two points a and b such that O is the set consisting of all points between a and b.*

Definition 5. *The statement that the point set I is a* **closed interval** *means that there are two points a and b such that I is the set consisting of the points a and b and all points between a and b.*

In set notation,

$$(a, b) = \{x : x \text{ is a point and } a < x < b\}$$

and

$$[a, b] = \{x : x \text{ is a point and } a \leq x \leq b\}.$$

We do not use (a, b) or $[a, b]$ in the case $a = b$, although many mathematicians and texts do. We refer to a and b as the **endpoints** of the interval.

Definition 6. *If M is a point set and p is a point, the statement that p is a* **limit point** *of the point set M means that* every *open interval containing p contains a point of M different from p.*

Problem 1. *Show that if M is the open interval (a, b), and p is in M, then p is a limit point of M.*

Problem 2. *Show that if M is the closed interval $[a, b]$, and p is not in M, then p is not a limit point of M.*

Problem 3. *Show that if M is a point set having a limit point, then M contains 2 points. Must M contain 3 points? 4 points?*

Problem 4. *Show that if M is the set of all positive integers, then no point is a limit point of M.*

Problem 5. *Assume M is a point set and p is a point of M. Create a definition for "q is the first point to the left of p in M" by completing the following. "If M is a point set and p is a point in M..."*

Problem 6. *Assume M is a point set such that if p is a point of M, there is a first point to the left of p in M and a first point to the right of p in M. Is it true that M cannot have a limit point?*

Definition 7. *If each of A and B is a set, then the set defined by A* **union** *B is the set consisting of all elements that are in A or in B.*

Definition 8. *If each of A and B are sets, then the set defined by A* **intersection** *B is the set consisting of all elements that are in A and in B.*

If M is a set and m is a point then the notation $m \in M$ translates as "m is in M." $A \cup B$ is written in set notation as $A \cup B = \{x | x \in A \text{ or } x \in B\}$ and $A \cap B$ is written in set notation as $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Problem 7. *Show that if H is a point set and K is a point set and p is a limit point of $H \cap K$, then p is a limit point of H and p is a limit point of K.*

Problem 8. *Show that if H is a point set and K is a point set and every point of H is a limit point of K and p is a limit point of H, then p is a limit point of K.*

Problem 9. *If H is a point set and K is a point set and p is a limit point of $H \cup K$, then p is a limit point of H or p is a limit point of K.*

Problem 10. *Show that if M is the set of all reciprocals of positive integers, then* 0 *(zero) is a limit point of M.*

Up until now, the word *point* has meant a real number. From here forward, it may also be used to mean an ordered pair of real numbers, i.e. a point in the plane.

Definition 9. *The statement that f is a* **function** *means that f is a collection of points in the plane, no two of which have the same first coordinates.*

Definition 10. *If f is a function, then by the* **domain** *of f is meant the point set of all first coordinates of the ordered pairs in f, and by the* **range** *of f is meant the set of all second coordinates of the ordered pairs in f.*

We use the usual notation that if f if a function and x is a number in the domain of f, then $f(x)$ is the number which is the 2^{nd} coordinate of the point of f whose 1^{st} coordinate is x.

Definition 11. *A* ***sequence*** *is a function with domain the natural numbers and with range a subset of real numbers.*

If p is a sequence, then $p = \{(1, p(1)), (2, p(2)), (3, p(3)), \ldots\}$. Since writing p this way is cumbersome and the domain is always the natural numbers, we will denote sequences by listing only the points in the range of the sequence, $p(1), p(2), p(3), \ldots$. We'll further abbreviate this as: $p_1, p_2, p_3, \ldots$. The set $\{p_i : i = 1, 2, 3, \ldots\}$ denotes the range of the sequence. That is, $\{p_i : i = 1, 2, 3, \ldots\}$ denotes the point set to which the point x belongs if and only if there is a positive integer n such that $x = p_n$.

Definition 12. *The statement that the point sequence $p_1, p_2, \ldots$* **converges to the point x** *means that if S is an open interval containing x then there is a positive integer N such that if n is a positive integer and $n \geq N$ then $p_n \in S$.*

Definition 13. *The statement that the sequence $p_1, p_2, p_3, \ldots$* **converges** *means that there is a point x such that $p_1, p_2, p_3, \ldots$ converges to x.*

Problem 11. *For each positive integer n, let $p_n = 1 - 1/n$. Show that the sequence $p_1, p_2, p_3, \ldots$ converges to* 1.

Problem 12. *For each positive integer n, let $p_{2n-1} = 1/(2n-1)$ and let $p_{2n} = 1 + 1/2n$. Does the sequence $p_1, p_2, p_3, \ldots$ converge to 0?*

Problem 13. *For each positive integer n, let $p_{2n} = 1/(2n-1)$, and let $p_{2n-1} = 1/2n$. Show that the sequence $p_1, p_2, p_3, \ldots$ converges to* 0.

Problem 14. *Show that if the sequence $p_1, p_2, p_3, \ldots$ converges to the point x, and, for each positive integer n, $p_n \neq p_{n+1}$, then x is a limit point of the set which is the range of the sequence.*

Problem 15. *Show that if $p \neq 0$, then p is not a limit point of the set* $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

Problem 16. *Show that if c is a number and $p_1, p_2, p_3, \ldots$ is a sequence which converges to the point x, then the sequence $c \cdot p_1, c \cdot p_2, c \cdot p_3, \ldots$ converges to $c \cdot x$.*

Problem 17. *Show that if the sequence $p_1, p_2, p_3, \ldots$ converges to x and the sequence $q_1, q_2, q_3, \ldots$ converges to y, then the sequence $p_1 + q_1, p_2 + q_2, p_3 + q_3, \ldots$ converges to $x + y$.*

Definition 14. *The statement that p is the* **first point to the right of the point set M** *means that p is greater than every point of M and if q is a point less than p, then q is not greater than every point of M.*

Definition 15. *The statement that p is the* **right-most point of M** *means that p is in M and no point of M is greater than p.*

Problem 18. *Show that if M is a point set, then there cannot be both a right-most point of M and a first point to the right of M.*

Problem 19. *Show that if M is a point set and there is a point p which is the first point to the right of M, then p is a limit point of M.*

Theorem 1. *If the sequence $p_1, p_2, p_3 \ldots$ converges to the point x and y is a point different from x, then $p_1, p_2, p_3, \ldots$ does not converge to y.*

Definition 16. *The statement that the point set M is* **finite** *means that there is a positive integer n such that M contains n points but M does not contain $n+1$ points.*

Definition 17. *The statement that the point set M is* **infinite** *means that M is not finite.*

Theorem 2. *If M is a finite point set, then M has a right-most point and a left-most point.*

Theorem 3. *If the point p is a limit point of the point set M and S is an open interval containing p, then $S \cap M$ is infinite.*

Theorem 4. *If the sequence $p_1, p_2, p_3, \ldots$ converges to the point x and y is a point different from x, then y is not a limit point of $\{p_i : i = 1, 2, 3, \ldots\}$, the range of the sequence.*

Definition 18. *If A and B are point sets, then we say that A is a* **subset** *of B if every point of A is also a point of B. This is typically denoted by $A \subseteq B$.*

Definition 19. *The statement that the point set M is an* **open** *point set means that for every point p of M there is an open interval which contains p and is a subset of M.*

Definition 20. *The statement that the point set M is a* **closed** *point set means that if p is a limit point of M, then p is in M.*

Note that if a set M has no limit point, then it is a closed point set. We could equivalently define closed by saying that M is closed if, and only if, there is no limit point of M that is not in M.

Theorem 5. *If M is a closed point set and M is not all points, then the set of all points not in M is an open point set.*

Theorem 6. *If M is an open point set, then the set of all points not in M is a closed point set.*

Theorem 7. *If p is a point, there is a sequence of open intervals $S_1, S_2, S_3, \ldots$ each containing p such that for each positive integer n, $S_{n+1} \subseteq S_n$, and p is the only point that is in every open interval in the sequence.*

Definition 21. *The statement that the point set M is* **bounded** *means that M is a subset of some closed interval.*

Definition 22. *Let M be a point set. The statement that M is* **bounded below** *means that there is a point z such that z is less than or equal to m for every m in M.* **Bounded above** *is defined similarly.*

Theorem 8. *If the sequence $p_1, p_2, p_3, \ldots$ converges to the point x, then $M = \{p_1, p_2, p_3 \ldots\}$ is bounded.*

Axiom 7. The Completeness Axiom *If M is a point set and there is a point to the right of every point of M, then there is either a right-most point of M or a first point to the right of M.*

Similarly, if there is a point to the left of every point of M, then there is either left-most point of M or a first point to the left of M.

Theorem 9. *(CA) If M is a closed and bounded point set, then there is a left-most point of M and a right-most point of M.*

Definition 23. *The statement that the sequence $p_1, p_2, p_3, \ldots$ is an* **increasing** *sequence means that for each positive integer n, $p_n < p_{n+1}$.*

Definition 24. *The statement that the sequence $p_1, p_2, p_3, \ldots$ is* **non-decreasing** *means that for each positive integer n, $p_n \leq p_{n+1}$.*

Decreasing and **non-increasing** sequences are defined similarly.

Theorem 10. *(CA) If $p_1, p_2, p_3, \ldots$ is a non-decreasing sequence and there is a point, x, to the right of each point of the sequence, then the sequence converges to some point.*

Problem 20. *Show that if M is a point set and p is a point and every closed interval containing p contains a point of M different from p, then p is a limit point of M.*

Problem 21. *Show that it is not true that if p is a limit point of a point set M, then every closed interval containing p must contain a point of M different from p.*

Problem 22. *True or false? If $[a,b]$ is a closed interval and G is a collection of open intervals with the property that every point in $[a,b]$ is in some open interval in G then there is a finite subcollection of G with the same property.*

Theorem 11. *If M has p as a limit point, then there exists either an increasing or a decreasing sequence of points of M converging to p.*

Chapter 2

Continuity

It is quite common for mathematicians to come up with more than one definition for a concept. Two definitions are said to be *equivalent* if a mathematical object satisfying either one of the definitions must also satisfy the other. The following are three equivalent definitions for continuity, one geometrical, one topological (based on open intervals), and one analytical (probably similar to one you saw in a calculus course).

You might review Definitions 9 and 10 and the discussion following these definitions before reading the next definition.

Definition 25. *The statement that the function f is* **continuous** *at the point* $p = (x,y)$ *means that*

1. *p is a point on f, and*
2. *if H and K are any two horizontal lines with p between them, then there are two vertical lines, h and k with p between them so that if t is any point in the domain of* f *between* h *and* k, *then* $(t, f(t))$ *is in the rectangle bounded by* $h, k, H,$ *and* K.

Definition 26. *The statement that the function f is* **continuous** *at the point* $p = (x,y)$ *means that*

1. *p is a point on f, and*
2. *if S is* **any** *open interval containing the number f(x), then there is an open interval T containing the number x such that if* $t \in T$, *and t is in the domain of* f, *then* $f(t) \in S$.

Definition 27. *The statement that the function f is* **continuous** *at the point* $p = (x,y)$ *means that*

1. *p is a point on f, and*
2. *if* ε *is any positive number, then there is a positive number* δ *so that if t is in the domain of* f *and* $|t-x| < \delta$, *then* $|f(t) - f(x)| < \varepsilon$.

Definition 28. *The statement that the function f is* **continuous** *at the number x means that x is in the domain of f and f is continuous at the point $(x, f(x))$.*

Definition 29. *The statement that f is a* **continuous function** *means that f is a function which is continuous at each of its points.*

Problem 23. *Let f be the function such that $f(x) = 2$ for all numbers $x > 5$, and $f(x) = 1$ for all numbers $x \leq 5$.*

1. *Show that f is not continuous at the point (5,1).*
2. *Show that if t is a number and $t > 5$, then f is continuous at $(t, 2)$.*

Problem 24. *Show that if f is a function and $(x, f(x))$ is a point on f, and x is not a limit point of the domain of f, then f is continuous at $(x, f(x))$.*

Problem 25. *Let f be the function such that $f(x) = x^2$ for all numbers x. Show that f is continuous at the point (2,4).*

Problem 26. *If f is a function which is continuous on $[a, b]$ and $x \in (a, b)$ such that $f(x) > 0$ then there exists an open interval, T, containing x such that $f(t) > 0$ for all $t \in T$.*

Theorem 12. *If f is a function and $x_1, x_2, x_3, \ldots$ is a sequence of points in the domain of f converging to the number x in the domain of f, and f is continuous at $(x, f(x))$, then $f(x_1), f(x_2), \ldots$ converges to $f(x)$.*

The converse of this statement is that if f is a function so that for every sequence $x_1, x_2, x_3, \ldots$ in the domain of f converging to a point x we have that $f(x_1), f(x_2), \ldots$ converges to $f(x)$ then f is continuous at x. This gives us a fourth equivalent definition for continuity of a function.

Definition 30. *We say that a function f is* **continuous** *at the point x if and only if for every sequence $p_1, p_2, p_3, \ldots$ in the domain of f converging to x we have that $f(p_1), f(p_2), \ldots$ converges to $f(x)$.*

Definition 31. *If f and g are functions and there is a point common to the domain of f and the domain of g, then $f + g$ denotes the function h such that for each number x in the domain of both of f and g, $h(x) = f(x) + g(x)$.*

Theorem 13. *If each of f and g is a function, x is a point in the domain of each of f and g, f is continuous at the point $(x, f(x))$, g is continuous at the point $(x, g(x))$, and $h = f + g$, then h is continuous at the point $(x, h(x))$.*

We can't prove everything in the given time, but we'll assume additional theorems as needed about continuity. For example we'll assume that under appropriate conditions, the product, quotient, and composition of continuous functions are continuous and that all polynomials are continuous.

Theorem 14. *Suppose f and g are functions having domain M and each is continuous at the point p in M. Suppose that h is a function with domain M such that f(p)=h(p)=g(p) and for each number x in M,* $f(x) \leq h(x) \leq g(x)$*. Prove h is continuous at p.*

Theorem 15. *(CA) If* $I_1, I_2, I_3, \ldots$ *is a sequence of closed intervals such that for each positive integer n,* $I_{n+1} \subseteq I_n$*, then there is a point p such that if n is any positive integer, then p is in* I_n*. In other words, there is a point p which is in all the closed intervals of the sequence* $I_1, I_2, I_3, \ldots$*.*

Theorem 16. *(CA) If* $I_1, I_2, I_3, \ldots$ *is a sequence of closed intervals so that for each positive integer n,* $I_{n+1} \subseteq I_n$*, and the length of* I_n *is less than* $\frac{1}{n}$*, then there is only one point p such that for each positive integer n,* $p \in I_n$*.*

Theorem 17. *If f is a continuous function whose domain includes the closed interval* $[a,b]$ *and there is a point x in* $[a,b]$ *so that* $f(x)$ *is greater than or equal to zero, then the set of all numbers* $x \in [a,b]$ *such that* $f(x) \geq 0$ *is a closed point set.*

Theorem 18. *If f is a continuous function whose domain includes a closed interval* $[a,b]$ *and* $p \in [a,b]$*, then the set of all numbers* $x \in [a,b]$ *such that* $f(x) = f(p)$ *is a closed point set.*

Definition 32. *The statement that the point sets H and K are* **disjoint** *or* **mutually exclusive** *means that they have no point in common.*

Theorem 19. *(CA) No closed interval is the union of two mutually exclusive closed point sets.*

Problem 27. *(CA) If f is a function with domain the closed interval* $[a,b]$ *and the range of f is* $\{-1,1\}$*, then there is a number x in* $[a,b]$ *at which f is not continuous.*

Theorem 20. *(CA) Let f be a continuous function whose domain includes the closed interval* $[a,b]$*. If* $f(a) < 0$ *and* $f(b) > 0$*, then there is a number x between a and b such that* $f(x) = 0$*.*

Theorem 21. *If f is a continuous function whose domain includes a closed interval* $[a,b]$*, and L is a horizontal line, and* $(a, f(a))$ *is below L, and* $(b, f(b))$ *is above L, then there is a number x between a and b such that* $(x, f(x))$ *is on L.*

Chapter 3

Differentiability

As with continuity, we offer three equivalent definitions of derivative, one geometric, topological, and one analytical.

Definition 33. *The non-vertical line L is* **tangent to the function f at the point** $P = (x, y)$ *means that:*

1. *x is a limit point of the domain of f,*
2. *P is a point of L, and*
3. *if A and B are non-vertical lines containing P with the line L between them (except at P), then there are two vertical lines H and K with P between them such that if Q is a point of f between H and K which is not P, then Q is between A and B.*

In the previous definition we write that we have three distinct lines, A, B, and L with L *between* A and B (except at P). By this we mean that for any point l on L (except P) there is a point a on A and a point b on B so that either a is below l which is below b or that b is below l which is below a.

Definition 34. *If f is a function, then the statement that f has a* **derivative** *at the number a in the domain of f means that f has a non-vertical tangent line at the point $(a, f(a))$. We use the notation $f'(a)$ to denote the slope of the line tangent to f at the point $(a, f(a))$ and $f'(a)$ is called the* **derivative of** f **at** *a.*

Definition 35. *If f is a function, the statement that f has* **derivative** *D at the number x in the domain of f means that*

1. *x is a limit point of the domain of f, and*
2. *if S is an open interval containing D, then there is an open interval T containing x such that if t is a number in T and in the domain of f and $t \neq x$, then*
$$\frac{f(t) - f(x)}{t - x} \in S.$$

As an alternative to this definition:

Definition 36. *If f is a function, the statement that f has* **derivative** *D at the number x in the domain of f means that*

1. *x is a limit point of the domain of f, and*
2. *if ε is a positive number, then there is a positive number δ such that if t is in the domain of f and $|t-x| < \delta$ then $\left|\frac{f(t)-f(x)}{t-x} - D\right| < \varepsilon$.*

Problem 28. *Use any of the definitions of* **derivative** *to show that if $f(x) = x^2 + 1$ then $f'(3) = 6$.*

Problem 29. *Use the definition of tangent to show that if f is a function whose domain includes $(-1,1)$, and for each number x in $(-1,1)$, $-x^2 \leq f(x) \leq x^2$, then the x-axis is tangent to f at the point $(0,0)$.*

Problem 30. *Use any of the definitions of derivative to show that if f is a function whose domain includes $(-1,1)$ and for each number x in $(-1,1)$, $-x^2 \leq f(x) \leq x^2$, then the derivative of f at the point $(0,0)$ is 0.*

Theorem 22. *If f is a function, and x is in the domain of f, then f does not have two tangent lines at the point $(x, f(x))$.*

Definition 37. *If f is a function which has a derivative at some point, then the* **derivative of f** *is the function denoted by f', such that for each number x at which f has a derivative, $f'(x)$ is the derivative of f at x.*

Definition 38. *If M is a point set, then the* **closure** *of M is the set consisting of M together with any limit points of M. It is denoted by $Cl(M)$ or by $\overline{M}$.*

Theorem 23. *If M is a point set then $Cl(M)$ is a closed point set.*

From this point forward we may use $\mathbb{R}$ to represent the set of real numbers and D_f to denote the domain of f.

Theorem 24. *Suppose that f is a function that is differentiable at the point p and that $c \in \mathbb{R}$. Show that the function g defined by $g(x) = cf(x)$ for all $x \in D_f$ is also differentiable at the point p.*

Theorem 25. *Suppose that each of f and g are functions that are differentiable at the point p and that h is the function defined by $h(x) = f(x) + g(x)$ for all $x \in D_f$. Show that h is also differentiable at the point p.*

Theorem 26. *If f is a function, x is in the domain of f, and f has a derivative at $(x, f(x))$, then f is continuous at $(x, f(x))$.*

Theorem 27. *If f is a function, $D_f \subseteq [a,b]$, $x \in (a,b)$, $f(x) \geq f(t)$ for all $t \in (a,b)$, and f has a derivative at x, then $f'(x) = 0$.*

Problem 31. *Does there exist a function f defined and continuous on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$ and $f'(x) = 0$ at all but countably many points of $[0,1]$?*

We can't prove everything we need, but at this point, you could prove that all polynomials are differentiable. You could also prove all the theorems about differentiability: the power rule, constant rule, sum rule, product rule, and quotient rules.

Chapter 4

Riemann Integration

We have already shown that if M is a bounded point set, then either M has a right-most point or there is a first point to the right of M. We shall call this number, whichever it is, the **least upper bound of** M**,** and we will denote it by **lub(**M**)**. Similarly if a set M has a left-most point or a first point to the left of M, then we will refer to this point as the **greatest lower bound of** M and denote it by **glb(**M**)**. Some mathematicians use the notation, **supremum of** M and **infimum of** M respectively.

We won't present the next two problems, but you may use them if you need them.

Problem 32. *If H and K are bounded sets and $H \subseteq K$ then $glb(K) \leq glb(H)$.*

Problem 33. *If each of H and K are bounded sets and $H \oplus K = \{h+k : h \in H, k \in K\}$ then $glb(H) + glb(K) = glb(H \oplus K)$.*

Definition 39. *A* **bounded** *function is a function with bounded range.*

Definition 40. *If $[a,b]$ is a closed interval, by a* **partition** *of $[a,b]$ is meant a set of points $\{t_0, t_1, \ldots, t_n\}$ satisfying $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$.*

For the next four definitions, assume that f is a bounded function with domain the closed interval $[a,b]$.

Definition 41. *The statement that the number S is a* **Riemann sum** *for f on $[a,b]$ means that there is a partition $\{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ and a sequence $x_1, x_2, \ldots, x_n$ of numbers such that $x_i \in [t_{i-1}, t_i]$ for $i = 1, 2, 3, \ldots, n$ and $S = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1})$.*

Definition 42. *The statement that the number S is the* **upper Riemann sum** *for f on $[a,b]$ means that there is a partition $\{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ and a sequence $y_1, y_2, \ldots, y_n$ of numbers such that $y_i = lub\{f(x) | x \in [t_{i-1}, t_i]\}$ for $i = 1, 2, \ldots, n$ and $S = \sum_{i=1}^{n} y_i(t_i - t_{i-1})$.*

Definition 43. *We define the* **lower Riemann sum** *in the same way except that* $y_i = glb\{f(x)|x \in [t_{i-1}, t_i]\}$ *for each positive integer* $i = 1, 2, \ldots, n$.

If f is a bounded function with domain the closed interval $[a, b]$ and P is a partition of $[a, b]$, then $U_P(f)$ and $L_P(f)$ denote the upper and lower Riemann sums of f.

Problem 34. *Let* $f(x) = 0$ *for each number* x *in* $[0, 1]$ *except* $x = 0$, *and let* $f(0) = 1$. *Show that:*

1. *if* P *is a partition of* $[0, 1]$, *then* $0 < U_P f$,
2. *if* $\varepsilon > 0$, *then there is a partition* P *of* $[0, 1]$ *such that* $U_P f < \varepsilon$, *and*
3. *zero is the only lower Riemann sum for* f *on* $[0, 1]$.

Theorem 28. *If* $p_1, p_2, p_3, \ldots$ *is a sequence of points in the closed interval* $[a, b]$, *then there is a point in* $[a, b]$ *which is not in the sequence* $p_1, p_2, p_3, \ldots$.

Theorem 29. *If* x *is a limit point of the point set* M, *then there is a sequence of points* $p_1, p_2, p_3, \ldots$ *of* M, *all different and none equal to* x *which converge to* x.

Theorem 30. *If* $x_1, x_2, x_3, \ldots$ *is a sequence of distinct points in the closed interval* $[a, b]$, *then the range of the sequence has a limit point.*

A consequence of Theorem 30 is that every infinite bounded set has a limit point.

Theorem 31. *If* f *is a function with domain* $[a, b]$, *and* f *is continuous at each number in* $[a, b]$, *then the range of* f *is a closed point set.*

Theorem 32. *If* f *is a function with domain* $[a, b]$ *and* f *is continuous at each number in* $[a, b]$, *then the range of* f *is bounded.*

Theorem 33. *If* f *is a continuous function with domain* $[a, b]$, *then there is a number* $x \in [a, b]$ *such that if* $t \in [a, b]$, *then* $f(t) \leq f(x)$.

Definition 44. *If* f *is a bounded function with domain the closed interval* $[a, b]$, *then the* **upper integral** *from* a *to* b *of* f *is the greatest lower bound of the set of all upper Riemann sums for* f *on* $[a, b]$ *and is denoted by* ${}_U\int_a^b f$. *The* **lower integral** *from* a *to* b *of* f *is the least upper bound of the set of all lower Riemann sums for* f *on* $[a, b]$ *and is denoted by* ${}_L\int_a^b f$.

Definition 45. *If* f *is a bounded function with domain* $[a, b]$, *then the statement that* f *is* **Riemann integrable** *on* $[a, b]$ *means that* ${}_L\int_a^b f = {}_U\int_a^b f$.

When a function is Riemann integrable, we drop the subscripts U and L and refer to $\int_a^b f$ as the **Riemann integral** of f.

Theorem 34. *Show that if f is a function whose domain includes the closed interval $[a,b]$, and for each number x in $[a,b]$, $m \leq f(x) \leq M$, and $P = \{t_0, t_1, \ldots, t_n\}$ is any partition of $[a,b]$, then $U_P f \leq M(b-a)$ and $L_P(f) \geq m(b-a)$.*

Theorem 35. *If f is a bounded function with domain the closed interval $[a,b]$, and P is a partition of $[a,b]$, then $L_P(f) \leq U_P(f)$.*

Theorem 36. *If f is bounded on $[a,b]$ then the set of all Riemann sums of f is bounded.*

Theorem 37. *If f is a bounded function with domain $[a,b]$, and for each number x in $[a,b]$, $f(x) \geq 0$, and for some number z in $[a,b]$, $f(z) > 0$ and f is continuous at z, then ${}_U\int_a^b f > 0$.*

Definition 46. *The statement that the partition Q of the closed interval $[a,b]$ is a* **refinement** *of the partition P of $[a,b]$ means that $P \subseteq Q$.*

Theorem 38. *If P_1 and P_2 are partitions of [a,b] then there exists a partition Q of [a,b] so that Q is a refinement of both P_1 and P_2.*

Theorem 39. *If f is a bounded function with domain the closed interval $[a,b]$, P is a partition of $[a,b]$, Q is a partition of $[a,b]$, and Q is a refinement of P, then $L_P(f) \leq L_Q(f)$ and $U_P(f) \geq U_Q(f)$.*

Theorem 40. *If f is a bounded function with domain $[a,b]$, then ${}_L\int_a^b f \leq {}_U\int_a^b f$.*

Theorem 41. *If f is a continuous function with domain the closed interval $[a,b]$, and ε is a positive number, then there is a partition $\{x_0, x_1, x_2, \ldots, x_n\}$ of the closed interval $[a,b]$ such that for each positive integer i not larger than n, if u and v are two numbers in the closed interval $[x_{i-1}, x_i]$, then $|f(u) - f(v)| \leq \varepsilon$.*

Theorem 42. *If f is a continuous function with domain a closed interval, then the range of f contains only one value or it is a closed interval.*

Theorem 43. *If f is a bounded function with domain the closed interval $[a,b]$ and for each positive number ε, there is a partition P of $[a,b]$ such that $U_P(f) - L_P(f) < \varepsilon$, then f is Riemann integrable on $[a,b]$.*

Theorem 44. *If f is a continuous function with domain the closed interval $[a,b]$, then f is Riemann integrable on $[a,b]$.*

Definition 47. *A function f is* **increasing** *if for each pair of points x and y in the domain of f satisfying $x < y$ we have $f(x) < f(y)$. The function is* **non-decreasing** *if under the same assumptions we have $f(x) \leq f(y)$.*

Theorem 45. *Every non-decreasing bounded function on $[a,b]$ is Riemann integrable on $[a,b]$.*

Theorem 46. *If $[a,b]$ is a closed interval and $c \in (a,b)$ and f is integrable on $[a,c]$ and on $[c,b]$ and on $[a,b]$, then $\int_a^c f + \int_c^b f = \int_a^b f$.*

Definition 48. *If $[a,b]$ is a closed interval and f is integrable on $[a,b]$ then we define $\int_b^a f = -\int_a^b f$ and $\int_a^a f = 0$.*

Theorem 47. *If f is a continuous function with domain the closed interval $[a,b]$, then there is a number c in $[a,b]$ such that $\int_a^b f = f(c)(b-a)$.*

Theorem 48. *If f is a continuous function with domain the closed interval $[a,b]$ and F is the function such that for each number x in $[a,b]$, $F(x) = \int_a^x f$, then for each number c in $[a,b]$ F has a derivative at c and $F'(c) = f(c)$.*

Theorem 49. *If f is a function with domain the closed interval $[a,b]$ and f has a derivative at each point of $[a,b]$ and f' is continuous at each point in $[a,b]$, then $\int_a^b f' = f(b) - f(a)$.*

Chapter 5

Miscellany

Some material in this chapter is pre-requisite for the next chapter.

Theorem 50. *If f is continuous at the point p and K is a subset of the domain of f and p is a limit point of K, then $p \in Cl(f(K))$.*

Theorem 51. *If f is an integrable function, then $|\int_a^b f| \leq \int_a^b |f|$.*

Lemma 52. *Suppose f is a function whose domain includes $[a,b]$, $f(a) = 0 = f(b)$, and f has a derivative at each of its points. Then there is a number c in (a,b) such that $f'(c) = 0$.*

Theorem 53. *If f is continuous on $[a,b]$ and $g(x) = \int_a^x f$ for all $x \in [a,b]$ then g is continuous on $[a,b]$.*

Theorem 54. *Suppose f is a non-decreasing function whose domain includes $[a,b]$ and f has a derivative at each of its points, then there is a number c in (a,b) such that $f'(c)$ is the same as the slope of the line joining the two points $(a, f(a))$ and $(b, f(b))$.*

Although we stated the previous theorem only for non-decreasing functions, it is valid for any differentiable function defined on $[a,b]$ and differentiable on (a,b). You may use the more general statement if you require it later.

Definition 49. *A set is* **countable** *if it is the range of some sequence.*

Theorem 55. *All finite sets, the natrual numbers, the integers and the rationals numbers are countable.*

Theorem 56. *Every countable closed set has a point that is not a limit point of the set.*

Theorem 57. *The real numbers are not countable.*

Theorem 58. *If f is continuous on the closed interval $[a,b]$ and $M \subseteq [a,b]$ is closed then $f(M)$ is closed.*

Problem 35. *Show there exists a function f that is continuous at a point x which is a limit point of points at which f is not continuous.*

Problem 36. *Show that there exists a function f that is nowhere continuous on $[0,1]$.*

Definition 50. *If A and B are sets, then $A - B = \{x \in A : x \text{ is not in } B\}$.*

Theorem 59. *If M is a countable subset of $[a,b]$ then every point of M is a limit point of $[a,b] - M$.*

Definition 51. *A function f is* **uniformly continuous** *on the set M if for every $\varepsilon > 0$ there exists a number $\delta > 0$ so that if $u, v \in M$ and $|u - v| < \delta$ then $|f(u) - f(v)| < \varepsilon$.*

Theorem 60. *A function f is continuous on $[a,b]$ if and only if f is uniformly continuous on $[a,b]$.*

A stronger results holds: a function defined and continuous on any closed and bounded (compact) subset of the reals is uniformly continuous on that domain.

Problem 37. *Show that there exists a function f that is continuous on (a,b) but not uniformly continuous on (a,b).*

A shorthand for the sequence $a_1, a_2, a_3, \ldots$ is $(a_n)_{n=1}^{\infty}$.

Definition 52. *If $(a_n)_{n=1}^{\infty}$ is a sequence then the sequence of* **partial sums** *of $(a_n)_{n=1}^{\infty}$ is the (new) sequence defined by $S_N = \sum_{n=1}^{N} a_n$, $N = 1, 2, 3, \ldots$. If the sequence of partial sums $(S_N)_{N=1}^{\infty}$ converges then we define the point to which this sequence converges to be the* **infinite series** *associated with $(a_n)_{n=1}^{\infty}$ and denote it by $\sum_{i=1}^{\infty} a_i$.*

Theorem 61. *If $(a_n)_{n=1}^{\infty}$ is a sequence and $\sum_{i=1}^{\infty} |a_i|$ converges then $\sum_{i=1}^{\infty} a_i$ converges.*

Definition 53. *A function f is called a* **Lipschitz** *function if there exists $c \geq 0$ such that for every pair u, v in the domain of f, $|f(u) - f(v)| \leq c|u - v|$.*

Problem 38. *Show that there exists a function that is Lipschitz on $[a,b]$ but not differentiable on $[a,b]$.*

Theorem 62. *Show that every Lipschitz function is uniformly continuous.*

Definition 54. *If $f_1, f_2, f_3, \ldots$ is a sequence of functions with a common domain D then we say that $f_1, f_2, f_3,$* **converges pointwise** *on D if there is a function f defined on D so that for each $x \in D$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$.*

Definition 55. *If $f_1, f_2, f_3, \ldots$ is a sequence of functions with a common domain D then we say that $f_1, f_2, f_3,$* **converges uniformly** *on D if there is a function f defined on D so that for all $\varepsilon > 0$ there is a natural number N so that for all natural numbers $n > N$ and for all $x \in D$ we have $|f(x) - f_n(x)| < \varepsilon$.*

Problem 39. *Show there is a sequence of continuous functions, $f_1, f_2, f_3, \ldots$ converging pointwise to a function that is not continuous.*

Problem 40. *Show there is a sequence of differentiable functions, $f_1, f_2, f_3, \ldots$ converging pointwise to a function that is not continuous.*

Theorem 63. *Show that if $f_1, f_2, f_3, \ldots$ is a sequence of continuous functions converging uniformly to the function f then f is continuous.*

Chapter 6

Successive Approximations

In this section, we use successive approximations to demonstrate the existence of a unique solution to the differential equation, $y' = y, y(0) = 1$, which you will recall from calculus is the function, $E(x) = e^x$. There are many ways to define the "exponential function." Here are a few.

1. Define sequences and convergence, then show that the sequence, $a_n = (1+\frac{1}{n})^n$ for $n = 1, 2, \ldots$ is increasing and bounded above. Then apply the completeness axiom to assure that it converges to some number. Call that number e. Define general exponential functions of the form $f(x) = b^x$. When $b = e$ you have the natural exponential function.

2. Develop differential and integral calculus and then define the integral $L(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$. Show that this function is strictly increasing, hence one-to-one and then define a function E, the natural exponential function, to be the inverse of L.

3. Develop sequences, series, and convergence and show that for each real number, x, the series $\displaystyle\sum_{i=0}^{\infty} \frac{x^i}{i!}$ converges. Now define $E(x) = \displaystyle\sum_{i=0}^{\infty} \frac{x^i}{i!}$.

4. Develop differential and integral calculus and then consider the question, does there exist a function f that satisfies:

   i. $f(0) = 1$ and

   ii. $f'(t) = f(t)$ for all $t \in \mathbb{R}$?

All of these approaches lead to the functions $E(x) = e^x$ and $L(x) = \ln(x)$ that you are familiar with. It is the latter path that we take because it makes use of much of the analysis that you have already developed and serves as a brief introduction to series.

Problem 41 is a "warm-up" for the next sequence of problems. For this problem, assume that you do know that the function $E(x) = e^x$ exists and that you remember all your calculus(!) and that the usual rules of differentiation and integration apply. For *this problem only*, if you need a reminder of Taylor series, you may look at the web or a book.

Problem 41. *Successive approximations, Picard's iterates.*

1. *Compute the Taylor Series for* $E(x) = e^x$.
2. *Show that if* y *is differentiable on* $[0,1]$ *and* $y'(t) = y(t)$ *for all* $t \in [0,1]$ *and* $y(0) = 1$ *then* $y(t) = 1 + \int_0^t y$.
3. *Show that if* y *is differentiable on* $[0,1]$ *and* $y(t) = 1 + \int_0^t y$ *then* $y' = y$ *and* $y(0) = 1$.
4. *Let* $y_0 = 1$ *and* $y_n = y_0 + \int_0^t y_{n-1}$ *for all* $n = 1,2,\ldots$ *and compute by hand* $y_0, y_1, y_2, \ldots$.

Now that you've completed the "warm-up" exercise, forget that you know that there exists a function $E(x) = e^x$ and close your calculus book or website.

Theorem 64. *Suppose* $0 < r < 1$ *and define a sequence by* $S_n = \sum_{i=0}^{n} r^i$ *for all* $n = 1,2,\ldots$. *Show that* $S_1, S_2, S_3, \ldots$ *converges to* $\dfrac{1}{1-r}$.

Theorem 65. *Suppose that* c *is a number between* 0 *and* 1 *and* $a_0, a_1, a_2, \ldots$ *is a sequence of positive numbers and* $a_i < ca_{i-1}$ *for all* $i = 1,2,\ldots$ *and* $S_n = \sum_{i=0}^{n} a_i$ *for all* $n = 1,2,\ldots$. *Show that the sequence* $S_1, S_2, \ldots$ *converges.*

Theorem 66. *For each natural number* n *define the function* f_n *by* $f_n(x) = \sum_{k=0}^{n} \dfrac{x^k}{k!}$ *for every* $x \in [0,1]$. *Show that* f_n *is continuous on* $[0,1]$.

Theorem 67. *Let* $f_1, f_2, \ldots$ *be the sequence of functions defined in the previous problem and show that if* $x \in [0,1]$ *then* $f_1(x), f_2(x), f_3(x)\ldots$ *converges to some number.*

Since for each $x \in [0,1]$ the sequence $f_1(x), f_2(x), f_3(x)\ldots$, converges we may define a function f on $[0,1]$ as follows: for each $x \in [0,1]$ let $f(x)$ be the number to which $f_1(x), f_2(x), f_3(x)\ldots$ converges. Now we have that the sequence $f_1, f_2, f_3, \ldots$ converges *pointwise* to f on $[0,1]$.

Theorem 68. *Show that the sequence of functions defined in the previous problem converges uniformly on* $[0,1]$.

Theorem 69. *If $f_1, f_2, \ldots$ converges uniformly to f on $[0,1]$ and $s \in [0,1]$ and $\int_0^s f_n$ exists for all $n = 1, 2, \ldots$ then the sequence of numbers $\int_0^s f_1, \int_0^s f_2, \ldots$ converges to the number $\int_0^s f$.*

Theorem 70. *Let f be the function defined by Theorem 66 and g be the function defined by $g(s) = 1 + \int_0^s f$. Show that $f = g$ on $[0,1]$ and $f(0) = 1$.*

Theorem 71. *Show that if L is the function with domain all differentiable functions and defined by $L(u) = u' - u$ then $y = 0$ is the unique solution to $L(y) = 0$ and $y(0) = 0$.*

Theorem 72. *Suppose $t_0, x_0 \in \mathbb{R}$ and show that there are not two solutions to $L(y) = 0$ and $y(t_0) = x_0$.*

Theorem 73. *Show there is a unique solution to the initial value problem $y'' + y = 0, y(0) = 0, y'(0) = 1$ as follows:*

1. *Convert the second order equation to a first order system, $\begin{pmatrix} u \\ v \end{pmatrix}' = A \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where A is a 2×2 matrix.*
2. *Apply Picard's iteration to obtain sequences of functions, $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$*
3. *Show that there are functions u and v so that $(u_n)_{n=1}^{\infty} \to u$ and $(v_n)_{n=1}^{\infty} \to v$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution to the .*

Chapter 7

Subsequences and Cauchy Sequences

Definition 56. *The statement that $q_1, q_2, q_3, \ldots$ is a* **subsequence** *of $p_1, p_2, p_3, \ldots$ means that there is an increasing sequence of natural numbers, $n_1, n_2, n_3, \ldots$ such that for each natural number i, we have $p_{n_i} = q_i$.*

Example: Suppose $p_1, p_2, p_3, \ldots$ is a sequence and n is a function with domain the natural numbers defined by $n(k) = 2k$. Then n defines the subsequence: $p_2, p_4, p_6, \ldots$.

Theorem 74. *Suppose that $q_1, q_2, q_3, \ldots$ is a subsequence of $p_1, p_2, p_3, \ldots$. Show that if there is a number x so that $p_1, p_2, p_3, \ldots$ converges to x then $q_1, q_2, q_3, \ldots$ converges to x.*

Problem 42. *Suppose that $q_1, q_2, q_3, \ldots$ is a subsequence of $p_1, p_2, p_3, \ldots$ and there is a number x so that $q_1, q_2, q_3, \ldots$ converges to x. Is it true that $p_1, p_2, p_3, \ldots$ converges to x?*

Problem 43. *Suppose that $(p_n)_{n=1}^{\infty}$ is a sequence of points in the closed interval $[a, b]$. Is it true that every subsequence of $(p_n)_{n=1}^{\infty}$ converges to some point in $[a, b]$?*

Definition 57. *A set of numbers K is* **compact** *if every sequence of points in K has a subsequence that converges to some point in K.*

Theorem 75. *Show that every closed interval is compact.*

Theorem 76. *If x is a limit point of $\{p_0, p_1, p_2, \ldots\}$ and every subsequence of $(p_n)_{n=1}^{\infty}$ converges then $(p_n)_{n=1}^{\infty}$ converges to x.*

Theorem 77. *Show that every closed and bounded set in $\mathbb{R}$ is compact.*

Previously, we proved that every infinite bounded set has a limit point. Now we have the equivalent to this statement for sequences, that every sequence with infinite bounded range has a convergent subsequence.

Definition 58. *The statement that the sequence $p_1, p_2, p_3, \ldots$ is a* **Cauchy sequence** *means that if ε is a positive number, then there is a positive integer*

N such that if n is a positive integer and m is a positive integer, $n \geq N$, and $m \geq N$, then the distance from p_n to p_m is less than ε.

Theorem 78. *The sequence $p_1, p_2, p_3, \ldots$ is a Cauchy sequence if and only if it is true that for each positive number ε, there is a positive integer N such that if n is a positive integer and $n \geq N$, then $|p_n - p_N| < \varepsilon$.*

Theorem 79. *If the sequence $p_1, p_2, p_3, \ldots$ converges to a point x, then $p_1, p_2, p_3, \ldots$ is a Cauchy sequence.*

Theorem 80. *If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the set $\{p_1, p_2, p_3, \ldots\}$ is bounded.*

Theorem 81. *If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the set $\{p_1, p_2, p_3, \ldots\}$ does not have two limit points.*

Theorem 82. *If $p_1, p_2, p_3, \ldots$ is a Cauchy sequence, then the sequence $p_1, p_2, p_3, \ldots$ converges to some point.*

Chapter 8

Basic Set Theory

We now modify our notion of *function* so that the domain and range are not restricted to subsets of the real numbers. From this point on, we will also allow the possibility that a set is empty.

Definition 59. *Given two sets X and Y, $X \times Y = \{(x,y) : x \in X, y \in Y\}$. A* **relation** *on $X \times Y$ is a subset of $X \times Y$. A* **function** *on $X \times Y$ is a relation on $X \times Y$ with the property that no two elements have the same first coordinates. The set of all first coordinates is called the* **domain** *of the function and the set of all second coordinates is called the* **range** *of the function.*

For a function f on $X \times Y$ we will write $f : X \to Y$ and if (u,v) is an element of f then we will use the notation, $f(u) = v$. In this case, we say that f **maps** u to v.

Definition 60. *If $f : X \to Y$ is a function, then f is* **injective** *(one-to-one) if no two elements of X map to the same element in Y. We say that f is* **surjective** *(onto) if for each element $y \in Y$ there is some element $x \in X$ such that $f(x) = y$. We call an injective function an* **injection***, a surjective function a* **surjection***, and a function that is both injective and surjective a* **bijection***.*

Every function $f : X \to Y$ is a surjection onto its range.

Theorem 83. *Let $f : X \to Y$ be a surjection. Show that f is injective if and only if there is a function $g : Y \to X$ so that $g(f(x)) = x$ for all $x \in X$.*

Definition 61. *Given a function $f : X \to Y$, the relation f^{-1} is defined by $f^{-1} = \{(v,u) : (u,v) \in f\}$.*

The set f^{-1} might not be a function. If f is injective then by Theorem 83 we have that f^{-1} is a function.

From this point forward, we may use (i) "$\exists$" to mean "there exists," (ii) "$\notin$" to mean "is not in" and (iii) "$\ni$" to mean "such that."

Definition 62. *If $f : X \to Y$ and $A \subseteq X$ then the* **image of** A **under** f *is* $\{f(x) : x \in A\}$ *and is denoted by* $f(A)$*. More precisely,* $f(A) = \{y \in Y : \exists x \in A \ni f(x) = y\}$.

Definition 63. *If $f : X \to Y$ and $A \subseteq Y$ then the* **inverse image of** A **under** f *is* $\{x \in X : f(x) \in A\}$ *and is denoted by* $f^{-1}(A)$*. This is often called the* **pre-image of** A.

Question 1. *Are problems 62 and 63 acceptable definitions or are they an abuse of notation?*

Theorem 84. *Show that if $f : X \to Y$ then f is surjective if and only if the inverse image of every non-empty subset of Y is non-empty.*

Definition 64. *Assume that each of A and B are subsets of the set X. Assume that Λ is a set and that A_λ is a subset of X for each $\lambda \in \Lambda$. Λ is called an index set.*

1. $\emptyset =$ *the empty set*
2. $A^c = \{x \in X : x \notin A\}$
3. $A \cup B = \{x \in X : x \in A \text{ or } x \in B\}$
4. $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$
5. $\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$
6. $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda \text{ for every } \lambda \in \Lambda\}$

We won't present the next theorem. You only need to write down a proof if you cannot write down a proof.

Theorem 85. *Assume that each of A, B and C are subsets of the set X.*

1. $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$
2. $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$
3. $A \cup X = X$ *and* $A \cap X = A$
4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. $(A^c)^c = A$, $A \cap A^c = \emptyset$, $\emptyset^c = X$, *and* $X^c = \emptyset$
6. $A \subseteq B \iff B^c \subseteq A^c$

Theorem 86. *Assume that Λ is a set and that A_λ is a set for each $\lambda \in \Lambda$. Show that*

1. $(\bigcup_{\lambda \in \Lambda} A_\lambda)^c = \bigcap_{\lambda \in \Lambda} (A_\lambda)^c$,
2. $(\bigcap_{\lambda \in \Lambda} A_\lambda)^c = \bigcup_{\lambda \in \Lambda} (A_\lambda)^c$,

3. $A \cap (\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcup_{\lambda\in\Lambda}(A \cap A_\lambda)$, *and*
4. $A \cup (\bigcap_{\lambda\in\Lambda} A_\lambda) = \bigcap_{\lambda\in\Lambda}(A \cup A_\lambda)$.

Theorem 87. *Assume that $f : X \to Y$ is a function, Λ is a set, and B_λ is a subset of Y for each $\lambda \in \Lambda$. Show that*

1. $f^{-1}(\bigcup_{\lambda\in\Lambda} B_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(B_\lambda)$, *and*
2. $f^{-1}(\bigcap_{\lambda\in\Lambda} B_\lambda) = \bigcap_{\lambda\in\Lambda} f^{-1}(B_\lambda)$.

Theorem 88. *Assume that $f : X \to Y$ is a function and D is a subset of Y. Show that $\left(f^{-1}(D)\right)^c = f^{-1}(D^c)$.*

Theorem 89. *Assume that $f : X \to Y$ is a function, $A \subseteq X$, and $D \subseteq Y$. Show that*

1. $f(f^{-1}(D)) \subseteq D$,
2. $f^{-1}(f(A)) \subseteq A$, *and*
3. *if f is surjective, then* $f(f^{-1}(D)) = D$.

Chapter 9

Measure Theory

Theorem 90. *If O is a bounded open set and p is a point of O then there is a unique open interval containing p which is a subset of O whose endpoints do not lie in O.*

Now that we know that the intervals described in Theorem 90 exist, we can formally define them and give them a name.

Definition 65. *If O is a bounded open set and $p \in O$ then the open interval containing p which is a subset of O and whose endpoints are not in O is called the* **component of O containing p**.

Theorem 91. *If O is a bounded open set, then the set of all components of O is a countable collection of mutually disjoint open intervals whose union is O.*

Definition 66. *If S is any interval (open, closed, or half-open), then we define L(S) to be the* ***length*** *of S. For example, if $S = [a,b]$, then $L(S) = b-a$. If G is a finite collection of mutually disjoint open intervals then L(G) denotes the sum of the lengths of the elements of G. If $G = \{g_1, g_2, g_3, \dots\}$ is a countable collection of mutually disjoint open intervals lying in an open interval, then $L(G) = \sum_{i=1}^{\infty} L(g_i)$.*

Theorem 92. *If G is a finite collection of mutually disjoint open intervals lying in the open interval (a,b), then $L(G) \leq b-a$.*

Theorem 93. *If G is a countable collection of mutually disjoint open intervals lying in the open interval (a,b) then $L(G) \leq b-a$.*

Definition 67. *If G is a collection of point sets then G^* denotes the set which is the union of the members of G; that is, $G^* = \bigcup_{g \in G} g$.*

Theorem 94. *If G and H are countable collections of mutually disjoint open intervals and $G^* \subseteq H^*$, then $L(G) \leq L(H)$.*

Definition 68. *A point set M is said to be* **closed** *if and only if no point not in M is a limit point of M.*

Theorem 95. *If O is an open set which is a subset of the closed interval [a,b], then [a,b]-O is a closed point set, and if M is a closed point set which lies in an open interval (a,b), then (a,b)-M is an open set.*

Definition 69. *The statement that* **the set G of open intervals properly covers the set M** *means that every point of M lies in a member of G and every member of G contains a point of M.*

Definition 70. *If M is a bounded point set then by the* **outer measure of M**, *denoted $m_o(M)$, is meant the greatest lower bound of the set of all $L(G)$ where G is any collection of mutually disjoint open intervals which properly cover M.*

Theorem 96. *Show that if M is a countable point set then the outer measure of M is zero.*

Theorem 97. *If O is a bounded open set and G is the set of all components of O, then $m_o(O) = L(G)$.*

Theorem 98. *If $[a,b]$ is a closed interval, then $m_o([a,b]) = b - a$.*

Theorem 99. *If M is a bounded point set and I and J are two closed intervals containing M, and $I \subset J$, then $m_o(I) - m_o(I - M) = m_o(J) - m_o(J - M)$.*

Theorem 100. *If M is a bounded point set and I and J are two closed intervals containing M, then $m_o(I) - m_o(I - M) = m_o(J) - m_o(J - M)$.*

Definition 71. *If M is a bounded point set, then* **the inner measure of M**, *denoted $m_i(M)$, means $m_o(I) - m_o(I - M)$ for some closed interval, I, containing M.*

Definition 72. *The statement that the point set M is* **measurable** *means that $m_o(M) = m_i(M)$. When M is measurable, $m_o(M)$ is called the measure of M and denoted by m(M).*

Theorem 101. *If M is a bounded, measurable set, and I is a closed interval containing M, then I-M is measurable.*

Theorem 102. *If H and K are disjoint, bounded sets, then $m_o(H \cup K) \leq m_o(H) + m_o(K)$.*

Theorem 103. *If H and K are disjoint, bounded measurable sets then $H \cup K$ is measurable.*

Theorem 104. *If H and K are disjoint, bounded measurable sets then $m(H \cup K) = m(H) + m(K)$.*

Theorem 105. *If M is a closed interval or an open interval, then M is measurable and $L(M) = m(M)$.*

Theorem 106. *If G is a collection of open intervals covering the closed interval $[a,b]$ then there is a finite subcollection of G which also covers $[a,b]$.*

Chapter 10

Conclusion

Congratulations! You have come a long way since the definition of a limit point. The theorems that you proved on your journey are essential to many areas of mathematics including topology, complex analysis, functional analysis, real variables and measure theory. Perhaps as important as the results is the fact that you proved many of them on your own. Sadly, many an undergraduate has graduated with a degree in mathematics without the ability to either prove theorems on his or her own, or even understand the proof of a theorem as presented by another. Because of this, I know first-hand of graduate programs where a course equivalent to this course is taken for *graduate* credit because incoming students are unprepared to prove theorems. Even a student who graduates in mathematics without this skill should at least have a deep appreciation for this process that is so fundamental to the nature of the subject. Many undergraduate programs have avoided the issue of teaching students to prove theorems because of the difficulty of this daunting task. Applied programs often have minimal courses designed to train students in creating mathematics, rather they emphasize learning and applying mathematical results. While both have value, many of the best applied mathematicians are also pure mathematicians because not every problem is a direct application of a theorem. Sometimes the theorems must be modified or built to fit the application at hand.

Now let's talk about some of the important results that you have developed during this semester. I will speak loosely here without the precision to which you have been accustomed. Consider this a furtherance of your mathematical training. Many of the mathematicians you will encounter in the future will not be as precise with the language as we have been in this class. Just translate their work into nice precise mathematics just as you translated my rough proofs into precise mathematics when you let (made?) me present material on those rare days when you did not have mathematics to show off to the class. You can start practicing on what I have written below.

We started with limit points and convergence, two important underlying concepts in analysis and topology. And we played a bit with the study of open and closed sets. This is at the very heart of topology and analysis because we define continuity in terms of open sets. Therefore, if we change the definition of an open set, then we change the continuous functions. Go ahead, change "open interval" to "closed interval" in the definition of continuity and ask, "Which functions are continuous now?" The theorems that we proved that were topological in nature were 15 and 16 which show that the intersection of a nested sequence of closed intervals results in a point or a closed interval. Together these are referred to as the *Nested Interval Theorem*. In topology you will see generalizations of this – that the arbitrary union of open sets is open and the arbitrary intersection of closed sets is closed. In Problem 27 you showed that no sequence could fill the closed interval, $[0,1]$. This shows that the real line is not countable since a set is countable precisely when there is some sequence whose range is that set. Problem 30 is the *Bolzano-Wierstrauss Theorem* and states that every infinite bounded set has a limit point. A consequence of this result that we used regularly was that every bounded sequence has a convergent subsequence.

We discussed four equivalent definitions of continuity, Definitions 25, 26, 27 and 30. The second will be generalized in a topological setting by writing that *a function is continuous if and only if the inverse image of an open set under f is open*. The third is the definition most often shown to calculus students. The fourth is the analyst's definition, that a function f is continuous if for every sequence converging to x, the sequence obtained by applying f to the original sequence converges to $f(x)$. You also proved that the sum of two continuous functions is continuous in Theorem 13. Along the way, we showed several properties of continuous functions. Together Theorems 32 and 33 showed that every continuous function on a closed interval has a maximum and a minimum value and attains those values. This is known as the *Extreme Value Theorem*. This theorem along with the *Intermediate Value Theorem*, Theorem 21, yielded that the range of a continuous function on a closed interval is either a point or a closed interval, Theorem 42.

Knowing that the maximum and minimum exist is not good enough. We must be able to find them, and for that we need derivatives. In calculus you talked a lot about tangent lines, but you probably did not accurately define a tangent line from a geometric point of view. If it was defined at all, it might have been defined by first defining the derivative in terms of limits and then defining the tangent line to f at $(x, f(x))$ to be the line with slope $f'(x)$ passing through $(x, f(x))$. Our approach was to offer a geometric definition of a tangent line in Definition 33 and then, if a function has a tangent line at a point, we offered a geometric definition of the derivative based on this tangent line. In total, you saw three equivalent definitions of the derivative, Definitions 34, 35, and 36. You then proved that the derivative is unique in Theorem 22 and that every differentiable function is continuous in Theorem 26. As soon as you mastered derivatives in Calculus I, you started applying them to find the maxima and minima of differentiable functions and you always sought out points where the derivative was zero. In Theorem 27 we proved what you used in your calculus course: the derivative is zero at both local maxima and local minima.

The capstone theorem of the first semester was the fact that every continuous function is Riemann integrable, Theorem 44, a result we extended in a few ways. In fact, a function that is continuous on an interval except on a subset of that interval with measure zero is still integrable. This explains the introduction to measure theory. In the section on measure theory, we proved the *Heine-Borel Theorem*, Theorem 106, which states that any open cover of a closed interval has a finite subcover. We proved the *Fundamental Theorem of Calculus*, Theorems 48 and 49, the *Mean Value Theorem for Integrals*, Theorem 47, *Rolle's Theorem*, Theorem 52, and the *Mean Value Theorem for Derivatives*, Theorem 54. We showed that uniform continuity and continuity are equivalent on the interval, a result extending to any compact domain. We extended our notion of convergence to sequences of functions, defining pointwise convergence and uniform convergence. This led us to powerful theorems such as 63 and 69, which state that the uniform limit of continuous functions is continuous and that if the limit of a sequence of functions is uniform, then we may interchange the integral and the limit. We introduced uniform limits, Lipschitz functions, and series (*Ratio Test*, Theorem 65) to prepare us for a nice application of analysis, the existence and uniqueness of solutions to differential equations. While we showed this only for two very elementary differential equations, the process used illustrates the underlying concept for proving *Picard's Existence Theorem*.

Still, even with all the theorems we proved, we left out a few. Did you miss them? What about the continuity of the product and composition of continuous functions? What about the product, quotient, and composition (chain) rules for derivatives? What about the fact that we can factor a constant out of an integral? There are still many more such theorems to prove, but once you have proven a handful of theorems about continuity, derivatives, and integration, the rest fall in much the same way using the techniques that you learned this semester.

As I revisited these ideas in graduate school, the way all these theorems served as tools for other areas of mathematics, and the way they all were extended and generalized to spaces other than the real line, was part of the beauty of the subject. I hope that I have shared a part of the beauty of the subject with you and that it serves as a springboard to higher mathematics.